Lyapunov Exponents. IV

Steven Finch

March 18, 2008

We are interested in the effects of multiplicative noise (continuing our study [1]). Let E_n denote matrix $N(0,1)$ white noise, that is, E_1, E_2, E_3, ... is a sequence of independent $m \times m$ matrices and all m^2 entries of E_n, for each n, are independent standard normal variables. Cohen & Newman [2] proved that the recurrence

$$X_n = E_n \, X_{n-1}, \qquad X_0 \neq 0 \text{ arbitrary}$$

gives rise to Lyapunov exponent

$$\tfrac{1}{n} \ln |X_n| \to \tfrac{1}{2}\left(\ln(2) + \psi(\tfrac{m}{2})\right) \qquad \text{almost surely as } n \to \infty$$

where $\psi(x)$ is the digamma function and $\gamma = -\psi(1)$ is the Euler-Mascheroni constant [3]. In particular, for $m = 1$,

$$x_n = \varepsilon_n x_{n-1}$$

has Lyapunov exponent $\lambda = -(\ln(2) + \gamma)/2$ and the following Central Limit Theorem holds:

$$\frac{\ln |x_n| - n\,\lambda}{\pi\sqrt{n/8}} \to N(0,1) \qquad \text{as } n \to \infty;$$

for $m = 2$,

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} \varepsilon_n & \varepsilon_n' \\ \varepsilon_n'' & \varepsilon_n''' \end{pmatrix} \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix}$$

has Lyapunov exponent $\lambda = (\ln(2) - \gamma)/2$ and

$$\frac{\ln \sqrt{x_n^2 + y_n^2} - n\,\lambda}{\pi\sqrt{n/24}} \to N(0,1) \qquad \text{as } n \to \infty.$$

Upon constraining certain entries of E_n, relevant Lyapunov exponent calculations become more complicated. Wright & Trefethen [4] found that $\lambda = \ln(1.0574735537...)$ when

$$\begin{pmatrix} x_n \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \varepsilon_{n+1} & 1 \end{pmatrix} \begin{pmatrix} x_{n-1} \\ x_n \end{pmatrix},$$

$\lambda = \ln(1.1149200917...)$ when

$$\begin{pmatrix} x_n \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & \varepsilon_{n+1} \end{pmatrix} \begin{pmatrix} x_{n-1} \\ x_n \end{pmatrix},$$

and $\lambda = \ln(0.9949018837...)$ when

$$\begin{pmatrix} x_n \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \varepsilon'_{n+1} & \varepsilon_{n+1} \end{pmatrix} \begin{pmatrix} x_{n-1} \\ x_n \end{pmatrix}.$$

Upon replacing standard normal variables ε_n by symmetric Bernoulli variables

$$\mathrm{P}(\varepsilon_n = 1) = \mathrm{P}(\varepsilon_n = -1) = 1/2,$$

the three preceding examples no longer possess distinct Lyapunov exponents. Viswanath [5, 6] proved that the three **random Fibonacci sequences** each have $\lambda = v$, where

$$v = \ln(1.1319882487...) = 0.1239755988...$$

was computed via a fractal invariance measure on the Stern-Brocot division of the real line. A high-precision estimate of v, due to Bai [7], was based on the cycle expansion method applied to a corresponding Ruelle dynamical zeta function [8, 9, 10]. It is interesting to compare the "almost-sure growth rate"

$$\tfrac{1}{n}\mathrm{E}(\ln|x_n|) \to v = \ln(1.1319882487...)$$

against the "average growth rate" [11, 12]

$$\tfrac{1}{n}\ln(\mathrm{E}|x_n|) \to \ln(\xi) = \ln(1.2055694304...)$$

where ξ has minimal polynomial $\xi^3 + \xi^2 - \xi - 2$. The latter value is larger due to outlying sequences that occur with very small probability. It is difficult to detect the difference experimentally since [13]

$$\tfrac{1}{n}\ln(\mathrm{Var}|x_n|) \to \ln(1+\sqrt{5})$$

and hence $\sim (1+\sqrt{5})^n$ datapoints are needed to estimate $\mathrm{E}|x_n|$ adequately.

Embree & Trefethen [14] examined the more general linear recurrence

$$x_{n+1} = x_n + \beta\,\varepsilon_{n+1} x_{n-1}$$

and determined that the critical threshold β^* (below which solutions decay exponentially almost surely; above which solutions grow exponentially almost surely) is

$\beta^* = 0.70258....$ It also appears that the value $\tilde{\beta}$ corresponding to maximal decay is $\tilde{\beta} = 0.36747...$ with Lyapunov exponent $\ln(0.8951...)$.

Chassaing, Letac & Mora [15] examined a different kind of random Fibonacci sequence:

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{cases} \begin{pmatrix} x_{n-1} + y_{n-1} \\ y_{n-1} \end{pmatrix} & \text{with probability } 1/2, \\ \begin{pmatrix} x_{n-1} \\ x_{n-1} + y_{n-1} \end{pmatrix} & \text{with probability } 1/2 \end{cases}$$

which reduces to the study of random products of the two nonnegative matrices:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Bai [16] computed that $\lambda = \ln(1.4861851938...) = 0.3962125642....$ Let $\varphi = (1 + \sqrt{5})/2$ denote the Golden mean [17]. Another variation is the random sequence:

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{cases} \begin{pmatrix} x_{n-1} + y_{n-1} \\ x_{n-1} \end{pmatrix} & \text{with probability } \varphi - 1 \approx 0.62, \\ \begin{pmatrix} y_{n-1} \\ x_{n-1} + y_{n-1} \end{pmatrix} & \text{with probability } 2 - \varphi \approx 0.38 \end{cases}$$

with associated nonnegative matrices:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

In this case, λ turns out to be $2v/(\varphi - 1)$, which constitutes another occurrence of Viswanath's constant [7].

Fix $\alpha > 0$. Chassaing, Letac & Mora [15, 18] proved that

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & \varepsilon_n \end{pmatrix} \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix}$$

has Lyapunov exponent

$$\lambda = \frac{K_0(\alpha)}{\alpha\, K_1(\alpha)}$$

where ε_n is distributed according to $\mathrm{Exp}(\alpha/2)$ and K_0, K_1 are modified Bessel functions [19]. If $\alpha = 2$, then $2\lambda = K_0(2)/K_1(2) = 0.8143077587....$ A related ratio $I_1(2)/I_0(2)$ appears in [20]; see also [1].

Lyons [21, 22] studied

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & \varepsilon_n \\ 1 & 1+\varepsilon_n \end{pmatrix} \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix},$$

where $\varepsilon_n = 0$ with probability $1/2$ and $\varepsilon_n = \tau$ otherwise. It turns out that $\tau \mapsto \lambda(\tau)$ is a strictly increasing function of $\tau > 0$. An important threshold value $\tau = 0.2688513727...$ is the solution of the equation [16]

$$2\lambda(\tau) = \ln(2)$$

and is connected with the distribution of certain random continued fractions.

Ishii [23, 24] proved that

$$\begin{pmatrix} x_n \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & c-\varepsilon_n \end{pmatrix} \begin{pmatrix} x_{n-1} \\ x_n \end{pmatrix}$$

has Lyapunov exponent

$$\lambda(c) = \operatorname{arccosh}\left(\frac{\sqrt{(2+c)^2+\delta^2}+\sqrt{(2-c)^2+\delta^2}}{4}\right)$$

where ε_n is distributed according to Cauchy(δ). If instead ε_n follows a Unif$(-\sqrt{3}\sigma, \sqrt{3}\sigma)$ distribution or a $N(0,\sigma^2)$ distribution, then asymptotic results of Derrida & Gardner [25, 26] apply:

$$\lim_{\sigma\to 0^+} \frac{\lambda(c,\sigma)}{\sigma^{2/3}} = \frac{6^{1/3}\sqrt{\pi}}{2\Gamma(1/6)} = 0.2893082598... \qquad \text{if } c = 2,$$

$$\lim_{\sigma\to 0^+} \frac{\lambda(c,\sigma)}{\sigma^2} = \begin{cases} 1/6 & \text{if } c = 1, \\ \dfrac{\Gamma(3/4)^2}{\Gamma(1/4)^2} = 0.1142366452... = \dfrac{12}{105.0451015308...} & \text{if } c = 0. \end{cases}$$

The constants 0.2893082598... and 0.1142366452... also appear in [27, 28], respectively, but reasons for these connections are unclear.

Fix an odd integer $k \geq 3$. Pincus [29, 30] and Lima & Rahibe [31] examined

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{cases} \begin{pmatrix} \cos(\frac{\pi}{k})x_{n-1} + \sin(\frac{\pi}{k})y_{n-1} \\ -\sin(\frac{\pi}{k})x_{n-1} + \cos(\frac{\pi}{k})y_{n-1} \end{pmatrix} & \text{with probability } 1-\eta, \\ \begin{pmatrix} x_{n-1} \\ 0 \end{pmatrix} & \text{with probability } \eta \end{cases}$$

and proved that

$$\lambda(k) = \frac{\eta^2}{1-(1-\eta)^{2k}} \sum_{j=1}^{2k-1} (1-\eta)^j \ln \left| \cos\left(\frac{j\,\pi}{k}\right) \right|.$$

The identical expression emerges if we replace the definition of the latter portion by

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} \ell\, x_{n-1} \\ (1/\ell) y_{n-1} \end{pmatrix} \qquad \text{with probability } \eta$$

for a fixed integer $\ell \geq 2$, and compute the asymptotic difference between $\lambda(k,\ell)$ and $\eta \ln(\ell)$ in the limit as $\ell \to \infty$. A precise numerical estimate of $\lambda(3,2) = 0.1794...$, however, is evidently open [16].

Ben-Naim & Krapivsky [32] studied two variations of random Fibonacci sequences:

$$x_n = \begin{cases} x_{n-1} + x_{n-2} & \text{with probability } 1-\eta \\ x_{n-1} + x_{n-3} & \text{with probability } \eta \end{cases}, \qquad x_0 = 0, \qquad x_1 = x_2 = 1;$$

$$x_n = \begin{cases} x_{n-1} + x_{n-2} & \text{with probability } 1-\eta \\ 2x_{n-1} & \text{with probability } \eta \end{cases}, \qquad x_1 = x_2 = 1$$

and determined that

$$\lim_{\eta \to 0^+} \lambda(\eta) = \ln(\varphi)$$

for both cases. Second-order asymptotic terms differ, however:

$$\lim_{\eta \to 0^+} \frac{\lambda(\eta) - \ln(\varphi)}{\eta} = \begin{cases} \ln\left(\dfrac{2\varphi}{\varphi+2}\right) & \text{for case 1,} \\ \ln\left(\dfrac{2\varphi+1}{\varphi+2}\right) & \text{for case 2} \end{cases}$$

and a third-order term is possible for the latter.

Consider the random geometric sequence [33]

$$x_n = 2x_p, \qquad x_0 = 1, \qquad p \in \{0, 1, \ldots, n-1\}$$

where each of the n possible indices is given equal weight. The sequence is not necessarily increasing, but enjoys average growth $n+1$ and almost-sure growth

$$2^{\gamma} n^{\ln(2)} = (1.4919670404...) \exp\left(\ln(2)\ln(n)\right).$$

Consider instead two additional random Fibonacci models [34, 35]:

$$x_n = x_{n-1} + x_q, \qquad x_0 = 1, \qquad q \in \{0, 1, \ldots, n-1\};$$

$$x_n = x_p + x_q, \qquad x_0 = 1, \qquad p, q \in \{0, 1, \ldots, n-1\}.$$

Model 1 enjoys average growth

$$\frac{1}{2\sqrt{e\,\pi}} n^{-1/4} \exp(2\sqrt{n})$$

and almost-sure growth

$$C \exp\left((1.889...)\sqrt{n}\right)$$

where $C > 0$ is unknown. Model 2 is not necessarily increasing but enjoys average growth $n+1$; unlike the random geometric sequence, it seems not to display almost-sure behavior of any kind.

Kenyon & Peres [36] studied random products associated with two sets of matrices:

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

and

$$\begin{pmatrix} 3 & 0 \\ 2 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 2 \\ 0 & 3 \end{pmatrix}.$$

The three matrices in the first set are equiprobable, with Lyapunov exponent $\ln(2)/3 = 0.2310490601....$ The four matrices in the second set are likewise equiprobable, with Lyapunov exponent [37]

$$\frac{1}{6}\ln\left(\frac{2}{3}\right) + \sum_{i=0}^{\infty} 4^{-i-1} \ln\left(\frac{(3\cdot 2^i)!}{(2^{i+1})!}\right) = 0.7974350484....$$

We wonder whether $\exp(0.7974350484...)$ is transcendental. Moshe [38] studied random products associated with two equiprobable 3×3 matrices:

$$\begin{pmatrix} 1 & 3 & 1 \\ 1 & 2 & 0 \\ -3 & -6 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 4 & 2 & 8 \\ -2 & -1 & -4 \\ 3 & 1 & 4 \end{pmatrix}$$

and computed Lyapunov exponent

$$\frac{1}{16}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{2^{j+k}} \ln\left|3\cdot 2^{3j} - 2(-1)^j - \tfrac{22}{9}2^{3j+k} + \tfrac{22}{9}(-1)^j 2^k\right| = 0.5897925607....$$

Many more similar examples are found in [39, 40, 41, 42].

Up to now, the random mechanisms underlying sequences have been very simple. Here is a more complicated but well-known example [43, 44]:

$$x_{n+1} = a_n x_n + x_{n-1}, \qquad x_0 = 0, \qquad x_1 = 1$$

where the cofficients a_n are obtained by selecting a random $\theta \in [0,1]$ and computing its continued fraction digits:

$$\theta = \frac{1|}{|a_1} + \frac{1|}{|a_2} + \frac{1|}{|a_3} + \cdots .$$

For instance, if $\theta = \pi - 3$, then

$$\{a_1, a_2, a_3, a_4\} = \{7, 15, 1, 292\}, \qquad \{x_2, x_3, x_4, x_5\} = \{7, 106, 113, 33102\};$$

note that x_n is simply the denominator of the n^{th} partial convergent to θ. Lévy [45] proved that this recurrence gives rise to Lyapunov exponent

$$\frac{\pi^2}{12\ln(2)} = 1.1865691104....$$

Another example involves the recurrence [46]

$$x_{n+1} = 2^{b_n} x_n + 2^{b_{n-1}} x_{n-1}, \qquad x_0 = 0, \qquad x_1 = 1$$

where the cofficients b_n are obtained via

$$\theta = \frac{2^{-b_1}\big|}{|1} + \frac{2^{-b_2}\big|}{|1} + \frac{2^{-b_3}\big|}{|1} + \cdots .$$

The corresponding Lyapunov exponent is

$$\frac{1}{\ln(4/3)}\left(\frac{\pi^2}{12} + \operatorname{Li}_2\left(-\frac{1}{2}\right)\right) = 1.3002298798...$$

where $\operatorname{Li}_2(y)$ is the dilogarithm function [47]. (This constant also appears in [48] without explanation.) Generalization to base $k \geq 2$ is possible, as well as formulation for Khintchine-type and Lochs-type constants in this broad setting.

0.1. Addendum. The subject continues to expand [49, 50, 51, 52, 53, 54, 55]. Two earlier works deserve mention. Hope [56] examined

$$x_{n+1} = a_n x_n + x_{n-1}, \qquad x_0 = 0, \qquad x_1 = 1$$

like Lévy, but with a simple rule

$$\operatorname{P}(a_n = 1) = \operatorname{P}(a_n = 2) = 1/2$$

and independence assumed. The Lyapunov exponent is

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{a=1 \text{ or } 2} \ln\left(a_1 + \frac{1|}{|a_2} + \frac{1|}{|a_3} + \cdots + \frac{1|}{|a_{n-1}} + \frac{1|}{|a_n}\right) \approx 0.673 \approx \ln(1.96).$$

Davison [57] studied the same except with the rule

$$a_n = 1 + (\lfloor \theta\, n \rfloor \bmod 2)$$

for a random $\theta \in [0, 2]$, showing that

$$1.931 < \sqrt{2+\sqrt{3}} \leq \liminf_{n\to\infty} x_n^{1/n} \leq \limsup_{n\to\infty} x_n^{1/n} \leq \sqrt{\left(\frac{1+\sqrt{5}}{2}\right)\left(1+\sqrt{2}\right)} < 1.977.$$

We wonder how closely these examples might be connected.

The sequence of polynomials giving Pascal's rhombus [39] arises from a second-order recurrence

$$p_n(x) = (1+x+x^2)p_{n-1}(x) + x^2 p_{n-2}(x), \qquad p_1(x) = 1+x+x^2, \qquad p_0(x) = 1.$$

Let u_n to be the number of odd coefficients in $p_n(x)$. A numerical method gives "typical growth" $\lambda = 0.57331379313....$ While $\limsup_{n\to\infty} \ln(u_n)/\ln(n) = 1$ is trivial, the following was proved only recently [58]:

$$\begin{aligned} \liminf_{n\to\infty} \frac{\ln(u_n)}{\ln(n)} &= \rho\left(A^3 B^3\right)^{1/6} \\ &= \frac{\ln(1.6376300574...)}{\ln(2)} = 0.7116094872... \end{aligned}$$

where A, B are known 5×5 integer matrices and ρ denotes spectral radius (the maximal modulus of eigenvalues). Consider instead the Fibonacci polynomials [39]

$$q_n(x) = x\, q_{n-1}(x) + q_{n-2}(x), \qquad q_1(x) = x, \qquad q_0(x) = 1.$$

The number v_n of odd coefficients in $q_n(x)$ is the n^{th} term of Stern's sequence [59]:

$$v_{2n+1} = v_n, \qquad v_{2n} = v_n + v_{n-1}.$$

Again, $\lambda = 0.3962125642...$ via numerics; "typical dispersion" $\sigma^2 = 0.0221729451...$ can be found similarly [40]. The limit superior and limit inferior do not present any difficulties for $\{v_n\}$. An evaluation of σ^2 corresponding to $\{u_n\}$, however, remains open.

References

[1] S. R. Finch, Lyapunov exponents. III, unpublished note (2007).

[2] J. E. Cohen and C. M. Newman, The stability of large random matrices and their products, *Annals of Probab.* 12 (1984) 283–310; MR0735839 (86a:60013).

[3] S. R. Finch, Euler-Mascheroni constant, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 28–40.

[4] T. G. Wright and L. N. Trefethen, Computing Lyapunov constants for random recurrences with smooth coefficients, *J. Comput. Appl. Math.* 132 (2001) 331–340; http://people.maths.ox.ac.uk/trefethen/papers.html; MR1840632 (2002d:65010).

[5] D. Viswanath, Random Fibonacci sequences and the number 1.13198824..., *Math. Comp.* 69 (2000) 1131–1155; http://www.math.lsa.umich.edu/~divakar/; MR1654010 (2000j:15040).

[6] B. Hayes, The Vibonacci numbers, *Amer. Scientist,* v. 87 (1999) n. 4, 296-301; http://www.americanscientist.org/issues/pub/the-vibonacci-numbers.

[7] Z.-Q. Bai, On the cycle expansion for the Lyapunov exponent of a product of random matrices, *J. Phys. A* 40 (2007) 8315–8328; MR2371235.

[8] R. Mainieri, Zeta function for the Lyapunov exponent of a product of random matrices, *Phys. Rev. Lett.* 68 (1992) 1965–1968; chao-dyn/9301001.

[9] R. Mainieri, Cycle expansion for the Lyapunov exponent of a product of random matrices, *Chaos* 2 (1992) 91–97; MR1158540 (93e:82029).

[10] J. L. Nielsen, Lyapunov exponent for products of random matrices (1997), http://chaosbook.org/projects/index.shtml.

[11] B. Rittaud, On the average growth of random Fibonacci sequences, *J. Integer Seq.* 10 (2007) 07.2.4; MR2276788 (2007j:11018).

[12] É. Janvresse, B. Rittaud and T. de la Rue, How do random Fibonacci sequences grow?, *Probab. Theory Related Fields* 142 (2008) 619–648; math.PR/0611860; MR2438703 (2009m:37148).

[13] E. Makover and J. McGowan, An elementary proof that random Fibonacci sequences grow exponentially, *J. Number Theory* 121 (2006) 40–44; math.NT/0510159; MR2268754 (2008a:11087).

[14] M. Embree and L. N. Trefethen, Growth and decay of random Fibonacci sequences, *Royal Soc. Lond. Proc. Ser. A* 455 (1999) 2471–2485; http://people.maths.ox.ac.uk/trefethen/papers.html; MR1807827 (2001i:11098).

[15] P. Chassaing, G. Letac and M. Mora, Brocot sequences and random walks in $\mathrm{SL}(2,\mathbb{R})$, *Probability Measures on Groups. VII*, Proc. 1983 Oberwolfach conf., ed. H. Heyer, Lect. Notes in Math. 1064, Springer-Verlag, 1984, pp. 36–48; MR0772400 (86g:60012).

[16] Z.-Q. Bai, Calculations of certain Lyapunov exponents, unpublished note (2007).

[17] S. R. Finch, The Golden mean, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 5–11.

[18] D. Mannion, Products of 2×2 random matrices, *Annals Appl. Probab.* 3 (1993) 1189–1218; MR1241041 (94k:60021).

[19] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, 1972, pp. 374–377; MR1225604 (94b:00012).

[20] S. R. Finch, Euler-Gompertz constant, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 423–428.

[21] R. Lyons, Singularity of some random continued fractions, *J. Theoret. Probab.* 13 (2000) 535–545; http://mypage.iu.edu/~rdlyons/; MR1778585 (2002c:60138).

[22] K. Simon, B. Solomyak and M. Urbański, Invariant measures for parabolic IFS with overlaps and random continued fractions, *Trans. Amer. Math. Soc.* 353 (2001) 5145–5164; http://www.math.unt.edu/~urbanski/papers.html; MR1852098 (2003c:37030a).

[23] K. Ishii, Localization of eigenstates and transport phenomena in the one-dimensional disordered system, *Prog. Theor. Phys. Suppl.* 53 (1973) 77–138.

[24] I. M. Lifshits, S. A. Gredeskul and L. A. Pastur, *Introduction to the Theory of Disordered Systems*, Wiley, 1988, pp. 124–163; MR1042095 (90k:82073).

[25] B. Derrida and E. Gardner, Lyapounov exponent of the one-dimensional Anderson model: weak disorder expansions, *J. Physique* 45 (1984) 1283–1295; MR0763431 (85m:82098).

[26] F. M. Izrailev, S. Ruffo and L. Tessieri, Classical representation of the one-dimensional Anderson model, *J. Phys. A* 31 (1998) 5263–5270; MR1634869 (99c:82037).

[27] S. T. Ariaratnam and W. C. Xie, Lyapunov exponent and rotation number of a two-dimensional nilpotent stochastic system, *Dynam. Stability Systems* 5 (1990) 1–9; MR1057870 (91h:60086).

[28] C. Sire and P. L. Krapivsky, Random Fibonacci sequences, *J. Phys. A* 34 (2001) 9065–9083; cond-mat/0106457; MR1876126 (2003b:60018).

[29] S. Pincus, Furstenberg-Kesten results: asymptotic analysis, *Random Matrices and their Applications*, Proc. 1984 Brunswick conf., ed. J. E. Cohen, H. Kesten and C. M. Newman, Amer. Math. Soc., 1986, pp. 79–86; MR0841083 (87m:60069).

[30] S. Pincus, Strong laws of large numbers for products of random matrices, *Trans. Amer. Math. Soc.* 287 (1985) 65–89; MR0766207 (86i:60087).

[31] R. Lima and M. Rahibe, Exact Lyapunov exponent for infinite products of random matrices, *J. Phys. A* 27 (1994) 3427–3437; MR1282183 (95d:82004).

[32] E. Ben-Naim and P. L. Krapivsky, Weak disorder in Fibonacci sequences, *J. Phys. A* 39 (2006) L301–L307; cond-mat/0603117; MR2238100 (2007c:82047).

[33] E. Ben-Naim and P. L. Krapivsky, Random geometric series, *J. Phys. A* 37 (2004) 5949–5957; cond-mat/0403157; MR2074617 (2005i:11098).

[34] E. Ben-Naim and P. L. Krapivsky, Growth and structure of stochastic sequences, *J. Phys. A* 35 (2002) L557–L563; cond-mat/0208072; MR1946873.

[35] I. Krasikov, G. J. Rodgers and C. E. Tripp, Growing random sequences. *J. Phys. A* 37 (2004) 2365–2370; http://people.brunel.ac.uk/~mastgjr/sequences.htm; MR2045930 (2005k:60167).

[36] R. Kenyon and Y. Peres, Intersecting random translates of invariant Cantor sets, *Invent. Math.* 104 (1991) 601–629; MR1106751 (92g:28018).

[37] P. Sebah, Series evaluation via Kummer acceleration, Stirling approximation and Euler-Maclaurin summation, unpublished note (2007).

[38] Y. Moshe, Random matrix products and applications to cellular automata, *J. d'Analyse Math.* 99 (2006) 267–294; MR2279553.

[39] S. Finch, P. Sebah and Z.-Q. Bai, Odd entries in Pascal's trinomial triangle, arXiv:0802.2654.

[40] S. Finch, Z.-Q. Bai and P. Sebah, Typical dispersion and generalized Lyapunov exponents, arXiv:0803.2611.

[41] M. Pollicott, Maximal Lyapunov exponents for random matrix products, *Invent. Math.* 181 (2010) 209–226; http://homepages.warwick.ac.uk/~masdbl/lyapunov.pdf; MR2651384.

[42] V. Yu. Protasov and R. M. Jungers, Lower and upper bounds for the largest Lyapunov exponent of matrices, *Linear Algebra Appl.* 438 (2013) 4448–4468; MR3034543.

[43] K. Dajani and C. Kraaikamp, *Ergodic Theory of Numbers*, Math. Assoc. Amer., 2002, pp. 20–31; MR1917322 (2003f:37014).

[44] S. R. Finch, Continued fraction transformation, unpublished note (2007).

[45] H.-C. Chan, The asymptotic growth rate of random Fibonacci type sequences, *Fibonacci Quart.* 43 (2005) 243–255; MR2171636 (2006f:11011).

[46] H.-C. Chan, The asymptotic growth rate of random Fibonacci type sequences. II, *Fibonacci Quart.* 44 (2006) 73–84; MR2209557 (2006m:11014).

[47] S. R. Finch, Apéry's constant, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 40–53.

[48] S. R. Finch, Continued fraction transformation. IV, unpublished note (2007).

[49] É. Janvresse, B. Rittaud and T. de la Rue, Growth rate for the expected value of a generalized random Fibonacci sequence, *J. Phys. A* 42 (2009) 085005; MR2525481 (2010e:11011).

[50] É. Janvresse, B. Rittaud and T. de la Rue, Almost-sure growth rate of generalized random Fibonacci sequences, *Annales de l'Institut Henri Poincaré Probab. Stat.* 46 (2010) 135–158; arXiv:0804.2400; MR2641774 (2012a:37106).

[51] E. B. Cureg and A. Mukherjea, Numerical results on some generalized random Fibonacci sequences, *Comput. Math. Appl.* 59 (2010) 233–246; MR2575510 (2010j:11024).

[52] Z.-Q. Bai, A transfer operator approach to random Fibonacci sequences, *J. Phys. A* 44 (2011) 115002; MR2773858 (2012a:37102).

[53] Y. Lan, Novel computation of the growth rate of generalized random Fibonacci sequences, *J. Stat. Phys.* 142 (2011) 847–861; MR2773789 (2012b:82056).

[54] C. Zhang and Y. Lan, Computation of growth rates of random sequences with multi-step memory, *J. Stat. Phys.* 150 (2013) 722–743; MR3024154.

[55] P. J. Forrester, Lyapunov exponents for products of complex Gaussian random matrices, *J. Stat. Phys.* 151 (2013) 796–808; arXiv:1206.2001; MR3055376.

[56] P. Hope, Exponential growth of random Fibonacci sequences, *Fibonacci Quart.* 33 (1995) 164–168; MR1329024 (96b:11115).

[57] J. L. Davison, A class of transcendental numbers with bounded partial quotients, *Number Theory and Applications*, Proc. 1988 Banff NATO Adv. Study Instit., ed. R. A. Mollin, Kluwer, 1989, pp. 365–371; MR1123082 (92h:11060).

[58] N. Guglielmi and V. Yu. Protasov, Exact computation of joint spectral characteristics of linear operators, *Found. Comput. Math.* 13 (2013) 37–97; MR3009529.

[59] S. R. Finch, Stolarsky-Harborth constant, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 145–151.