Lyapunov Exponents. II

Steven Finch

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Before discussing continuous-time systems, let us emphasize the definition of Lyapunov exponent λ for discrete-time systems in one-dimension [1]. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and

$$x_n = f(x_{n-1}), \qquad x_0 = u,$$

then λ quantifies the exponential rate at which two initially close points u, u_0 separate under the iteration:

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \ln |f^n(u) - f^n(u_0)| = \lim_{n\to\infty} \frac{1}{n} \ln |D_u f^n(u_0)\,(u - u_0)|$$

almost always. For example, $\lambda = \ln(2)$ is experimentally verified for the logistic case $f(x) = 4x\,(1-x)$ and $u_0 = 1/3$. This definition is meaningful as well for multi-dimensional maps $f : \mathbb{R}^m \to \mathbb{R}^m$. It is not true, however, that the norm of a product of Jacobian matrices is equal to the product of their norms; thus the calculational technique (based on the chain rule) used in [2] fails for $m > 1$.

Consider the **classical Lorenz system** [3, 4, 5, 6, 7]

$$\begin{cases} dx/dt = -10(x-y), & x(0) = 0, \\ dy/dt = 28x - y - x\,z, & y(0) = 1, \\ dz/dt = x\,y - \frac{8}{3}z, & z(0) = 0 \end{cases}$$

and define, for convenience,

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad F(X) = \begin{pmatrix} -10(x-y) \\ 28x - y - x\,z \\ x\,y - \frac{8}{3}z \end{pmatrix}.$$

Let $U = (u, v, w)$ denote a point that is close to the initial state $U_0 = (0, 1, 0)$. The solution of the perturbed system

$$dX/dt = F(X), \qquad X(0) = U$$

is written as $X(t; u, v, w)$. Differentiating both sides with respect to U, we obtain the **variational equation** [8, 9, 10, 11]

$$d\Phi/dt = D_X F(X)\,\Phi(t), \qquad \Phi(0) = I$$

where $\Phi(t;u,v,w)=D_U X(t;U)$ is a 3×3 matrix. The Lyapunov exponent λ for the Lorenz system satisfies [1]

$$\lambda=\lim_{t\to\infty}\frac{1}{t}\ln\left|X(t;U)-X(t;U_0)\right|=\lim_{t\to\infty}\frac{1}{t}\ln\left|D_U X(t;U_0)\,(U-U_0)\right|$$

almost always. It follows that [12]

$$\lambda=\lim_{t\to\infty}\frac{1}{t}\ln\left|\Phi(t;U_0)\right|$$

where $|M|$ denotes the 2-norm (largest singular value) of a matrix M. Equivalently, $|M|$ is the square root of the largest eigenvalue of $M^T M$. To compute λ, therefore, we must possess not only $(x(t),y(t),z(t))$ but also

$$\Phi(t)=\left(\varphi_{i,j}(t)\right)_{1\leq i\leq 3,\,1\leq j\leq 3}$$

where

$$\left\{\begin{array}{lll} d\varphi_{1,j}/dt=-10(\varphi_{1,j}-\varphi_{2,j}), & \varphi_{1,1}(0)=1, & \varphi_{1,2}(0)=\varphi_{1,3}(0)=0,\\ d\varphi_{2,j}/dt=(28-z)\varphi_{1,j}-\varphi_{2,j}-x\,\varphi_{3,j}, & \varphi_{2,2}(0)=1, & \varphi_{2,1}(0)=\varphi_{2,3}(0)=0,\\ d\varphi_{3,j}/dt=y\,\varphi_{1,j}+x\,\varphi_{2,j}-\frac{8}{3}\varphi_{3,j}, & \varphi_{3,3}(0)=1, & \varphi_{3,1}(0)=\varphi_{3,2}(0)=0 \end{array}\right.$$

for $j=1,2,3$. Difficulties arising from integrating this 12×12 ODE system include numerical overflow and numerical rank deficiency [12]. We obtain experimentally that $\lambda\approx 0.9$ via this approach; approximating $\Phi(t)$ as $t\to\infty$ to higher precision seems hopeless.

Using alternative approaches, Viswanath [12, 13, 14] and Sprott [15, 16, 17] independently computed that $\lambda=0.90563....$ It is known via rigorous numerics that the classical Lorenz system is chaotic [18, 19, 20, 21, 22, 23, 24] and that, indeed, almost all points in state space tend to a strange attractor (the famous *Lorenz butterfly*) [25, 26, 27]. No such behavior can possibly occur for continuous flows in one or two dimensions. The literature on calculating Lypanouv exponents is huge; we merely mention a few helpful surveys [28, 29, 30, 31, 32, 33, 34].

Although the Lorenz system was originally derived from a meteorological model of fluid convection, it can be more easily formulated in connection with the Malkus water wheel [4, 5, 6]. The wheel is free to rotate about a horizontal axis and its circumference is composed of small leaky cells. Water pours into the cells near the top of the wheel at a constant rate. Water leaks out of each cell at a rate proportional to the density of water inside. The mass of the wheel consists entirely of water confined to the circumference. As the wheel starts to rotate, new cells will move into position to receive the water. With the right balance between rates of water in-flow and out-flow, as well as frictional damping and gravitational acceleration, the Lorenz system emerges (governing, for example, angular velocity of the wheel). This is a fascinatingly simple illustration of chaos!

What is the algebraically simplest example of a dissipative chaotic flow? Sprott [35] conjectured that

$$\frac{d^3\xi}{d\,t^3}+\frac{2017}{1000}\frac{d^2\xi}{d\,t^2}-\left(\frac{d\,\xi}{d\,t}\right)^2+\xi=0,$$

with Lyapunov exponent 0.0551..., is one such case. For conservative flows,

$$\frac{d^3\xi}{d\,t^3}+\frac{d\,\xi}{d\,t}-\xi^2+\frac{1}{100}=0$$

may be algebraically simplest [36]. Upon setting $\eta=5\xi+1/2$, an equation resembling the logistic equation:

$$\frac{d^3\eta}{d\,t^3}+\frac{d\,\eta}{d\,t}+\frac{1}{5}\eta(1-\eta)=0$$

is the interesting outcome, with Lyapunov exponent 0.0964.... A survey of this line of thought is found in [15, 37, 38]. These examples deserve further analysis.

A single pendulum [39]

$$\begin{cases} d\theta/dt=\omega, \\ d\omega/dt=-(g/\ell)\sin(\theta) \end{cases}$$

cannot exhibit chaos. Only with the introduction of a nonautonomous driving term (and possibly a viscous damping term) can chaos arise: see [40, 41, 42, 43, 44, 45]. By contrast, a double pendulum [39, 46, 47, 48]

$$\begin{cases} \dfrac{d\theta_1}{dt}=\omega_1, \\ \dfrac{d\theta_2}{dt}=\omega_2, \\ \dfrac{d\omega_1}{dt}=\dfrac{-m_2\sin(\theta_1-\theta_2)\left(\ell_1\cos(\theta_1-\theta_2)\omega_1^2+\ell_2\omega_2^2\right)-\frac{g}{2}\left((2m_1+m_2)\sin(\theta_1)+m_2\sin(\theta_1-2\theta_2)\right)}{\ell_1\left(m_1+m_2-m_2\cos(\theta_1-\theta_2)^2\right)} \\ \dfrac{d\omega_2}{dt}=\sin(\theta_1-\theta_2)\dfrac{(m_1+m_2)\left(g\cos(\theta_1)+\ell_1\omega_1^2\right)+\ell_2 m_2\cos(\theta_1-\theta_2)\omega_2^2}{\ell_2\left(m_1+m_2-m_2\cos(\theta_1-\theta_2)^2\right)} \end{cases}$$

exhibits chaos if, for example, θ_1 is initially large ($\pi/2<\theta_1<\pi$) and $\theta_2=\omega_1=\omega_2=0$. (Point-mass m_1 determines angle θ_1 relative to a downward vertical axis at ceiling suspension; point-mass m_2 determines angle θ_2 relative to a downward vertical axis at m_1; the connecting rods of length ℓ_1, ℓ_2 are massless and no friction or forcing occurs; g is acceleration due to gravity.) The value of a Lyapunov exponent computed in [49] awaits confirmation.

We merely mention the interesting feedback control-theoretic problem of stabilizing an inverted pendulum on a moving cart [50, 51, 52, 53, 54, 55, 56, 57, 58]. Under the most ideal conditions, chaos cannot occur. If, however, we include realistic effects like time delay [59, 60], discrete sampling [61] or system friction [62], then chaos becomes possible again. More on a torque-driven pendulum (not cart-driven) and optimal control is found in [63].

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