Lyapunov Exponents

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We are interested in iterates of the **logistic map** $T : [0,1] \to [0,1]$ defined by

$$T(x) = a\,x\,(1-x)$$

where $0 \leq a \leq 4$ is constant. Actually, only the values $a = 4$ and

$$a = \frac{2}{3}\left(\left(19+3\sqrt{33}\right)^{1/3} + 4\left(19+3\sqrt{33}\right)^{-1/3} + 1\right) = 3.6785735104...$$

will be examined (the latter has minimal polynomial $a^3 - 2a^2 - 4a - 8$). Both correspond to chaotic maps for which invariant probability densities $f(x)$ provably exist. An important feature of chaos is sensitivity to initial conditions. The **Lyapunov exponent** for each map quantifies the exponential rate at which two initially close points x, y separate [1]:

$$|T(x) - T(y)| \approx |T'(x)| \cdot |x-y|$$

after the first iteration,

$$|T^n(x) - T^n(y)| \approx \prod_{0 \leq j < n} \left|T'(T^j x)\right| \cdot |x-y|$$

after the n^{th} iteration, and hence

$$\frac{1}{n}\ln|T^n(x) - T^n(y)| \approx \frac{1}{n}\sum_{0 \leq j < n} \ln\left|T'(T^j x)\right|.$$

For X distributed according to f, let us write

$$\hat{\mu}_n(X) = \frac{1}{n}\sum_{0 \leq j < n} T^j X, \qquad \hat{\lambda}_n(X) = \frac{1}{n}\sum_{0 \leq j < n} \ln|T'(T^j X)|$$

which converge as $n \to \infty$ almost surely, by ergodicity, to

$$\mathrm{E}(X) = \int_0^1 x\,f(x)\,dx, \qquad \mathrm{E}\,|\ln(T'X)| = \int_0^1 \ln|T'(x)|\,f(x)\,dx.$$

Our study will encompass not only means, but also variances and autocovariances of arbitrary time lag. A complete solution is possible for $a = 4$; only partial results exist for $a = 3.678...$. The approach we take is similar to [2].

0.1. Ulam-von Neumann Map. When $a = 4$, the invariant density has a closed-form expression [3]:

$$f(x) = \frac{1}{\pi\sqrt{x(1-x)}}$$

and thus

$$\mathrm{E}(T^j X) = \frac{1}{2}, \qquad \mathrm{Var}(T^j X) = \frac{1}{8}, \qquad \mathrm{Cov}(T^j X, T^k X) = 0$$

for all $j < k$. Also [4],

$$\mathrm{E}\left|\ln(T'(T^j X))\right| = \ln(2), \qquad \mathrm{Var}\left|\ln(T'(T^j X))\right| = \frac{\pi^2}{12},$$

$$\mathrm{Cov}\left(\left|\ln(T'(T^j X))\right|, \left|\ln(T'(T^k X))\right|\right) = -\frac{\pi^2}{24}\frac{1}{2^{k-j}}$$

for all $j < k$. Clearly

$$\lim_{n\to\infty} \mathrm{E}\left(\hat{\mu}_n(X)\right) = \frac{1}{2},$$

$$\lim_{n\to\infty} n\,\mathrm{Var}\left(\hat{\mu}_n(X)\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{\substack{0\leq j<n,\\ 0\leq k<n}} \mathrm{Cov}(T^j X, T^k X) = \frac{1}{8}$$

and the Central Limit Theorem holds:

$$\lim_{n\to\infty} \mathrm{P}\left(2\sqrt{2n}\left(\hat{\mu}_n(X) - \frac{1}{2}\right) \leq t\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp\left(-\frac{u^2}{2}\right) du.$$

By contrast,

$$\lim_{n\to\infty} \mathrm{E}\left(\hat{\lambda}_n(X)\right) = \ln(2),$$

$$\begin{aligned} \lim_{n\to\infty} n^2\,\mathrm{Var}\left(\hat{\lambda}_n(X)\right) &= \lim_{n\to\infty} \sum_{\substack{0\leq j<n,\\ 0\leq k<n}} \mathrm{Cov}\left(\left|\ln(T'(T^j X))\right|, \left|\ln(T'(T^k X))\right|\right) \\ &= \lim_{n\to\infty} \frac{\pi^2}{6}\left(1 - \frac{1}{2^n}\right) = \frac{\pi^2}{6}. \end{aligned}$$

Estimates $\hat{\lambda}_n(X)$ of the Lyapunov exponent are anomalously precise [5]: they possess a standard deviation that scales as $1/n$ rather than $1/\sqrt{n}$. In this case, evidence points to a revised Central Limit Theorem of the form [4, 6]:

$$\lim_{n\to\infty} \mathrm{P}\left(n\left(\hat{\lambda}_n(X) - \ln(2)\right) \leq t\right) = \frac{2}{\pi^2} \int_{-\infty}^{t} \ln\left(\coth\left(\frac{u}{2}\right)\right) du$$

but a rigorous proof seems to be open.

0.2. Ruelle-Misiurewicz Map. When $a = 3.678...$, no closed-form expression for the invariant density is known, even though its existence is certain [7, 8, 9]. A numerical approach is necessary. Let $y = \frac{1}{2}a - a\,x$, then under the change of variables, T becomes

$$S(y) = y^2 - c$$

where

$$c = \frac{1}{4}a^2 - \frac{1}{2}a = 1.5436890126...$$

(with minimal polynomial $c^3 - 2c^2 + 2c - 2$). Now let [10]

$$\theta = \frac{1}{\pi}\arccos\left(\frac{y}{c - c^2}\right);$$

under this second change of variables, S^2 becomes

$$\tau(\theta) = \frac{1}{\pi}\arccos\left(\cos(2\pi\theta) + \kappa\sin(2\pi\theta)^2\right)$$

where

$$\kappa = \frac{c(c^2 - c) - 1}{2} = 0.1477988712....$$

The invariant density φ associated with $\tau : [0, 1] \to [0, 1]$ satisfies the functional equation [10]

$$\frac{\varphi(\tau^{-1}(\theta))}{|\tau'(\tau^{-1}(\theta))|} + \frac{\varphi(1 - \tau^{-1}(\theta))}{|\tau'(1 - \tau^{-1}(\theta))|} = \varphi(\theta)$$

where

$$\tau^{-1}(\theta) = \frac{1}{2\pi}\arccos\left(\frac{1 - \sqrt{1 + 4\kappa^2 - 4\kappa\cos(\pi\theta)}}{2\kappa}\right),$$

$$\tau'(\theta) = \frac{2\sin(2\pi\theta)\left(1 - 2\kappa\cos(2\pi\theta)\right)}{\sqrt{\sin(2\pi\theta)^2\left(1 - 2\kappa\cos(2\pi\theta) - \kappa^2\sin(2\pi\theta)^2\right)}}.$$

The left-hand side of this equation is a special case of the Frobenius-Perron operator $P_\tau\varphi(\theta)$. Starting with an initial guess $\varphi_0 \equiv 1$, the uniform density, iterates $\varphi_{n+1} = P_\tau\varphi_n$ converge to a limiting density φ. Backtracking through the two coordinate transformations, we obtain the desired invariant density f. It turns out to be supported on the intervals $[\frac{1}{a}, 1 - \frac{1}{a}]$ and $[1 - \frac{1}{a}, \frac{a}{4}]$, which are exchanged by T, with three vertical asymptotes.

Recall that $x = \frac{1}{2} - \frac{1}{a}y$ and $y = (c - c^2)\cos(\pi\theta)$. For X distributed according to f, we compute

$$\mathrm{E}(X) = \frac{1}{2}\int_0^1 (x + T(x))\,\varphi(\theta)d\theta = 0.6717404535...,$$

$$\mathrm{E}\left|\ln(T'X)\right| = \frac{1}{2}\int_0^1 \left(\ln|2y| + \ln|2S(y)|\right)\varphi(\theta)d\theta = 0.3421726886....$$

No one evidently has computed higher-order moments of X and $\ln(T'X)$, let alone $\hat{\mu}_n(X)$ and $\hat{\lambda}_n(X)$. Does the Central Limit Theorem need revision here too?

The value 3.678... is the simplest *Misiurewicz point*. For any $a \leq 3.678...$, the logistic map T admits no periodic point x of odd order > 1, i.e., it has no odd cycles. For any $a > 3.678...$, T has odd cycles [11, 12].

A graph of $\mathrm{E}(X)$, as a function of a, appears in [13]; the more familiar graph of $\mathrm{E}\left|\ln(T'X)\right|$ appears in [14]. In a sense, such plotting is meaningless, because there always exists finer detail than captured in whatever scale we choose [15].

Jakobson [16, 17] proved that the set $A = \{a \in [0,4] : T$ has an absolutely continuous invariant density$\}$ has positive measure. Both $4 \in A$ and $3.678... \in A$, but the status of values like 3.6, 3.7, 3.8 or 3.9 is unknown. Note: the condition that a density be *absolutely continuous* is important, yet outside our scope of study. What can be said about T for $a \notin A$? This question was satisfactorily answered only recently [18, 19].

The *metric entropy* of T can be proved to be equal to the Lyapunov exponent, but the *topological entropy* is altogether a different characterization [20, 21, 22, 23]. For the regular continued fraction transformation $T_{\mathrm{RCF}}(x) = \{1/x\}$, the metric entropy is $\pi^2/(6\ln(2))$ while the topological entropy is infinite [24]. The limit of $\mathrm{E}\left(\hat{\lambda}_n(X)\right)$ as $n \to \infty$ is $\pi^2/(6\ln(2))$; the limit of $n\,\mathrm{Var}\left(\hat{\lambda}_n(X)\right)$ as $n \to \infty$ is equal to $4(0.8621470373...)$ and, in fact, the Central Limit Theorem holds [2].

One-dimensional maps of the interval have inspired much computation [25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35]. We mention, for example, the maps $T_\ell : [0,1] \to [0,1]$ defined by

$$T_\ell(x) = 1 - |2x - 1|^\ell$$

for real $\ell > 1$. Clearly the case $\ell = 2$ gives the Ulam-von Neumann map. Each T_ℓ has an absolutely continuous invariant density with metric entropies (Lyapunov exponents) equal to [32, 34]

$$\begin{cases} \ln(2) = 0.6931471805... & \text{if } \ell = 2, \\ 0.6908569334... & \text{if } \ell = 3, \\ 0.6844935750... & \text{if } \ell = 4, \\ 0.6756910613... & \text{if } \ell = 5. \end{cases}$$

As another example, consider the map $S_0 : [0,1] \to [0,1]$ defined by

$$S_0(x) = \left\{2x + \frac{1}{4\pi}\sin(2\pi x)\right\}.$$

The absolutely continuous invariant density of S_0 has entropy equal to 0.6837719602.... It would be good someday to see such high-precision results for the logistic map, given values of a other than 3.678... and 4.

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