Mathieu Eigenvalues

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Consider the differential equation [1, 2, 3]

$$y''(x) + (\lambda - 2\mu\cos(2x))\,y(x) = 0$$

which admits periodic solutions of (least) period π and 2π for four countably infinite sets of eigenvalues, for each value of μ.

0.1. Even Solutions of Period π. Given boundary conditions $y'(0) = y'(\pi/2) = 0$, the eigenvalues $\lambda = \alpha_{2k}$ for $k \geq 0$ satisfy the infinite tridiagonal determinant equation [4]

$$\begin{vmatrix} 0^2-\lambda & \sqrt{2}\mu & 0 & 0 & 0 & \\ \sqrt{2}\mu & 2^2-\lambda & \mu & 0 & 0 & \\ 0 & \mu & 4^2-\lambda & \mu & 0 & \\ 0 & 0 & \mu & 6^2-\lambda & \mu & \\ 0 & 0 & 0 & \mu & 8^2-\lambda & \ddots \\ & & & & \ddots & \ddots \end{vmatrix} = 0$$

as well as the continued fraction equation [5]

$$-\frac{\lambda}{2} = \frac{\mu^2 \quad |}{|2^2-\lambda} - \frac{\mu^2 \quad |}{|4^2-\lambda} - \frac{\mu^2 \quad |}{|6^2-\lambda} - \frac{\mu^2 \quad |}{|8^2-\lambda} - \frac{\mu^2 \quad |}{|10^2-\lambda} - \cdots$$

For example, if $\mu = 1$, then [6] $\alpha_0 = -0.4551386041...$ and $\alpha_2 = 4.3713009827....$ The corresponding eigenfunctions are written as $\mathrm{ce}_{2k}(x)$. Only for complex μ can the equality $\alpha_0 = \alpha_2$ occur; the first such example [7, 8, 9, 10, 11] happens when $\mu = (1.4687686137...)i$, at which $\alpha_0 = \alpha_2 = 2.0886989027....$

0.2. Odd Solutions of Period π. Given boundary conditions $y(0) = y(\pi/2) = 0$, the eigenvalues $\lambda = \beta_{2k+2}$ for $k \geq 0$ satisfy the infinite tridiagonal determinant equation

$$\begin{vmatrix} 2^2-\lambda & \mu & 0 & 0 & 0 & \\ \mu & 4^2-\lambda & \mu & 0 & 0 & \\ 0 & \mu & 6^2-\lambda & \mu & 0 & \\ 0 & 0 & \mu & 8^2-\lambda & \mu & \\ 0 & 0 & 0 & \mu & 10^2-\lambda & \ddots \\ 0 & 0 & 0 & 0 & \ddots & \ddots \end{vmatrix} = 0$$

as well as the continued fraction equation

$$4-\lambda = \frac{\mu^2 \quad |}{|4^2-\lambda} - \frac{\mu^2 \quad |}{|6^2-\lambda} - \frac{\mu^2 \quad |}{|8^2-\lambda} - \frac{\mu^2 \quad |}{|10^2-\lambda} - \frac{\mu^2 \quad |}{|12^2-\lambda} - \cdots$$

For example, if $\mu = 1$, then $\beta_2 = 3.9170247729...$ and $\beta_4 = 16.0329700814...$. The corresponding eigenfunctions are written as $\mathrm{se}_{2k+2}(x)$. Only for complex μ can the equality $\beta_2 = \beta_4$ occur; the first such example [9, 10, 11, 12] happens when $\mu = (6.9289547587...)i$, at which $\beta_2 = \beta_4 = 11.1904735991...$.

0.3. Even Solutions of Period 2π. Given boundary conditions $y'(0) = y(\pi/2) = 0$, the eigenvalues $\lambda = \alpha_{2k+1}$ for $k \geq 0$ satisfy the infinite tridiagonal determinant equation

$$\begin{vmatrix} 1+\mu-\lambda & \mu & 0 & 0 & 0 & \\ \mu & 3^2-\lambda & \mu & 0 & 0 & \\ 0 & \mu & 5^2-\lambda & \mu & 0 & \\ 0 & 0 & \mu & 7^2-\lambda & \mu & \\ 0 & 0 & 0 & \mu & 9^2-\lambda & \ddots \\ 0 & 0 & 0 & 0 & \ddots & \ddots \end{vmatrix} = 0$$

as well as the continued fraction equation

$$1+\mu-\lambda = \frac{\mu^2 \quad |}{|3^2-\lambda} - \frac{\mu^2 \quad |}{|5^2-\lambda} - \frac{\mu^2 \quad |}{|7^2-\lambda} - \frac{\mu^2 \quad |}{|9^2-\lambda} - \frac{\mu^2 \quad |}{|11^2-\lambda} - \cdots$$

For example, if $\mu = 1$, then $\alpha_1 = 1.8591080725...$ and $\alpha_3 = 9.0783688472...$. The corresponding eigenfunctions are written as $\mathrm{ce}_{2k+1}(x)$. Only for complex μ can the equality $\alpha_1 = \alpha_3$ occur; the first such example [9, 10, 11, 13] happens when

$$\mu = 1.93139250... + (3.23763841...)i = (3.7699574940...)e^{i\theta},$$

$$\theta = \arccos(0.51231148...) \approx 59.182°$$

at which

$$\alpha_1 = \alpha_3 = 6.17649... + (1.23174...)i.$$

0.4. Odd Solutions of Period 2π. Given boundary conditions $y(0) = y'(\pi/2) = 0$, the eigenvalues $\lambda = \beta_{2k+1}$ for $k \geq 0$ satisfy the infinite tridiagonal determinant equation

$$\begin{vmatrix} 1-\mu-\lambda & \mu & 0 & 0 & 0 & \\ \mu & 3^2-\lambda & \mu & 0 & 0 & \\ 0 & \mu & 5^2-\lambda & \mu & 0 & \\ 0 & 0 & \mu & 7^2-\lambda & \mu & \\ 0 & 0 & 0 & \mu & 9^2-\lambda & \ddots \\ 0 & 0 & 0 & 0 & \ddots & \ddots \end{vmatrix} = 0$$

as well as the continued fraction equation

$$1-\mu-\lambda = \frac{\mu^2 \quad |}{|3^2-\lambda} - \frac{\mu^2 \quad |}{|5^2-\lambda} - \frac{\mu^2 \quad |}{|7^2-\lambda} - \frac{\mu^2 \quad |}{|9^2-\lambda} - \frac{\mu^2 \quad |}{|11^2-\lambda} - \cdots$$

For example, if $\mu = 1$, then $\beta_1 = -0.1102488169...$ and $\beta_3 = 9.0477392598....$ The corresponding eigenfunctions are written as $\mathrm{se}_{2k+1}(x)$. No new constants emerge in connection with $\beta_1 = \beta_3$ because $\beta_1(\mu) = \alpha_1(-\mu)$ and $\beta_3(\mu) = \alpha_3(-\mu)$; hence this case reduces to the preceding.

0.5. Double Points. The values $|\mu| = 1.468...$, $6.928...$, $3.769...$ are first terms of the three sequences [10, 11]

- $\{a_k\}$, where $a_k = |\mu|$ and μ is the complex point closest to 0 satisfying $\alpha_{2k}(\mu) = \alpha_{2k+2}(\mu)$
- $\{b_k\}$, where $b_k = |\mu|$ and μ is the complex point closest to 0 satisfying $\beta_{2k+2}(\mu) = \beta_{2k+4}(\mu)$
- $\{c_k\}$, where $c_k = |\mu|$ and μ is the complex point closest to 0 satisfying $\alpha_{2k+1}(\mu) = \alpha_{2k+3}(\mu)$ if k is even and $\beta_{2k+1}(\mu) = \beta_{2k+3}(\mu)$ if k is odd.

It is conjectured (among other things) that

$$a_k \sim b_k \sim c_k$$

asymptotically as $k \to \infty$ and $a_k \approx (2.042)k^2$ for large k. Conceivably $\pi^{-1/4}e = 2.04177...$ could be an exact expression for the leading coefficient [11]: no one knows.

0.6. Hill and Ince. Let n be a positive integer. Hill's equation is the following generalization [14]

$$y''(x) + \left(\lambda - 2\sum_{j=1}^{n} \mu_j \cos(2j\,x)\right) y(x) = 0$$

of Mathieu's equation (for which $n = 1$ was assumed). A special case of Hill's equation is Ince's equation [4, 15]

$$y''(x) + c\sin(2x)y'(x) + (\lambda - \mu\, c\cos(2x))\, y(x) = 0$$

after a suitable transformation (assuming here that $n = 2$). Let λ denote the leftmost eigenvalue of the above. We merely mention that the derivatives $\lambda'(0)$ and $\lambda''(0)$ of the function $\mu \mapsto \lambda(\mu)$, for fixed c, play an interesting role in [16]. By contrast, $\alpha_0'(0) = 0$ and $\alpha_0''(0) = -1$ for Mathieu's equation, which are comparatively straightforward.

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