Moduli of Continuity

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December 22, 2004

0.1. Bernstein Polynomials. Bernstein's proof of the Weierstrass approximation theorem makes use of the operator

$$B_n f(x) = \sum_{j=0}^{n} \binom{n}{j} x^j (1-x)^{n-j} f\left(\frac{j}{n}\right),$$

given any continuous function $f : [0,1] \to \mathbb{R}$. To demonstrate that

$$\lim_{n\to\infty} B_n f(x) = f(x)$$

uniformly on $[0,1]$ requires a bound of the form

$$\sup_{0\leq x\leq 1} |B_n f(x) - f(x)| \leq c \cdot \omega(f, n^{-1/2}),$$

where $\omega(f,\delta)$ is the **first modulus of continuity**

$$\omega(f,\delta) = \sup_{|u-v|<\delta} |f(u) - f(v)|$$

and $0 \leq \delta \leq 1$. What is the best possible constant c that works for all $n \geq 1$? Starting from [1, 2], Sikkema [3, 4, 5] proved that

$$\sup_{n\geq 1} \sup_{f} \sup_{0\leq x\leq 1} \frac{|B_n f(x) - f(x)|}{\omega(f, n^{-1/2})} = \frac{4306 + 837\sqrt{6}}{5832} = 1.0898873310...$$

and this value is attained only for $n = 6$. Table 1 lists the best possible constants c_n that work for specified $n = 1, 2, \ldots, 8$.

Table 1. *Best Constants c_n: Exact Expressions and Decimal Approximations*

n	Exact	Decimal	n	Exact	Decimal
1	1	1	5	$\frac{21-7\sqrt{5}}{5}$	1.0695048315...
2	$\frac{5-2\sqrt{2}}{2}$	1.0857864376...	6	$\frac{4306+837\sqrt{6}}{5832}$	1.0898873310...
3	$\frac{27-10\sqrt{3}}{9}$	1.0754991027...	7	$\frac{35442+33754\sqrt{7}}{117649}$	1.0603293674...
4	$\frac{17}{16}$	1.0625	8	$\frac{3865512\sqrt{8}-1937991}{8388608}$	1.0723266591...

Esseen [6, 7, 8, 9] examined the limiting behavior of c_n as n grows without bound:

$$\begin{aligned}\limsup_{n\to\infty}\sup_f \sup_{0\le x\le 1}\frac{|B_n f(x)-f(x)|}{\omega(f,n^{-1/2})} &= 2\sum_{m=0}^{\infty}(m+1)\left(\Phi(2m+2)-\Phi(2m)\right)\\ &= 1.0455636083...\end{aligned}$$

where

$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt=\frac{1}{2}\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)+\frac{1}{2}.$$

Of course, we understand to omit constant functions f from the supremum (for which $\omega=0$).

Define the **second modulus of continuity**

$$\tilde{\omega}(f,\delta)=\sup_{|u-v|<\delta}\left|f(u)-2f\left(\frac{u+v}{2}\right)+f(v)\right|.$$

In contrast with the preceding results, the best constant and best asymptotic constant here coincide:

$$\sup_f \sup_{0\le x\le 1}\frac{|B_n f(x)-f(x)|}{\tilde{\omega}(f,n^{-1/2})}=1$$

for each $n\ge 1$. This was proved only recently by Paltanea [10], building on earlier results [11, 12, 13, 14, 15, 16]. Here, of course, we understand to omit linear functions f from the supremum (for which $\tilde{\omega}=0$).

Let us return to the first modulus ω for the remainder of this essay. Define Ω to be the set of all continuous functions $g:[0,1]\to\mathbb{R}$ that vanish at zero, are nondecreasing and subadditive (meaning $g(x+y)\le g(x)+g(y)$ always). Each member g of Ω satisfies $g(x)=\omega(g,x)$ and thus is itself a modulus of continuity. Define Ω^* to be the subset of Ω whose elements g are such that $x\mapsto x^{-1}g(x)$ is nonincreasing on $(0,1]$. Then [17, 18]

$$\sup_{n\ge 1}\sup_{0<x\le 1}\sup_{g\in\Omega}\frac{B_n g(x)}{g(x)}=2>1.1855905950...=\alpha=\sup_{0<x\le 1}\sup_{n\ge 1}\sup_{g\in\Omega^*}\frac{B_n g(x)}{g(x)}$$

where

$$\alpha=\sup_{k\ge 0}\sup_{k\le x\le k+1}1+e^{-x}\left(\frac{x^k}{k!}-1\right)=1+\frac{\xi^2}{2}e^{-\xi}$$

and $\xi=3.4920333011...$ is the unique real zero of the cubic equation $x^3-3x^2-6=0$.

A seemingly related problem involves the ratio of moduli [19, 20]

$$\rho_1(n)=\sup_{0<\delta\le 1}\sup_f\frac{\omega(B_n f,\delta)}{\omega(f,\delta)}=2$$

for each $n \geq 1$. There are interesting multivariate versions of this result. Consider the operator

$$B_n f(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{n}\binom{n}{i}x^i(1-x)^{n-i}\binom{n}{j}y^j(1-y)^{n-j}f\left(\frac{i}{n},\frac{j}{n}\right),$$

given any continuous function $f:[0,1]\times[0,1]\to\mathbb{R}$. This is also called the bivariate **tensor product** Bernstein polynomial on the unit square. De La Cal, Cárcamo & Valle [21, 22] proved that, in this two-dimensional case, the ratio

$$\rho_2(n) = \sup_{0<\delta\leq 1}\sup_{f}\frac{\omega(B_n f,\delta)}{\omega(f,\delta)}$$

depends on n and

$$\begin{aligned}\sup_{n\geq 1}\rho_2(n) &= 1-\frac{1}{e^2}+\sum_{t=0}^{\infty}\left[1-\frac{1}{e^2}\left(\sum_{s=0}^{t}\frac{1}{s!}\right)^2\right] \\ &= 2.3884423285... = 1-e^{-2}+\beta,\end{aligned}$$

where $\beta = 1.5237776118...$ is the mean of the maximum of two independent Poisson(1) random variables. One would expect the k-dimensional case, $k\geq 3$, to be even more complicated. In fact, $\rho_k(n) = k$ for all $n \geq 1$. Hence only the bivariate case gives n-dependent behavior as well as a new constant, which is quite surprising.

0.2. Müntz-Jackson theorem. Müntz's theorem gives that the power functions

$$\left\{x^{\lambda_j} : 0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots\right\}$$

generate a dense subspace of the space of all continuous functions on $[0,1]$ if and only if $\sum_{j=0}^{\infty} 1/\lambda_j = \infty$. Jackson's theorem is in the spirit of other results in this essay: It provides bounds on the error in approximating a continuous function f by polynomials in terms of ω. Newman [23, 24] combined the two theorems in the following way. Define

$$\Lambda = \{\lambda_j : 0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n\}$$

and generalized polynomials

$$Q_\Lambda = \left\{\sum_{j=0}^{n} a_j x^{\lambda_j} : a_j \in \mathbb{R} \text{ for all } 0 \leq j \leq n\right\}.$$

Then

$$\inf_{q\in Q_\Lambda}\sup_{0\leq x\leq 1}|q(x)-f(x)| \leq C\cdot\omega(f,\varepsilon_\Lambda)$$

where C is a constant independent of f and Λ, and

$$\varepsilon_\Lambda = \sup_{\mathrm{Re}(z)=1} \left| \frac{1}{z} \frac{z-\lambda_1}{z+\lambda_1} \frac{z-\lambda_2}{z+\lambda_2} \cdots \frac{z-\lambda_n}{z+\lambda_n} \right|.$$

Newman [23, 24] demonstrated that $1/50 < C < 368$ and Odogwu [25] improved the upper bound to 66. Over and beyond the value of C, the Blaschke product formula for ε_Λ is intriguing. Special cases (when consecutive λs are at least 2 apart, or when consecutive λs are at most 2 apart) with simpler formulas also exist.

An L_p-generalization of ω can be defined; the constants in this essay correspond only to the case $p = \infty$. It would be good to see their L_p-analogs for $p < \infty$. Clearly $\lim_{\delta \to 0} \omega(f, \delta) \cdot \delta^{-1} = 0$ implies that f is constant. Consequences of the weaker condition $\lim_{\delta \to 0} \omega(f, \delta) \cdot \ln(\delta) = 0$ are mentioned in [26].

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