Inflating an Inelastic Membrane

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Starting with two circular unit disks (made of Mylar, a thin material that does not stretch nor shrink), we sew these together along their boundaries and then fill the interior with a fluid (air or helium) to capacity. What is the shape of the resulting three-dimensional solid of revolution (Mylar balloon)? [1, 2, 3]

Without loss of generality, assume that the solid is centered at the origin and its axis of revolution is the z axis. In the plane $y = 0$, the boundary curve $z = z(x)$ solves the following calculus of variations problem: Maximize volume

$$4\pi \int_0^\rho x\, z(x) dx$$

subject to the constraint

$$\int_0^\rho \sqrt{1 + z'(x)^2}\, dx = 1$$

where $0 < \rho < 1$ is fixed. It turns out that the optimal value of ρ is

$$\rho = \frac{4\sqrt{2\pi}}{\Gamma(1/4)^2} = \frac{\sqrt{2}}{K[1/2]} = 0.7627597635... = (1.3110287771...)^{-1}$$

and the parametric representation for the associated boundary surface is

$$x = \rho \operatorname{cn}\left(u, \tfrac{1}{2}\right) \cos(v), \qquad y = \rho \operatorname{cn}\left(u, \tfrac{1}{2}\right) \sin(v),$$

$$z = \sqrt{2}\rho \left(E\left[\arcsin\left(\operatorname{sn}\left(u, \tfrac{1}{2}\right)\right), \tfrac{1}{2}\right] - \tfrac{1}{2}u\right)$$

for $-K[1/2] < u < K[1/2]$, $0 < v < 2\pi$. In the preceding, $K[m]$, $\operatorname{sn}(u, m)$, $\operatorname{cn}(u, m)$ are defined exactly as in [4] and

$$E[\phi, m] = \int_0^{\sin(\phi)} \sqrt{\frac{1 - m\, t^2}{1 - t^2}}\, dt$$

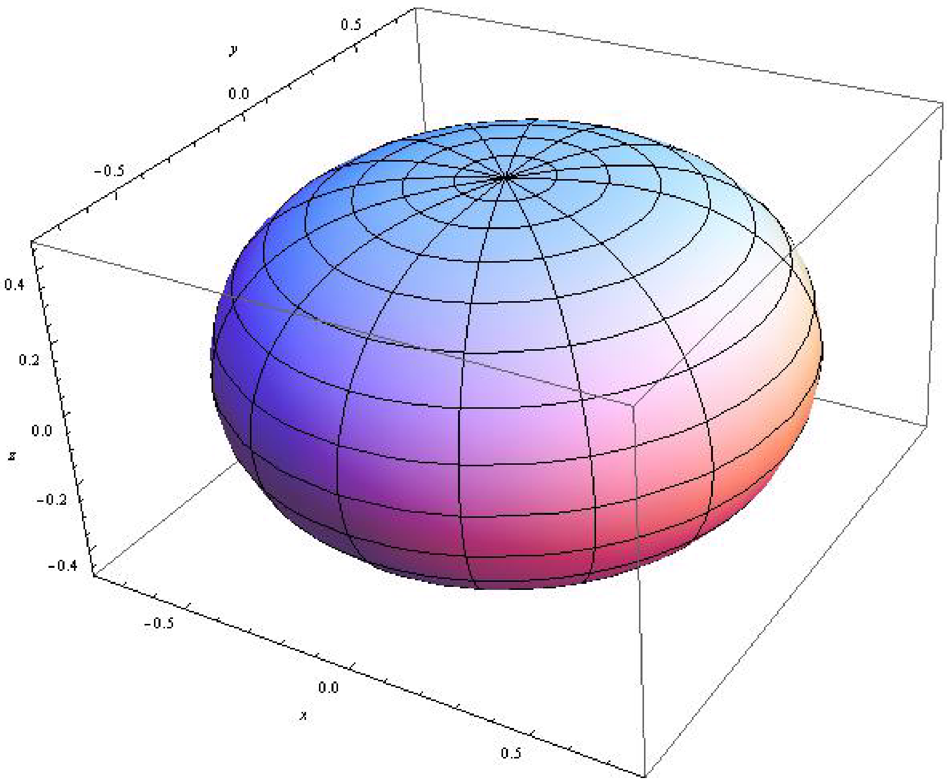

Figure 1: Mylar balloon, starting from two unit disks

denotes the incomplete elliptic integral of the second kind. These are admittedly incompatible with [5] but we purposefully choose formulas here to be consistent with the computer algebra package MATHEMATICA. See Figure 1. Let $E[m] = E[\pi/2, m]$. Clearly ρ is the equatorial radius and

$$\begin{aligned} \tau &= \frac{(2\pi)^{3/2}}{\Gamma(1/4)^2}\rho = 2\sqrt{2}\left(E\left[\tfrac{1}{2}\right] - \tfrac{1}{2}K\left[\tfrac{1}{2}\right]\right)\rho = (1.1981402347...)\rho \\ &= \frac{16\pi^2}{\Gamma(1/4)^4} = \frac{\pi}{K\left[1/2\right]^2} = 0.9138931620... \end{aligned}$$

is the polar diameter (thickness). Note that $\tau/(2\rho) = 0.5990701173...$, the ratio of extreme distances through the origin.

The volume

$$\begin{aligned} V &= \sqrt{\frac{\pi}{2}}\frac{\Gamma(1/4)^2}{6}\rho^3 = \frac{\sqrt{2}\pi}{3}K\left[\frac{1}{2}\right]\rho^3 = (2.7458122499...)\rho^3 \\ &= \frac{64\pi^2}{3\Gamma(1/4)^4} = \frac{4\pi}{3K[1/2]^2} = 1.2185242161... \end{aligned}$$

is considerably less than $(\sqrt{2}/3)\pi = 1.48...$, the volume of the sphere with surface area equal to that of the two original disks. ($4\pi r^2 = 2\pi$, hence $r = 1/\sqrt{2}$, hence $(4/3)\pi r^3 = (\sqrt{2}/3)\pi$.) It seems reasonable to call V the **Mylar balloon constant**.

The surface area A possesses an elementary expression: $\pi^2\rho^2$. Comparing the original area 2π with A:

$$\frac{2\pi}{A} = \frac{2}{\pi\rho^2} = \frac{1}{\tau} = 1.0942198076...$$

reveals a remarkable fact. We seem to have lost some of the 2D area, despite the 1D restriction on Mylar stretching/shrinking. There must be crimping or wrinkling of the inflated balloon in order to accommodate $\approx 9.42\%$ area of the deflated balloon. Most of the crimping occurs at the equator; none occurs at the poles. More precisely, the crimping is governed by a local distribution function [6]

$$\delta(x) = \frac{\rho^2}{x}\int_0^x \frac{dt}{\sqrt{\rho^4 - t^4}} = \frac{\rho}{\sqrt{2}x}\left(K\left[\frac{1}{2}\right] - F\left[\arccos\left(\frac{x}{\rho}\right), \frac{1}{2}\right]\right)$$

over $0 < x < \rho$, where $F[\phi, m]$ is defined exactly as in [4]. See Figure 2. We have $\delta(\rho) = 1/r = 1.311...$ whereas $\delta(0) = 1$. Implicit in all our analysis is an assumption that the wrinkles do not affect the volume of the balloon. We wonder about the realism of such, given that the wrinkles *do* affect the surface area significantly.

The unit square analog of the Mylar balloon gives rise to a teabag or paper bag [7, 8], whose optimal volume appears to be approximately 0.208 [9, 10]. More work will be needed to confirm that the actual **teabag constant** is no larger than this value.

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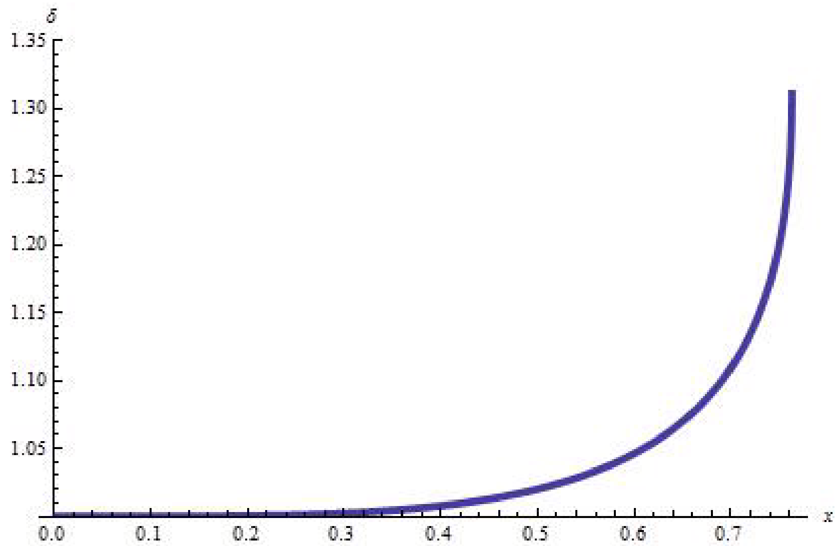

Figure 2: Local distribution of the 9.42% excess area

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