Map Asymptotics Constant

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A **map** on a compact surface S without boundary is an embedding of a graph G into S such that all components of $S-G$ are simply connected [1]. These components are thus homeomorphic to open disks and are called **faces**. The graph G is allowed to have both loops and multiple parallel edges (unlike those in [2]). A map is **rooted** when an edge, a direction along that edge, and a side of the edge, are distinguished. The edge is called the **root edge**, and the face on the distinguished side is the **root face**. Two rooted maps are **equivalent** if there is a homeomorphism between the underlying surfaces that preserves all graph incidences and rootedness.

In the case when S is orientable, two rooted maps are equivalent if and only they are related by an orientation-preserving homeomorphism that (merely) preserves all graph incidences. Such thinking doesn't apply, of course, when S is non-orientable. For orientable surfaces, the **genus** g is 0 for the sphere, 1 for the torus, 2 for the connected sum of two tori, and so forth. For non-orientable surfaces, the **type** h is $1/2$ for the projective plane, 1 for the Klein bottle, $3/2$ for the connected sum of three projective planes, and so forth.

The requirement that faces be simply connected implies that the graph G itself must be connected [3]. Proof: if G were to possess two components, then a curve drawn around one of the components could not be contracted to a point (because the other component would present an obstacle), which is a contradiction. The converse is true if the surface S is a sphere, but is false if S is a torus. Reason: consider the figure-eight graph G consisting of one vertex and two edges (orthogonal loops that together generate the torus). While $S-G$ is simply connected, this is not true for any proper subgraph of G.

Let $T_g(n)$ denote the number of rooted maps with n edges on an orientable surface of genus g. Let $P_h(n)$ denote the number of rooted maps with n edges on a non-orientable surface of type h. (T stands for "torus" and P stands for "projective plane".) It is known that $T_0(n)$ is the coefficient of x^n in the Maclaurin series expansion [1, 4, 5]

$$\begin{aligned}\frac{4(1+2r)}{3(1+r)^2} &= 1+2x+9x^2+54x^3+378x^4+2916x^5+24057x^6 \\ &\quad +208494x^7+1876446x^8+17399772x^9+165297834x^{10}+\cdots,\end{aligned}$$

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$T_1(n)$ is the coefficient of x^n in the expansion [6, 7]

$$\begin{aligned}\frac{(-1+r)^2}{12r^2(2+r)} &= x^2+20x^3+307x^4+4280x^5+56914x^6 \\ &\quad +736568x^7+9370183x^8+117822512x^9+1469283166x^{10}+\cdots,\end{aligned}$$

$P_{1/2}(n)$ is the coefficient of x^n in the expansion [1, 6]

$$\begin{aligned}\frac{-q}{(-1+r)(1+r)} &= x+10x^2+98x^3+982x^4+10062x^5+105024x^6 \\ &\quad +1112757x^7+11934910x^8+129307100x^9+1412855500x^{10}+\cdots\end{aligned}$$

and $P_1(n)$ is the coefficient of x^n in the expansion [8, 9, 10]

$$\begin{aligned}\frac{(1+r)q}{2r^2(2+r)} &= 4x^2+84x^3+1340x^4+19280x^5+263284x^6 \\ &\quad +3486224x^7+45247084x^8+579150012x^9+7338291224x^{10}+\cdots\end{aligned}$$

where $r=\sqrt{1-12x}$ and $q=2+4r-2\sqrt{3}\sqrt{r(2+r)}$ throughout. Moreover [11],

$$T_g(n)\sim t_g n^{5(g-1)/2}12^n, \qquad P_h(n)\sim p_h n^{5(h-1)/2}12^n$$

as $n\to\infty$, where t_g is the **orientable map asymptotics constant**:

$$t_0=\frac{2}{\sqrt{\pi}}, \qquad t_1=\frac{1}{24}, \qquad t_2=\frac{7}{4320\sqrt{\pi}}, \qquad t_3=\frac{245}{15925248}, \qquad t_4=\frac{37079}{96074035200\sqrt{\pi}}$$

and p_h is the **non-orientable map asymptotics constant**:

$$p_{1/2}=\frac{\sqrt{3}}{2\pi}\Gamma\left(1/4\right)=-\frac{2\sqrt{6}}{\Gamma(-1/4)}, \qquad p_1=\frac{1}{2}, \qquad p_{3/2}=\frac{\sqrt{6}}{3\Gamma(1/4)}=\frac{5}{8\sqrt{6}\Gamma(9/4)}.$$

Since the status of t_g is quite different from the status of p_h, we shall treat them separately.

For many years, the values of t_g for $g>2$ were unknown, owing to difficulties in their formulation. Impressive progress has been made recently. Define a sequence

$$u_0=1, \qquad u_n=\frac{25(n-1)^2-1}{48}u_{n-1}-\frac{1}{2}\sum_{k=1}^{n-1}u_k u_{n-k} \qquad \text{for } n\geq 1$$

then provably

$$t_g=-\frac{1}{2^{g-2}\Gamma\left((5g-1)/2\right)}u_g$$

for all integers $g \geq 0$. The formal power series $u(z) = \sum_{n=0}^{\infty} u_n z^{-(5n-1)/2}$ satisfies the Painlevé I differential equation

$$u''(z) = 6u(z)^2 - 6z$$

which makes possible the following asymptotics:

$$t_g \sim \frac{40 \sin(\pi/5) K}{\sqrt{2\pi}} \left(\frac{1440 g}{e} \right)^{-g/2}$$

as $g \to \infty$ and

$$K = \sqrt{\frac{3}{5}} \frac{\Gamma(1/5)\Gamma(4/5)}{4\pi^2} = 0.1048689877....$$

We explain further: Bender, Gao & Richmond [12] discovered the preceding approximation for t_g but with only a rough numerical estimate 0.1034 for K. The connection with Painlevé I, streamlined u_n recursion and exact K expression are due to Garoufalidis, Lê & Mariño [13]. A (somewhat different) full asymptotic series is also possible. We give the first term only:

$$u_n \sim -\frac{1}{2\pi} \frac{3^{1/4}}{\sqrt{\pi}} \left(\frac{8\sqrt{3}}{5} \right)^{-2n+\frac{1}{2}} \Gamma\left(2n - \frac{1}{2}\right)$$

as $n \to \infty$, quoting [14]. This is reminiscent of other quadratic recurrence studies [15, 16].

Likewise, the path to understanding p_h for $h > 2$ is fraught with peril. Define a sequence

$$v_0 = -\sqrt{3}, \qquad v_n = \frac{1}{2\sqrt{3}} \left(-3u_{n/2} + \frac{5n-6}{2} v_{n-1} + \sum_{k=1}^{n-1} v_k v_{n-k} \right) \qquad \text{for } n \geq 1$$

(the dependence of v_n on $u_{n/2}$ from before is striking: if n is odd, let $u_{n/2} = 0$). Conjecturally, we have [14]

$$p_h = \frac{1}{2^{h-2}\Gamma\left((5h-3)/2\right)} v_{2h-1}$$

for all integers/half-integers $h \geq 1/2$. Evidence for this equality comes from quantum physics. As consequences,

$$p_2 = \frac{5}{36\sqrt{\pi}}, \qquad p_{5/2} = \frac{1033}{1024\sqrt{6}\Gamma(19/4)}, \qquad p_3 = \frac{3149}{442368}, \qquad p_{7/2} = \frac{1599895}{294912\sqrt{6}\Gamma(29/4)}.$$

The formal power series $v(z)=\sum_{n=0}^{\infty} v_n z^{-(5n-1)/4}$ satisfies the differential equation

$$2v'(z)=v(z)^2-3u(z)$$

and a full asymptotic series is again possible. We give the first term only:

$$v_n \sim \frac{C}{2\pi}\left(\frac{4\sqrt{3}}{5}\right)^{-n}\Gamma(n)$$

as $n\to\infty$, where the Stokes constant C is conjectured to be $\sqrt{6}$. See [17, 18] for a bivariate analog of the preceding theory.

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