Quadratic Dirichlet L-Series

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Let $D = 1$ or D be a fundamental discriminant [1]. The **Kronecker-Jacobi-Legendre symbol** (D/n) is a completely multiplicative function on the positive integers:

$$\left(\frac{D}{n}\right) = \begin{cases} \prod_{j=1}^{k} \left(\frac{D}{p_j}\right)^{e_j} & \text{if } n \geq 2, \\ 1 & \text{if } n = 1 \end{cases}$$

where $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ is the prime factorization of n,

$$\left(\frac{D}{p}\right) = \begin{cases} 1 & \text{if } p \nmid D \text{ and } x^2 \equiv D \bmod p \text{ is solvable,} \\ -1 & \text{if } p \nmid D \text{ and } x^2 \equiv D \bmod p \text{ is not solvable,} \\ 0 & \text{if } p \mid D \end{cases}$$

assuming p is an odd prime, and

$$\left(\frac{D}{2}\right) = \begin{cases} 1 & \text{if } D \equiv 1, 7 \bmod 8, \\ -1 & \text{if } D \equiv 3, 5 \bmod 8, \\ 0 & \text{if } 2 \mid D. \end{cases}$$

The function $n \mapsto (D/n)$ is a real primitive Dirichlet character with modulus $|D|$. In particular, $(1/n) = 1$ always,

$$(-3/n)|_{n=1,2,3} = \{1, -1, 0\},$$

$$(-4/n)|_{n=1,2,3,4} = \{1, 0, -1, 0\},$$
$$(-7/n)|_{n=1,\ldots,7} = \{1, 1, -1, 1, -1, -1, 0\},$$
$$(-8/n)|_{n=1,\ldots,8} = \{1, 0, 1, 0, -1, 0, -1, 0\},$$
$$(5/n)|_{n=1,\ldots,5} = \{1, -1, -1, 1, 0\},$$
$$(8/n)|_{n=1,\ldots,8} = \{1, 0, -1, 0, -1, 0, 1, 0\},$$
$$(12/n)|_{n=1,\ldots,12} = \{1, 0, 0, 0, -1, 0, -1, 0, 0, 0, 1, 0\}.$$

Now define the **Dirichlet L-series associated to** (D/n):

$$L_D(z) = \sum_{n=1}^{\infty} \left(\frac{D}{n}\right) n^{-z}, \qquad \operatorname{Re}(z) > 1$$

which can also be written as an infinite product over primes:

$$L_D(z) = \prod_p \left(1 - \left(\frac{D}{p}\right) p^{-z}\right)^{-1}, \qquad \operatorname{Re}(z) > 1.$$

If $D = 1$, then $L_1(z) = \zeta(z)$, which can be analytically continued over the whole complex plane except for a simple pole at $z = 1$. For all other D, $L_D(z)$ can be made into an entire function with special values

$$L_D(1) = \begin{cases} \dfrac{\pi}{3\sqrt{3}} & \text{if } D = -3 \\ \dfrac{\pi}{4} & \text{if } D = -4 \\ \dfrac{\pi h(D)}{\sqrt{-D}} & \text{if } D < -4, \\ \dfrac{2h(D)\ln(\varepsilon)}{\sqrt{D}} & \text{if } D > 1 \end{cases} \qquad \text{(Dirichlet class number formula)}$$

where $h(D)$ is the ideal class number in the wide sense of the quadratic field $\mathbb{Q}(\sqrt{D})$, and ε is the fundamental unit of the integer subring $\mathbb{Z} + ((D + \sqrt{D})/2)\mathbb{Z}$. It follows that

$$L_{-7}(1) = \frac{\pi}{\sqrt{7}}, \qquad L_{-8}(1) = \frac{\pi}{2\sqrt{2}},$$

$$L_5(1) = \frac{2}{\sqrt{5}} \ln\left(\frac{1+\sqrt{5}}{2}\right), \qquad L_8(1) = \frac{\ln\left(1+\sqrt{2}\right)}{\sqrt{2}}, \qquad L_{12}(1) = \frac{\ln\left(2+\sqrt{3}\right)}{\sqrt{3}}.$$

The fact that $L_D(1) \neq 0$ leads to a proof of Dirichlet's theorem on arithmetic progressions r, $q+r$, $2q+r$, $\ldots$: There are infinitely many primes congruent to r modulo q if q, r are coprime [2].

A modification of an L-series $L_D(z)$, defined by [3]

$$L_D^*(z) = \begin{cases} (-D)^{z/2} \pi^{-z/2} \Gamma\left(\dfrac{z+1}{2}\right) L_D(z) & \text{if } D < 0, \\ D^{z/2} \pi^{-z/2} \Gamma\left(\dfrac{z}{2}\right) L_D(z) & \text{if } D > 0, \end{cases}$$

leads to the elegant functional equation $L_D^*(z) = L_D^*(1 - z)$.

We turn attention to the points $z = 2$, $z = 3$ and $z = 1/2$. If $D > 0$, closed-form expressions for $L_D(2)$ are known:

$$L_1(2) = \frac{\pi^2}{6}, \qquad L_5(2) = \frac{4\pi^2}{25\sqrt{5}},$$

$$L_8(2) = \frac{\pi^2}{8\sqrt{2}}, \qquad L_{12}(2) = \frac{\pi^2}{6\sqrt{3}}$$

but if $D < 0$, only numerical approximations apply:

$$L_{-3}(2) = 0.7813024128... \qquad ([4]),$$

$$L_{-4}(2) = G = 0.9159655941... \qquad (\text{Catalan's constant } [5]),$$

$$L_{-7}(2) = 1.1519254705..., \qquad L_{-8}(2) = 1.0647341710....$$

There is an unproven conjecture that [6, 7]

$$L_{-7}(2) = \frac{24}{7\sqrt{7}} \int_{\pi/3}^{\pi/2} \ln \left| \frac{\tan(t) + \sqrt{7}}{\tan(t) - \sqrt{7}} \right| dt$$

which has its origins in hyperbolic geometry and the Claussen function [8]. If $D < 0$, closed-form expressions for $L_D(3)$ are known:

$$L_{-3}(3) = \frac{4\pi^3}{81\sqrt{3}}, \qquad L_{-4}(3) = \frac{\pi^3}{32},$$

$$L_{-7}(3) = \frac{32\pi^3}{343\sqrt{7}}, \qquad L_{-8}(3) = \frac{3\pi^3}{64\sqrt{2}}$$

but if $D > 0$, only numerical approximations apply:

$$L_1(3) = \zeta(3) = 1.2020569031... \qquad (\text{Apéry's constant } [9]),$$

$$L_5(3) = 0.8548247666..., \qquad L_8(3) = 0.9583804545...,$$

$$L_{12}(3) = 0.9900400194....$$

By way of contrast, virtually nothing is known about $L_D(1/2)$ (regardless of the sign of D):

$$L_1(1/2) = -1.4603545088... \quad ([9, 10]),$$

$$L_{-3}(1/2) = 0.4808675576..., \qquad L_{-4}(1/2) = 0.6676914571... \quad ([10])$$

$$L_{-7}(1/2) = 1.1465856669..., \qquad L_{-8}(1/2) = 1.1004214095...,$$

$$L_5(1/2) = 0.2317509475......, \qquad L_8(1/2) = 0.3736917129...,$$
$$L_{12}(1/2) = 0.4985570024....$$

It is expected that $L_D(1/2) \neq 0$ always [11]; the Generalized Riemann Hypothesis (GRH) states that all zeroes of $L_D(z)$ in the strip $0 \leq \mathrm{Re}(z) \leq 1$ must lie on the central line $\mathrm{Re}(z) = 1/2$. A deeper conjecture, known as the Grand Simplicity Hypothesis [12], asserts that the nonnegative imaginary parts of all such zeroes, taken as D varies across $1 \cup \{\text{fundamental discriminants}\}$, form a linearly independent set over $\mathbb{Q}$.

0.1. Various Moments. A discussion of the first and second moments of $L_D(1)$, over all fundamental discriminants $-x < D < 0$ and $0 < D < x$, appears in [1]. We will focus on $L_D(1/2)$ here. Many of the numerical results are due to Conrey, Farmer, Keating, Rubinstein & Snaith [13, 14].

Jutila [15, 16] proved that

$$\begin{aligned} \sum_{0<-D<x} L_D(1/2) &\sim \frac{3}{\pi^2}\left(a_{1,1}\ln(x) + a_{1,0}^-\right)x \\ &\sim (0.1070623764...)x\ln(x) + (0.0806503246...)x, \end{aligned}$$

$$\begin{aligned} \sum_{0<D<x} L_D(1/2) &\sim \frac{3}{\pi^2}\left(a_{1,1}\ln(x) + a_{1,0}^+\right)x \\ &\sim (0.1070623764...)x\ln(x) - (0.2556960505...)x \end{aligned}$$

as $x \to \infty$, where

$$P_1(s) = \prod_p \left(1 - \frac{1}{(p+1)p^s}\right),$$

$$a_{1,1} = P_1(1)/2 = (0.7044422009...)/2 = 0.3522211004...,$$

$$\begin{aligned} a_{1,0}^- &= \frac{P_1(1)}{2}\left(-1 - \ln(\pi) + 4\gamma + \frac{\Gamma'(3/4)}{\Gamma(3/4)} + 4\frac{P_1'(1)}{P_1(1)}\right) = 0.2653289331... \\ &= 0.6175500336... - a_{1,1} = 1.2648891165... - (1+\ln(2\pi))a_{1,1}, \end{aligned}$$

$$\begin{aligned} a_{1,0}^+ &= \frac{P_1(1)}{2}\left(-1 - \ln(\pi) + 4\gamma + \frac{\Gamma'(1/4)}{\Gamma(1/4)} + 4\frac{P_1'(1)}{P_1(1)}\right) = -0.8412062886... \\ &= -0.4889851881... - a_{1,1} = 0.1583538947... - (1+\ln(2\pi))a_{1,1}. \end{aligned}$$

The fact that $a_{1,1} > 0$ confirms that $L_D(1/2) > 0$ for infinitely many $D < 0$ and for infinitely many $D > 0$. Interestingly, the expression

$$\frac{P_1'(1)}{P_1(1)} = \sum_p \frac{\ln(p)}{p^2 + p - 1} = 0.4187575787...$$

resembles an expression in [17] for which the denominator is p^2-p+1 instead of p^2+p-1.

Jutila [15] also proved that [13, 14]

$$\begin{aligned}\sum_{0<-D<x} L_D(1/2)^2 &\sim \frac{3}{\pi^2}\left(a_{2,3}\ln(x)^3+a_{2,2}^{-}\ln(x)^2+a_{2,1}^{-}\ln(x)+a_{2,0}^{-}\right)x\\ &\sim (0.0037642089...)x\ln(x)^3+(0.0436478230...)x\ln(x)^2\\ &\quad +(0.0239243562...)x\ln(x)-(0.0664474558...)x,\end{aligned}$$

$$\begin{aligned}\sum_{0<D<x} L_D(1/2)^2 &\sim \frac{3}{\pi^2}\left(a_{2,3}\ln(x)^3+a_{2,2}^{+}\ln(x)^2+a_{2,1}^{+}\ln(x)+a_{2,0}^{+}\right)x\\ &\sim (0.0037642089...)x\ln(x)^3+(0.0081709895...)x\ln(x)^2\\ &\quad -(0.1388692446...)x\ln(x)+(0.4058928120...)x\end{aligned}$$

as $x\to\infty$, where

$$P_2=\prod_p\left(1-\frac{4p^2-3p+1}{(p+1)p^3}\right)=0.2972100247,$$

$$a_{2,3}=P_2/24=0.0123837510...$$

($a_{2,2}^{-}$, $a_{2,2}^{+}$, $a_{2,1}^{-}$, $a_{2,1}^{+}$ formulas appear in the Addendum). The work of Soundararajan [11], Diaconu, Goldfeld & Hoffstein [18] and Zhang [19] gives rise to the conjecture [13, 14]:

$$\begin{aligned}\sum_{0<-D<x} L_D(1/2)^3 &\sim \frac{3}{\pi^2}\left(a_{3,6}\ln(x)^6+\sum_{k=0}^{5}a_{3,k}^{-}\ln(x)^k\right)x+b^{-}x^{3/4}\\ &\sim (0.0000046457...)x\ln(x)^6+(0.0002447286...)x\ln(x)^5\\ &\quad +(0.0039480538...)x\ln(x)^4+(0.0174395675...)x\ln(x)^3\\ &\quad -(0.0110235234...)x\ln(x)^2-(0.0487615392...)x\ln(x)\\ &\quad +(0.1926975162...)x-(0.07...)x^{3/4},\end{aligned}$$

$$\begin{aligned}\sum_{0<D<x} L_D(1/2)^3 &\sim \frac{3}{\pi^2}\left(a_{3,6}\ln(x)^6+\sum_{k=0}^{5}a_{3,k}^{+}\ln(x)^k\right)x+b^{+}x^{3/4}\\ &\sim (0.0000046457...)x\ln(x)^6+(0.0001571591...)x\ln(x)^5\\ &\quad +(0.0007916339...)x\ln(x)^4-(0.0094598480...)x\ln(x)^3\\ &\quad +(0.0136781642...)x\ln(x)^2+(0.1643132466...)x\ln(x)\\ &\quad -(0.5385378337...)x-(0.14...)x^{3/4}\end{aligned}$$

as $x \to \infty$, where

$$P_3 = \prod_p \left(1 - \frac{12p^5 - 23p^4 + 23p^3 - 15p^2 + 6p - 1}{(p+1)p^6}\right) = 0.0440172316...,$$

$$a_{3,6} = P_3/2880 = 0.0000152837...$$

($a_{3,5}^-$, $a_{3,5}^+$, $a_{3,4}^-$, $a_{3,4}^+$ formulas appear in the Addendum). The exceptional term $x^{3/4}$ has no analog in the first and second moment cases. It is believed that [19]

$$\begin{aligned} b^- + b^+ &= \frac{223\sqrt{2} - 253}{192}\left(\frac{\Gamma(1/8)^4}{\Gamma(3/8)^4} + \frac{\Gamma(1/8)\Gamma(5/8)^3}{\Gamma(3/8)\Gamma(7/8)^3}\right)\pi Q \\ &= \frac{4}{3}(-0.1615725999...) = -0.2154301332... \end{aligned}$$

where [20]

$$\begin{aligned} Q &= \left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)^3 \zeta\left(\tfrac{1}{2}\right)^7 \\ &\quad \times \prod_{p>2}\left(1 - \tfrac{14}{p^{3/2}} - \tfrac{1}{p^2} + \tfrac{78}{p^{5/2}} - \tfrac{84}{p^3} - \tfrac{58}{p^{7/2}} + \tfrac{154}{p^4} - \tfrac{70}{p^{9/2}} - \tfrac{49}{p^5} + \tfrac{64}{p^{11/2}} - \tfrac{22}{p^6} + \tfrac{1}{p^7}\right) \\ &= -0.0019314869... \end{aligned}$$

(might separate expressions for b^+ and b^- be possible?) For arbitrary $n \geq 1$, Conrey & Farmer [21] conjectured that

$$\sum_{|D|<x} L_D(1/2)^n \sim \frac{6}{\pi^2} a_{n,N}\, x \ln(x)^N$$

as $x \to \infty$, where $N = n(n+1)/2$ and

$$a_{n,N} = \prod_{j=1}^{n} \frac{j!}{(2j)!} \cdot \prod_p \frac{\left(1 - \frac{1}{p}\right)^N}{1 + \frac{1}{p}} \left\{\frac{1}{2}\left(\left(1 - \tfrac{1}{\sqrt{p}}\right)^{-n} + \left(1 + \tfrac{1}{\sqrt{p}}\right)^{-n}\right) + \frac{1}{p}\right\}.$$

This is based in part on research in random matrix theory by Keating & Snaith [22, 23].

0.2. Dedekind Zeta Function. Given a fundamental discriminant D, define the **Dedekind zeta function** of $\mathbb{Q}(\sqrt{D})$ to be

$$\begin{aligned} \zeta_D(z) &= \zeta(z) \cdot L_D(z) \\ &= \prod_{\left(\frac{D}{p}\right)=1} (1 - p^{-z})^{-2} \cdot \prod_{\left(\frac{D}{p}\right)=-1} (1 - p^{-2z})^{-1} \cdot \prod_{\left(\frac{D}{p}\right)=0} (1 - p^{-z})^{-1} \end{aligned}$$

(the latter formula is valid for $\mathrm{Re}(z) > 1$). For example, if $D = -4$ (which corresponds to the ring $\mathcal{O}_{-1}$ of Gaussian integers), we have

$$\zeta_{-4}(z) = \prod_{\substack{p\equiv 1 \\ \text{mod } 4}} (1-p^{-z})^{-2} \cdot \prod_{\substack{p\equiv 3 \\ \text{mod } 4}} (1-p^{-2z})^{-1} \cdot (1-2^{-z})^{-1}$$

and if $D = -3$ (which corresponds to the ring $\mathcal{O}_{-3}$ of Eisenstein-Jacobi integers), we have

$$\zeta_{-3}(z) = \prod_{\substack{p\equiv 1 \\ \text{mod } 3}} (1-p^{-z})^{-2} \cdot \prod_{\substack{p\equiv 2 \\ \text{mod } 3}} (1-p^{-2z})^{-1} \cdot (1-3^{-z})^{-1}.$$

Since $\zeta_D(z)$ has a simple pole at $z = 1$, its Laurent expansion at $z = 1$ is

$$\zeta_D(z) = c_{-1}(z-1)^{-1} + c_0 + c_1(z-1) + c_2(z-1)^2 + \cdots, \qquad c_{-1} \neq 0.$$

Define the **Euler-Kronecker constant** of $\mathbb{Q}(\sqrt{D})$ to be

$$\gamma_D = \frac{c_0}{c_{-1}} = \gamma + \frac{L_D'(1)}{L_D(1)},$$

which generalizes Euler's constant $\gamma = 0.5772156649...$ [24]. (In the case $D = 1$, we merely have $\zeta_1(z) = \zeta(z)$ and thus $c_{-1} = 1$, $c_0 = \gamma$). It follows that [25, 26, 27, 28]

$$\gamma_{-4} = \ln\left(2\pi e^{2\gamma}\frac{\Gamma(\frac{3}{4})^2}{\Gamma(\frac{1}{4})^2}\right) = 0.8228252496... \qquad \text{(Sierpinski's constant [29])},$$

$$\gamma_{-3} = \ln\left(2\pi e^{2\gamma}\frac{\Gamma(\frac{2}{3})^3}{\Gamma(\frac{1}{3})^3}\right) = 0.9454972808... \qquad \text{([30])};$$

alternatively, by the Kronecker limit formula [31],

$$\gamma_{-4} = \frac{\pi}{3} - \ln(4) + 2\gamma - 4\sum_{k=1}^{\infty} \ln\left(1 - e^{-2\pi k}\right) = \tfrac{1}{2}(1.1870859072...)\ln(4),$$

$$\gamma_{-3} = \frac{\pi}{2\sqrt{3}} - \ln(3) + 2\gamma - 4\sum_{k=1}^{\infty} \ln\left|1 - e^{-2\pi i\omega k}\right| = \tfrac{1}{2}(1.7212574274...)\ln(3)$$

where $\omega = -(1 + i\sqrt{3})/2$ and i is the imaginary unit. Further, we have

$$\gamma_{-7} = \ln\left(2\pi e^{2\gamma}\frac{\Gamma(\frac{3}{7})\Gamma(\frac{5}{7})\Gamma(\frac{6}{7})}{\Gamma(\frac{1}{7})\Gamma(\frac{2}{7})\Gamma(\frac{4}{7})}\right) = 0.5928513548... = \tfrac{1}{2}(0.6093306571...)\ln(7),$$

$$\gamma_{-8} = \ln\left(2\pi e^{2\gamma}\frac{\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})}{\Gamma(\frac{1}{8})\Gamma(\frac{3}{8})}\right) = 0.5565042591... = \tfrac{1}{2}(0.5352439565...)\ln(8).$$

In the event that $D > 0$, the only known formulas are [28, 32, 33]

$$\begin{aligned}\gamma_5 &= \ln(2\pi e^{2\gamma}) + \frac{R(\frac{1}{5}) - R(\frac{2}{5}) - R(\frac{3}{5}) + R(\frac{4}{5})}{2\ln\left(\frac{1+\sqrt{5}}{2}\right)} = 1.4048951416... \\ &= \tfrac{1}{2}(1.7458208617...)\ln(5),\end{aligned}$$

$$\begin{aligned}\gamma_8 &= \ln(2\pi e^{2\gamma}) + \frac{R(\frac{1}{8}) - R(\frac{3}{8}) - R(\frac{5}{8}) + R(\frac{7}{8})}{2\ln\left(1+\sqrt{2}\right)} = 1.2093306309... \\ &= \tfrac{1}{2}(1.1631302027...)\ln(8),\end{aligned}$$

$$\begin{aligned}\gamma_{12} &= \ln(2\pi e^{2\gamma}) + \frac{R(\frac{1}{12}) - R(\frac{5}{12}) - R(\frac{7}{12}) + R(\frac{11}{12})}{2\ln\left(2+\sqrt{3}\right)} = 1.0539656082... \\ &= \tfrac{1}{2}(0.8482939255...)\ln(12)\end{aligned}$$

where

$$R(x) = -\frac{\partial^2}{\partial z^2}\zeta(z,x)\bigg|_{z=0}$$

and $\zeta(z,x)$ is the second derivative of the Hurwitz zeta function, defined when $0 < x \leq 1$ by $\zeta(z,x) = \sum_{n=0}^{\infty}(n+x)^{-z}$ for $\mathrm{Re}(z) > 1$ and by analytic continuation elsewhere. Of course, $\zeta(z,1) = \zeta(z)$, $\zeta(z,1/2) = (2^z - 1)\zeta(z)$, $\zeta'(0) = -\ln(2\pi)/2$ and

$$R(1) = -\zeta''(0) = -\tilde{\gamma} - \frac{1}{2}\gamma^2 + \frac{1}{24}\pi^2 + \frac{1}{2}\ln(2\pi)^2 = 2.0063564559...,$$

$$\lim_{x\to 0^+} R(x) = -\infty, \qquad R\left(\frac{1}{2}\right) = \ln(2)\ln(2\pi) + \frac{1}{2}\ln(2)^2 = 1.5141458137...$$

where $\tilde{\gamma} = -0.0728158454...$ is the first Stieltjes constant [34], but little else is known about special values of $R(x)$.

For small $|D|$, γ_D is positive. The first $D < 0$ for which γ_D is negative is $D = -47$, and the first $D > 0$ for which γ_D is negative is $D = 337$. It can be shown that $\lim_{|D|\to\infty}\gamma_D/\ln\sqrt{|D|} = 0$. For arbitrary number fields (finite algebraic extensions of $\mathbb{Q}$), a corresponding limit superior is also 0, assuming the truth of GRH. The corresponding limit inferior, however, appears to lie between -0.26049 and -0.17849, and its exact value is open [31, 35]. We wonder if similar optimization problems can be studied involving higher-order coefficients c_j in the Laurent expansion of $\zeta_D(z)$.

0.3. Prime Products. Formulas such as [36, 37]

$$\prod_p \frac{p^2+1}{p^2-1} = \frac{5}{2}, \qquad \prod_p \frac{p^3+1}{p^3-1} = \frac{945\zeta(3)^2}{\pi^6},$$

$$\prod_{p\equiv 1 \bmod 4} \frac{p^2+1}{p^2-1} = \frac{12G}{\pi^2}, \qquad \prod_{p\equiv 1 \bmod 4} \frac{p^3+1}{p^3-1} = \frac{105\zeta(3)}{4\pi^3},$$
$$\prod_{p\equiv 3 \bmod 4} \frac{p^2+1}{p^2-1} = \frac{\pi^2}{8G}, \qquad \prod_{p\equiv 3 \bmod 4} \frac{p^3+1}{p^3-1} = \frac{28\zeta(3)}{\pi^3}$$

offer hope that prime products $\prod_{p\equiv k \bmod l}(p^m+1)/(p^m-1)$ might always be expressed via L-series values, where $m \geq 2$. Indeed, we have

$$\prod_{p\equiv 1 \bmod 3} \frac{p^2+1}{p^2-1} = \frac{27L_{-3}(2)}{2\pi^2}, \qquad \prod_{p\equiv 1 \bmod 3} \frac{p^3+1}{p^3-1} = \frac{15\sqrt{3}\zeta(3)}{\pi^3},$$
$$\prod_{p\equiv 2 \bmod 3} \frac{p^2+1}{p^2-1} = \frac{4\pi^2}{27L_{-3}(2)}, \qquad \prod_{p\equiv 2 \bmod 3} \frac{p^3+1}{p^3-1} = \frac{39\sqrt{3}\zeta(3)}{2\pi^3}.$$

More complicated examples include

$$\prod_{p\equiv 2 \text{ or } 3 \bmod 5} \frac{p^2+1}{p^2-1} = \sqrt{5}, \qquad \prod_{p\equiv 2 \text{ or } 3 \bmod 5} \frac{p^3+1}{p^3-1} = \frac{124\zeta(3)}{125L_5(3)},$$

$$\prod_{p\equiv 7 \bmod 8} \frac{p^2+1}{p^2-1} = \frac{\pi^2}{\sqrt{64\sqrt{2}G\,L_{-8}(2)}}, \qquad \prod_{p\equiv 7 \bmod 8} \frac{p^3+1}{p^3-1} = \frac{\sqrt{1792\sqrt{2}\zeta(3)L_8(3)}}{\sqrt{3}\pi^3}$$

and we wonder whether products over $p \equiv 2 \bmod 5$, or products over $p \equiv 3 \bmod 5$, dash the hope. Finally, series such as

$$\sum_{n=0}^{\infty} \frac{1}{(3n+1)^2} = \frac{1}{2}\left(\frac{4\pi^2}{27} + L_{-3}(2)\right), \qquad \sum_{n=0}^{\infty} \frac{1}{(3n+1)^3} = \frac{2}{81\sqrt{3}}\pi^3 + \frac{13}{27}\zeta(3),$$
$$\sum_{n=0}^{\infty} \frac{1}{(3n+2)^2} = \frac{1}{2}\left(\frac{4\pi^2}{27} - L_{-3}(2)\right), \qquad \sum_{n=0}^{\infty} \frac{1}{(3n+2)^3} = -\frac{2}{81\sqrt{3}}\pi^3 + \frac{13}{27}\zeta(3),$$

$$\sum_{n=0}^{\infty} \frac{1}{(5n+2)^2} + \sum_{n=0}^{\infty} \frac{1}{(5n+3)^2} = \frac{10-2\sqrt{5}}{125}\pi^2,$$
$$\sum_{n=0}^{\infty} \frac{1}{(5n+2)^3} + \sum_{n=0}^{\infty} \frac{1}{(5n+3)^3} = \frac{62}{125}\zeta(3) - \frac{1}{2}L_5(3),$$

$$\sum_{n=0}^{\infty}\frac{1}{(8n+7)^2}=\frac{1}{4}\left(\frac{1+\sqrt{2}}{8\sqrt{2}}\pi^2-G-L_{-8}(2)\right),$$

$$\sum_{n=0}^{\infty}\frac{1}{(8n+7)^3}=-\frac{1}{4}\left(\frac{3+2\sqrt{2}}{64\sqrt{2}}\pi^3-\frac{7}{8}\zeta(3)-L_8(3)\right)$$

raise similar issues.

0.4. Primitive Characters. Let $\mathbb{Z}_n^*$ denote the group (under multiplication modulo n) of integers relatively prime to n, and let $\mathbb{C}^*$ denote the group (under ordinary multiplication) of nonzero complex numbers. A **Dirichlet character modulo** n is a homomorphism $\chi : \mathbb{Z}_n^* \to \mathbb{C}^*$. It can be shown that $\chi(k)$ is a $\varphi(n)^{\text{th}}$ root of unity for any $k \in \mathbb{Z}_n^*$, where φ is the Euler totient function. In particular, if χ is real-valued, then $\chi(k) = \pm 1$ for any k. We have [38, 39, 40, 41]

$$\begin{array}{l}\#\text{ complex Dirichlet characters}\\ \text{of modulus }\le N\end{array} = \sum_{n\le N}\varphi(n) \sim \frac{3}{\pi^2}N^2,$$

$$\begin{aligned}\begin{array}{l}\#\text{ real Dirichlet characters}\\ \text{of modulus }\le N\end{array} &= \sum_{\substack{n\le N,\\ n\equiv 2,6 \bmod 8}} 2^{\omega(n)-1} + \sum_{\substack{n\le N,\\ n\equiv 1,3,4,5,7 \bmod 8}} 2^{\omega(n)} + \sum_{\substack{n\le N,\\ n\equiv 0 \bmod 8}} 2^{\omega(n)+1}\\ &\sim \frac{6}{\pi^2}N\cdot\ln(N)\end{aligned}$$

as $N \to \infty$, where $\omega(n)$ denotes the number of distinct prime factors of n. The constant $6/\pi^2$ appears in [42] as the probability that two randomly chosen integers are coprime; the above three-fold summation also counts the average number of solutions of $x^2 = 1$ in $\mathbb{Z}_n^*$. Why should a coprimality probability and square roots of unity mod n be at all related to real characters mod n?

Let m be a multiple of n. Extend the domain of χ to $\mathbb{Z}$ via the formula

$$\chi(k) = \begin{cases}\chi(j) & \text{if } \gcd(n,k)=1 \text{ and } j\equiv k \bmod n,\ 1\le j\le n\\ 0 & \text{otherwise}\end{cases}$$

and then define a new **induced character** mod m:

$$\hat{\chi}(k) = \begin{cases}\chi(k) & \text{if } \gcd(m,k)=1,\\ 0 & \text{otherwise.}\end{cases}$$

For example, if χ is the character mod 3 with $\chi(1) = 1$, $\chi(2) = -1$ and $\chi(3) = 0$, note that

$$\chi(k)|_{k=1,\ldots,6} = \{1,-1,0,1,-1,0\} \longmapsto \{1,0,0,0,-1,0\} = \hat{\chi}(k)|_{k=1,\ldots,6}\,.$$

As another example, if χ is the character mod 3 with $\chi(1) = 1$, $\chi(2) = 1$ and $\chi(3) = 0$, note that

$$\chi(k)|_{k=1,\ldots,6} = \{1,1,0,1,1,0\} \longmapsto \{1,0,0,0,1,0\} = \hat{\chi}(k)|_{k=1,\ldots,6}\,.$$

As a third and fourth example, if χ is the character mod 1 with $\chi(1) = 1$, note that

$$\chi(k)|_{k=1,2} = \{1,1\} \longmapsto \{1,0\} = \hat{\chi}(k)|_{k=1,2}\,,$$

$$\chi(k)|_{k=1,2,3,4} = \{1,1,1,1\} \longmapsto \{1,0,1,0\} = \hat{\chi}(k)|_{k=1,2,3,4}\,.$$

These are meant to prepare us for the following definition. A **primitive character mod** m is a character that is not induced by a character mod n for any divisor n of m other than m itself. The first two examples demonstrate that no primitive character mod 6 exists. Likewise, no primitive character mod 2 exists, but the mod 4 character χ with $\chi(1) = 1$, $\chi(2) = 0$, $\chi(3) = -1$ and $\chi(4) = 0$ is primitive.

Define a new multiplicative function

$$\psi(n) = \sum_{d|n} \varphi(d)\mu(n/d)$$

where μ is the Möbius mu function. Also, $\psi(p) = p - 2$ and $\psi(p^l) = p^{l-2}(p-1)^2$ for $l \geq 2$, for any prime p. We have [43, 44]

$$\begin{array}{l}\#\text{ complex primitive Dirichlet}\\ \text{characters of modulus } \leq N\end{array} = \sum_{n \leq N} \psi(n) \sim \frac{18}{\pi^4} N^2,$$

$$\begin{array}{l}\#\text{ real primitive Dirichlet}\\ \text{characters of modulus } \leq N\end{array} = \sum_{|D| \leq N} 1 \sim \frac{6}{\pi^2} N$$

as $N \to \infty$, where D varies across the set $1 \cup \{\text{fundamental discriminants}\}$. For future convenience, the latter sum can be written more explicitly as $\sum_{n \leq N} \delta(n)$, where

$$\delta(n) = \begin{cases} 1 & \text{if } n = 1, \\ 1 & \text{if either } n \text{ or } -n \text{ is a fundamental discriminant (not both)}, \\ 2 & \text{if } n \text{ and } -n \text{ are fundamental discriminants}, \\ 0 & \text{otherwise.} \end{cases}$$

In fact, $\delta(n)$ is multiplicative with $\delta(2) = 0$, $\delta(4) = 1$, $\delta(8) = 2$, $\delta(2^l) = 0$ for $l > 3$, $\delta(p) = 1$ for prime $p > 2$ and $\delta(p^l) = 0$ for $l > 1$; thus asymptotic techniques in [41] are applicable.

A less stringent version of primitiveness is also available. A **weakly primitive character mod** m is a character that does not coincide (as a function $\mathbb{Z} \to \mathbb{C}^*$) with a character mod n for any divisor n of m other than m itself. For example, $\{1,0\}$ is weakly primitive as a character mod 2, but not as a character mod 4, since $\{1,0,1,0\}$ is the same as $\{1,0\}$ concatenated with itself. Both earlier-mentioned characters mod 6 are weakly primitive as well, but not mod 12.

Define another multiplicative function $\xi(n)$ with $\xi(p^l) = \psi(p^l)$ for $l \geq 2$, but $\xi(p) = p-1$ instead. A Dirichlet convolution-type formula for $\xi(n)$ is also available:

$$\xi(n) = \sum_{d|\kappa'(n)} \psi(n/d)$$

where $\kappa'(n)$ is the product of primes that occur with multiplicity 1 when factoring n (using notation from [45]). We have [46, 47]

$$\begin{array}{l}\#\text{ complex, weakly primitive Dirichlet} \\ \text{characters of modulus } \leq N\end{array} = \sum_{n\leq N} \xi(n) \sim \frac{1}{2}\rho N^2$$

where

$$\begin{aligned}\rho &= \prod_p \left(1 - \frac{p^2+p-1}{p^4}\right) = \frac{6}{\pi^2}\prod_p \left(1 + \frac{1}{p^3+p^2-1}\right)^{-1} \\ &= \frac{6}{\pi^2}(1.1344121384...)^{-1} = 0.5358961538...\end{aligned}$$

as $N \to \infty$.

Define one last multiplicative function $\eta(n)$ with $\eta(2) = 1$, $\eta(4) = 1$, $\eta(8) = 2$, $\eta(2^l) = 0$ for $l > 3$, $\eta(p) = 2$ for prime $p > 2$ and $\eta(p^l) = 0$ for $l > 1$. A Dirichlet convolution-type formula for $\eta(n)$ is also available:

$$\eta(n) = \sum_{d|\kappa'(n)} \delta(n/d).$$

We have [41]

$$\begin{array}{l}\#\text{ real, weakly primitive Dirichlet} \\ \text{characters of modulus } \leq N\end{array} = \sum_{n\leq N} \eta(n) \sim \sigma N \ln(N)$$

where

$$\sigma = \frac{6}{\pi^2}\prod_p \left(1 - \frac{2}{p(p+1)}\right) = \frac{36}{\pi^4}\prod_p \left(1 - \frac{1}{(p+1)^2}\right) = 0.2867474284...$$

as $N \to \infty$. The constant σ appears in [42] as the probability that three randomly chosen integers are pairwise coprime; it is also unexpectedly connected to the asymptotics of the average number of solutions of $x^3 = 0$ in $\mathbb{Z}_n$. Why should a coprimality probability and cubic roots of nullity mod n be at all related to weakly primitive characters mod n?

0.5. Addendum. Writing exact expressions for $L_D(1/2)$ moments is difficult. We have, for example [48],

$$a_{2,2}^{\pm} = c^{\pm} - 3a_{2,3}, \qquad a_{2,1}^{\pm} = d^{\pm} - 2c^{\pm} + 6a_{2,3}$$

where

$$c^{-} = \tfrac{P_2}{4}\left(\tfrac{1}{2}\tfrac{\Gamma'(3/4)}{\Gamma(3/4)} + U\right) = 0.1807468351..., \qquad c^{+} = \tfrac{P_2}{4}\left(\tfrac{1}{2}\tfrac{\Gamma'(1/4)}{\Gamma(1/4)} + U\right) = 0.0640327313...,$$

$$d^{-} = \tfrac{P_2}{2}\left[\left(\tfrac{4}{P_2}c^{-}\right)^2 - V\right] = 0.3658991414..., \qquad d^{+} = \tfrac{P_2}{2}\left[\left(\tfrac{4}{P_2}c^{+}\right)^2 - V\right] = -0.4030985462...$$

and

$$U = -\frac{1}{2}\ln(\pi) + 3\gamma + \sum_p \frac{5p^2 - 6p + 3}{(p-1)(p^3 + 2p^2 - 2p + 1)}\ln(p),$$

$$V = \gamma^2 + 2\tilde{\gamma} + \sum_p \frac{p\,(5p^5 - 5p^4 + 4p^3 + 4p^2 - 5p + 1)}{(p-1)^2(p^3 + 2p^2 - 2p + 1)^2}\ln(p)^2.$$

To obtain $a_{2,0}^{-}$ or $a_{2,0}^{+}$ involves even more complicated formulas. As another example [48],

$$a_{3,5}^{\pm} = c^{\pm} - 6a_{3,6}, \qquad a_{3,4}^{\pm} = d^{\pm} - 5c^{\pm} + 30a_{3,6}$$

where

$$c^{-} = \tfrac{P_3}{240}\left(\tfrac{1}{2}\tfrac{\Gamma'(3/4)}{\Gamma(3/4)} + U\right) = 0.0008968276..., \qquad c^{+} = \tfrac{P_3}{240}\left(\tfrac{1}{2}\tfrac{\Gamma'(1/4)}{\Gamma(1/4)} + U\right) = 0.0006087355...,$$

$$d^{-} = \tfrac{P_3}{48}\left[\left(\tfrac{240}{P_3}c^{-}\right)^2 - V\right] = 0.0170142017..., \qquad d^{+} = \tfrac{P_3}{48}\left[\left(\tfrac{240}{P_3}c^{+}\right)^2 - V\right] = 0.0051895362...$$

and

$$U = -\frac{1}{2}\ln(\pi) + 4\gamma + \sum_p \frac{4\,(3p^3 - 3p^2 + 3p - 1)}{(p-1)(p^4 + 4p^3 - 3p^2 + 3p - 1)}\ln(p),$$

$$V = \gamma^2 + 2\tilde{\gamma} + \sum_p \frac{p\,(10p^7 + 5p^5 + 17p^4 - 31p^3 + 20p^2 - 6p + 1)}{(p-1)^2(p^4 + 4p^3 - 3p^2 + 3p - 1)^2}\ln(p)^2.$$

Again, $a_{3,k}^{-}$ or $a_{3,k}^{+}$ are increasingly complicated for decreasing $k \le 3$.

For arbitrary $n \geq 1$, the rational function in p for the infinite series within U, needed to compute $c^{\pm}$ and $a^{\pm}_{n,N-1}$, is [48]

$$g_n(p) = \frac{n+1}{p-1} + \frac{-\left(\sqrt{p}-1\right)^{-n-1} + \left(\sqrt{p}+1\right)^{-n-1}}{\left(\sqrt{p}-1\right)^{-n} + \left(\sqrt{p}+1\right)^{-n} + 2p^{-n/2-1}}.$$

For arbitrary $n \geq 2$, the rational function in p for the infinite series within V, needed to compute $d^{\pm}$ and $a^{\pm}_{n,N-2}$, is

$$\frac{p}{(p-1)^2} + g_n(p)^2 - \frac{\left(\sqrt{p}-1\right)^{-n-2} + \left(\sqrt{p}+1\right)^{-n-2}}{\left(\sqrt{p}-1\right)^{-n} + \left(\sqrt{p}+1\right)^{-n} + 2p^{-n/2-1}}.$$

See also [49, 50].

The constant $\sum \ln(p)/(p^2+p+1)$ appears explicitly in [51] with regard to the reciprocal sum of the Dedekind totient. The conjectured expression for $L_{-7}(2)$ is, in fact, a theorem due to Zagier [52]; other representations appear in [53, 54]. More on Euler-Kronecker constants is found in [55, 56]. We hope to report on Mathar's calculations later [57, 58].

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