Excursion Durations

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This essay bears some resemblance to [1], but comes from a different viewpoint. Let $\{X_t : 0 \leq t \leq 1\}$ denote standard Brownian motion and fix a time $0 < \tau < 1$. The **excursion** straddling τ is $\{X_t : \alpha_\tau \leq t \leq \beta_\tau\}$, where

$$\alpha_\tau = \sup\{t < \tau : X_t = 0\}, \qquad \beta_\tau = \inf\{t > \tau : X_t = 0\}.$$

We are interested in the **duration** $\beta_t - \alpha_\tau$ of this excursion, as well as all excursions staddling earlier times. More precisely, let

$$M_\tau - 1 = \#\{\text{excursions completed by time } \tau \text{ whose durations exceed } \tau - \alpha_\tau\},$$

$$N_\tau - 1 = \#\{\text{excursions completed by time } \tau \text{ whose durations exceed } \beta_\tau - \alpha_\tau\};$$

we wish to compute the probability that $M_\tau = 1$ (the current excursion, measured up to time τ, has a record duration) and the probability that $N_\tau = 1$ (the current excursion, measured to its completion, has a record duration). Since $\beta_\tau \geq \tau$, it is clear that $M_\tau \geq N_\tau$. Simple scaling arguments show that the distribution of M_τ and the distribution of N_τ are independent of τ.

Define functions

$$\varphi(x) = \frac{1}{2}\int_1^\infty e^{-x\,u}u^{-3/2}du = e^{-x} - \sqrt{\pi\,x}\,\operatorname{erfc}(\sqrt{x}),$$

$$\psi(x) = 1 + \frac{1}{2}\int_0^1 (1 - e^{-x\,u})u^{-3/2}du = e^{-x} + \sqrt{\pi\,x}\,\operatorname{erf}(\sqrt{x})$$

where

$$\operatorname{erf}(y) = \frac{2}{\sqrt{\pi}}\int_0^y \exp(-v^2)\,dv = 1 - \operatorname{erfc}(y);$$

then [2, 3]

$$\mathrm{P}(M_\tau = k) = \int_0^\infty e^{-x}\varphi(x)^{k-1}\psi(x)^{-k}dx,$$

$$\mathrm{P}(N_\tau = k) = \frac{1}{2}\int_0^\infty x^{-1}(1-e^{-x})\varphi(x)^{k-1}\psi(x)^{-k}dx.$$

Numerical integration gives

$$\mathrm{P}(M_\tau = k) = \begin{cases} 0.6265075987... & \text{if } k=1, \\ 0.1430092516... & \text{if } k=2, \\ 0.0630157050... & \text{if } k=3, \\ 0.0356483608... & \text{if } k=4 \end{cases}$$

$$\mathrm{P}(N_\tau = k) = \begin{cases} 0.8003100322... & \text{if } k=1, \\ 0.0812481569... & \text{if } k=2, \\ 0.0334196946... & \text{if } k=3, \\ 0.0184590943... & \text{if } k=4 \end{cases}$$

and asymptotic analysis gives, as $k \to \infty$,

$$\mathrm{P}(M_\tau = k) \sim \frac{2}{\pi\, k^2}, \qquad \mathrm{P}(N_\tau = k) \sim \frac{1}{\pi\, k^2}.$$

It is striking that the current excursion is, with fairly high probability, of duration greater than all preceding excursions!

Let $L_1 > L_2 > L_3 > \ldots > 0$ denote the ranked durations of excursions of X_t. Note that $\sum L_j = 1$ almost surely. The joint probability law of $(L_1, L_2, L_3, \ldots)$ follows what is called the Poisson-Dirichlet $(1/2, 0)$ distribution. If instead X_t is a Brownian bridge (meaning that $X_1 = 0$), then the Poisson-Dirichlet $(1/2, 1/2)$ distribution emerges. Can numerical results for $\mathrm{P}(M_\tau)$ and $\mathrm{P}(N_\tau)$ be found in this case? We also wonder what happens when X_t is an Ornstein-Uhlenbeck process [4].

Addendum The constant 0.6265... also appears in [5], as well as the Golomb-Dickman constant 0.6243... [6].

References

[1] S. R. Finch, Zero crossings, unpublished note (2003).

[2] C. L. Scheffer, The rank of the present excursion, *Stochastic Process. Appl.* 55 (1995) 101–118; MR1312151 (96m:60189).

[3] J. Pitman and M. Yor, The two-parameter Poisson-Dirichlet distribution derived from a stable subordinator, *Annals of Probab.* 25 (1997) 855–900; http://www.stat.berkeley.edu/users/pitman/433.pdf; MR1434129 (98f:60147).

[4] J. Pitman and M. Yor, On the lengths of excursions of some Markov processes, *Séminaire de Probabilités, XXXI*, ed. J. Azéma, M. Emery and M. Yor, Lect. Notes in Math. 1655, Springer, 1997, pp. 272–286; MR1478737 (98j:60108).

[5] S. N. Majumdar and R. M. Ziff, Universal record statistics of random walks and Lévy flights, *Phys. Rev. Lett.* 101 (2008) 050601; arXiv:0806.0057; MR2430298.

[6] S. R. Finch, Golomb-Dickman constant, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 284–292.