Mertens' Formula

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May 4, 2007

A generalization of Mertens' famous formula recently appeared [1]:

$$\lim_{x\to\infty}\ln(x)^{1/\varphi(k)}\prod_{\substack{p\leq x,\\ p\equiv\ell \bmod k}}\left(1-\frac{1}{p}\right)=\left[e^{-\gamma}\prod_{p}\left(1-\frac{1}{p}\right)^{\alpha(p;k,\ell)}\right]^{1/\varphi(k)}$$

where φ is the Euler totient function, γ is the Euler-Mascheroni constant [2], and $\alpha(p;k,\ell)$ is equal to $\varphi(k)-1$ if $p\equiv\ell \bmod k$ and is -1 otherwise. This constitutes a vast simplification of earlier such formulas [3, 4]. Computing the constant $e^{-\gamma}\Lambda_{k,\ell}=(0.5614594835...)\Lambda_{k,\ell}$ inside the square brackets, as well as the related limit [5]:

$$M_{k,\ell}=\lim_{x\to\infty}\left(\sum_{\substack{p\leq x\\ p\equiv\ell \bmod k}}\frac{1}{p}-\frac{1}{\varphi(k)}\ln(\ln(x))\right)$$

will occupy us for the remainder of this essay.

Let $\zeta(s)$ denote the Riemann zeta function and

$$P_{k,\ell}(s)=\sum_{p\equiv\ell \bmod k}\frac{1}{p^s}$$

denote the $(k,\ell)^{\text{th}}$ **prime zeta function** for $\operatorname{Re}(s)>1$. Clearly $\Lambda_{1,0}=1$; to efficiently compute $M_{1,0}$, we utilize the series [6, 7, 8]

$$P_{1,0}(s)=\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\ln\left(\zeta(ns)\right).$$

The numerical evaluation of other $P_{k,\ell}(s)$ will be discussed momentarily. For now, note that

$$\frac{-\gamma+\ln(\Lambda_{k,\ell})}{\varphi(k)}=\lim_{x\to\infty}\left(\sum_{\substack{p\leq x\\ p\equiv\ell \bmod k}}\ln\left(1-\frac{1}{p}\right)+\frac{1}{\varphi(k)}\ln(\ln(x))\right);$$

hence

$$\begin{aligned} M_{k,\ell} + \frac{\ln(\Lambda_{k,\ell}) - \gamma}{\varphi(k)} &= \sum_{p \equiv \ell \bmod k} \left(\ln\left(1 - \frac{1}{p}\right) + \frac{1}{p} \right) \\ &= -\sum_{p \equiv \ell \bmod k} \left(\frac{1}{2p^2} + \frac{1}{3p^3} + \frac{1}{4p^4} + \cdots \right) \\ &= -\sum_{n=2}^{\infty} \frac{P_{k,\ell}(n)}{n}; \end{aligned}$$

hence

$$M_{1,0} = \gamma - \sum_{n=2}^{\infty} \frac{P_{1,0}(n)}{n} = 0.2614972128....$$

In the following two sections, we will discuss the cases $k = 3$ and $k = 4$. In both cases, $\varphi(k) = 2$, which implies that

$$\begin{aligned} \Lambda_{k,1} &= \frac{k}{2} \prod_{p \equiv 1 \bmod k} \left(1 - \frac{1}{p}\right) \cdot \prod_{p \equiv -1 \bmod k} \left(1 - \frac{1}{p}\right)^{-1} \\ &= \frac{k}{2} \frac{1}{L_{-k}(1)} \prod_{p \equiv -1 \bmod k} \left(1 + \frac{1}{p}\right)^{-1} \cdot \prod_{p \equiv -1 \bmod k} \left(1 - \frac{1}{p}\right)^{-1} \\ &= \frac{k}{2} \frac{1}{L_{-k}(1)} \prod_{p \equiv -1 \bmod k} \left(1 - \frac{1}{p^2}\right)^{-1} \end{aligned}$$

where L_{-k} is Dirichlet's L-series associated to $(-k/\cdot)$. The infinite product can be evaluated via the $(k, -1)^{\text{th}}$ prime zeta function since

$$\begin{aligned} \ln\left(\prod_{p \equiv \ell \bmod k} \left(1 - \frac{1}{p^2}\right) \right) &= \sum_{p \equiv \ell \bmod k} \left(\ln\left(1 + \frac{1}{p}\right) + \ln\left(1 - \frac{1}{p}\right) \right) \\ &= -\sum_{p \equiv \ell \bmod k} \left(\frac{1}{p^2} + \frac{1}{2p^4} + \frac{1}{3p^6} + \cdots \right) \\ &= -\sum_{n=1}^{\infty} \frac{P_{k,\ell}(2n)}{n}. \end{aligned}$$

Thus we first compute $\Lambda_{3,1}$ and $\Lambda_{4,1}$, and then $M_{3,1}$ and $M_{4,1}$.

Let χ_0 denote the principal character $\bmod\, k$ and χ_1 denote the nonprincipal character $\bmod\, k$ (χ_1 is unique since $k = 3$ or $k = 4$). In order to evaluate $P_{k,1}(s)$ and

$P_{k,-1}(s)$, the associated Dirichlet L-series:

$$L_{\chi_j}(s) = \sum_{n=1}^{\infty} \frac{\chi_j(n)}{n^s} = \frac{1}{k^s}\left(\chi_j(1)\zeta\left(s,\tfrac{1}{k}\right) + \chi_j(-1)\zeta\left(s,1-\tfrac{1}{k}\right)\right), \qquad j = 0,1$$

are required, where $\zeta(s,a)$ is the Hurwitz zeta-function. It can be shown that [9]

$$P_{k,-1}(s) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{\mu(2n+1)}{2n+1} \ln\left(\frac{L_{\chi_0}((2n+1)s)}{L_{\chi_1}((2n+1)s)}\right),$$

$$P_{k,1}(s) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{\mu(2n+1)}{2n+1} \ln\left(\frac{L_{\chi_0}((2n+1)s)L_{\chi_1}((2n+1)s)}{L_{\chi_0}((4n+2)s)}\right).$$

We will additionally exhibit $\prod_{p\equiv 1 \bmod k}(1-p^{-2})$ and $\prod_{p\equiv -1 \bmod k}(1-p^{-2})$, since these are also easily available.

0.1. Residue Classes Mod 3. The two characters modulo 3 are

$$\chi_0(n)|_{n=1,2,3} = \{1,1,0\}, \qquad \chi_1(n)|_{n=1,2,3} = \{1,-1,0\};$$

thus $L_{\chi_0}(s) = \zeta(s)(1-1/3^s)$ and $L_{\chi_1}(s) = L_{-3}(s)$. We have [10]

$$\prod_{p\equiv 1 \bmod 3}\left(1-\frac{1}{p^2}\right) = 0.9671040753... = \frac{9\sqrt{3}}{8}(0.7044984335...)^2 = \frac{27\sqrt{3}}{2\pi^2}K_3^2,$$

$$\prod_{p\equiv 2 \bmod 3}\left(1-\frac{1}{p^2}\right) = 0.7071813747... = \frac{9\sqrt{3}}{2}(0.3012165544...)^2 = \frac{\sqrt{3}}{6}\frac{1}{K_3^2}$$

where $K_3 = 0.6389094054...$ is the Landau-Ramanujan constant for counting integers of the form a^2+3b^2 [11]. Also $\Lambda_{3,1} = 27K_3^2/\pi$ and therefore

$$\lim_{x\to\infty} \sqrt{\ln(x)} \prod_{\substack{p\le x \\ p\equiv 1 \bmod 3}} \left(1-\frac{1}{p}\right) = 3\sqrt{\frac{3}{\pi}}e^{-\gamma/2}K_3 = 1.4034774468...,$$

$$\lim_{x\to\infty} \sqrt{\ln(x)} \prod_{\substack{p\le x \\ p\equiv 2 \bmod 3}} \left(1-\frac{1}{p}\right) = \frac{1}{2}\sqrt{\frac{\pi}{3}}e^{-\gamma/2}\frac{1}{K_3} = 0.6000732161...,$$

$$M_{3,1} = \frac{\gamma}{2} - \ln\left(3\sqrt{\frac{3}{\pi}}K_3\right) + \sum_{p\equiv 1 \bmod 3}\left[\ln\left(1-\frac{1}{p}\right)+\frac{1}{p}\right] = -0.3568904795...,$$

$$M_{3,2} = \frac{\gamma}{2} - \ln\left(\frac{1}{2}\sqrt{\frac{\pi}{3}}\frac{1}{K_3}\right) + \sum_{p\equiv 2 \bmod 3}\left[\ln\left(1-\frac{1}{p}\right)+\frac{1}{p}\right] = 0.2850543590....$$

0.2. Residue Classes Mod 4. An alternative approach is given in [12]. The two characters modulo 4 are

$$\chi_0(n)|_{n=1,2,3,4} = \{1,0,1,0\}, \qquad \chi_1(n)|_{n=1,2,3,4} = \{1,0,-1,0\};$$

thus $L_{\chi_0}(s) = \zeta(s)(1-1/2^s)$ and $L_{\chi_1}(s) = L_{-4}(s)$. We have [10]

$$\prod_{p\equiv 1 \bmod 4}\left(1-\frac{1}{p^2}\right) = 0.9468064071... = 4(0.4865198883...)^2 = \frac{16}{\pi^2}K_1^2,$$

$$\prod_{p\equiv 3 \bmod 4}\left(1-\frac{1}{p^2}\right) = 0.8561089817... = 8\,(0.3271293669...)^2 = \frac{1}{2}\frac{1}{K_1^2}$$

where $K_1 = 0.7642236535...$ is the (classical) Landau-Ramanujan constant for counting integers of the form a^2+b^2 [11]. Also $\Lambda_{4,1} = 16K_1^2/\pi$ and therefore

$$\lim_{x\to\infty}\sqrt{\ln(x)}\prod_{\substack{p\le x\\ p\equiv 1 \bmod 4}}\left(1-\frac{1}{p}\right) = \frac{4}{\sqrt{\pi}}e^{-\gamma/2}K_1 = 1.2923041571...,$$

$$\lim_{x\to\infty}\sqrt{\ln(x)}\prod_{\substack{p\le x\\ p\equiv 3 \bmod 4}}\left(1-\frac{1}{p}\right) = \frac{\sqrt{\pi}}{2}e^{-\gamma/2}\frac{1}{K_1} = 0.8689277682...,$$

$$\begin{aligned} M_{4,1} &= \frac{\gamma}{2} - \ln\left(\frac{4}{\sqrt{\pi}}K_1\right) + \sum_{p\equiv 1 \bmod 4}\left[\ln\left(1-\frac{1}{p}\right)+\frac{1}{p}\right] = -0.2867420562...,\\ M_{4,3} &= \frac{\gamma}{2} - \ln\left(\frac{\sqrt{\pi}}{2}\frac{1}{K_1}\right) + \sum_{p\equiv 3 \bmod 4}\left[\ln\left(1-\frac{1}{p}\right)+\frac{1}{p}\right] = 0.0482392690.... \end{aligned}$$

Some low-precision results are known [13] for the residue classes mod 6 and 8; it would be good to repeat these calculations (using the prime zeta function, as above) to high accuracy.

0.3. Addendum. Languasco & Zaccagnini [14, 15, 16] have proved new formulas and greatly extended the preceding calculations, confirming our values for $M_{k,\ell}$ and for $(e^{-\gamma}\Lambda_{k,\ell})^{1/\varphi(k)}$ (what they call $C_{k,\ell}$) when $k=3$ and $k=4$.

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