Swing-Up Control of a Pendulum

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A pendulum is a bob of mass m, attached to a frictionless pivot point via a massless rod of length ℓ. The bob is free to swing from side to side in a vertical plane. Let g denote the acceleration due to gravity. Let θ denote the angle between the rod and a vertical axis. The pendulum has two equilibrium positions, a stable one at $\theta = 0$ (bottom) and an unstable one at $\theta = \pi$ (top). Assume further that we apply a torque τ to the pendulum, increasing θ (counterclockwise motion) when $\tau > 0$. Let τ be constrained by $|\tau| \leq \tau_0$. The angular equation of motion is [1, 2]

$$I \frac{d^2\theta}{ds^2} + m\, g\, \ell \sin(\theta) = \tau$$

where $I = m\,\ell^2$ is the moment of inertia and s is time. Define non-dimensional parameters

$$t = \sqrt{\frac{m\,g\,\ell}{I}}s, \qquad u = \frac{\tau}{\tau_0}, \qquad \kappa = \frac{\tau_0}{m\,g\,\ell}$$

so that

$$\frac{d\theta}{dt} = \frac{d\theta}{ds}\frac{ds}{dt} = \sqrt{\frac{I}{m\,g\,\ell}}\frac{d\theta}{ds}, \qquad \frac{d^2\theta}{dt^2} = \sqrt{\frac{I}{m\,g\,\ell}}\frac{d^2\theta}{ds^2}\frac{ds}{dt} = \frac{I}{m\,g\,\ell}\frac{d^2\theta}{ds^2}, \qquad \frac{\tau}{m\,g\,\ell} = \kappa\, u$$

and hence

$$\frac{d^2\theta}{dt^2} + \sin(\theta) = \kappa\, u$$

subject to $|u| \leq 1$. We shall first solve a simple problem with $u = 0$ before allowing more complicated controls in our study. For simplicity, let $\omega = d\theta/dt$.

Let $\kappa = 1$ for now. Let $\theta = \pi/2$ and $\omega = 0$ at $t = 0$. Under these initial conditions and the assumption that $u = 0$ for all t, the pendulum swings down due to gravity alone. What is the angular velocity when $\theta = 0$? Here an exact formula exists:

$$\theta(t) = -2 \arcsin\left(\frac{1}{\sqrt{2}}\operatorname{sn}\left(t - K\left(\frac{1}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right)\right)$$

where $K(x)$ is the complete elliptic integral of the first kind and $\operatorname{sn}(x,y)$ is one of the Jacobi elliptic functions [3]. Solving $\theta(t)=0$ gives [4]

$$t=K\left(\frac{1}{\sqrt{2}}\right)=\frac{1}{4\sqrt{\pi}}\Gamma\left(\frac{1}{4}\right)^2=1.8540746773...$$

and substituting this value into $\omega(t)$ gives $-\sqrt{2}=-1.4142135623...$ [5]. A simple outcome as such is possible only because $u=0$.

Assume either that $u=1$ for all t or that $u=-1$ for all t. Given initial conditions $\theta(t_0)=\theta_0$ and $\omega(t_0)=\omega_0$, we have

$$\omega\frac{d\omega}{d\theta}=\frac{d^2\theta}{dt^2}=-\sin(\theta)\pm\kappa$$

hence

$$\frac{1}{2}\omega^2=\cos(\theta)\pm\kappa\,\theta+c,\qquad c=\frac{1}{2}\omega_0^2-\cos(\theta_0)\mp\kappa\,\theta_0$$

hence

$$\left(\frac{d\theta}{dt}\right)^2=\omega^2=\omega_0^2+2\left[\cos(\theta)-\cos(\theta_0)\pm\kappa\,\theta\mp\kappa\,\theta_0\right]$$

hence

$$\frac{|d\theta|}{\sqrt{\omega_0^2+2\left[\cos(\theta)-\cos(\theta_0)\pm\kappa\,\theta\mp\kappa\,\theta_0\right]}}=dt.$$

Define

$$T_+(\theta_0,\omega_0,t_0,\kappa;\theta)=t_0+\left|\int_{\theta_0}^{\theta}\frac{d\varphi}{\sqrt{\omega_0^2+2\left[\cos(\varphi)-\cos(\theta_0)+\kappa\,\varphi-\kappa\,\theta_0\right]}}\right|$$

to be the time to reach θ, corresponding to $u=1$ and

$$T_-(\theta_0,\omega_0,t_0,\kappa;\theta)=t_0+\left|\int_{\theta_0}^{\theta}\frac{d\psi}{\sqrt{\omega_0^2+2\left[\cos(\psi)-\cos(\theta_0)-\kappa\,\psi+\kappa\,\theta_0\right]}}\right|$$

to be the time to reach θ, corresponding to $u=-1$. For example, $T_-(\pi/2,0,0,1;0)=1.2794771227...$ is the time required for the pendulum to swing down due to both gravity and a clockwise unit torque. This is unsurprisingly less than the time 1.854... calculated for gravity alone. As another example, $T_+(0,0,0,1;\pi/2)=2.1000505566...$ is the time required for the pendulum to swing halfway up due to a counterclockwise unit torque. This is greater than the preceding since here we are working against gravity. These constants are unrecognizable, as are the associated velocities $\omega_- = -2.2675080272...$ and $\omega_+ = 1.0684533932...$ obtained using a nonlinear ODE solver.

A more challenging problem is as follows [6, 7, 8]. Given $(\theta_0, \omega_0) = (0, 0)$, what is the unique strategy to drive the pendulum to $(\theta, \omega) = (\pi, 0)$ via a bang-bang control $u = \pm 1$ with one switching? The solution is to initially apply $u = 1$ until the precise time t_1 when

$$(\theta, \omega) = \left(\frac{\pi}{2} + \frac{1}{\kappa}, \sqrt{\kappa\,\pi - 2\sin\left(\frac{1}{\kappa}\right)} \right)$$

and subsequently apply $u = -1$ until the precise time t_∞ when $(\theta, \omega) = (\pi, 0)$. See Figure 1. For example, if $\kappa = 1$, then

$$t_1 = T_+ \left(0, 0, 0, \kappa; \frac{\pi}{2} + \frac{1}{\kappa} \right) = 3.0063538276...,$$

$$t_\infty = T_- \left(\frac{\pi}{2} + \frac{1}{\kappa}, \sqrt{\kappa\,\pi - 2\sin\left(\frac{1}{\kappa}\right)}, t_1, \kappa; \pi \right) = 4.0300186879...$$

but this is valid only since $\cos(\varphi) - 1 + \kappa\,\varphi > 0$ for all $0 < \varphi < \pi$. The minimizing value $\varphi_{\min}$ on the left-hand side of the inequality is $\pi - \arcsin(\kappa)$. After substituting $\varphi_{\min}$ into the expression, we solve

$$1 - \pi\,\kappa + \sqrt{1 - \kappa^2} + \kappa \arcsin(\kappa) = 0$$

and obtain $\kappa = 0.7246113537...$ as the smallest number for which t_1 is well-defined. Both this number and a related quantity $\pi - \arcsin(\kappa) = 2.3311223704...$ appear in [9] in connection not with *swing-up* control, but rather with *damping* (from unstable equilibrium position to stable). By contrast, the inequality $\cos(\psi) + 1 - \kappa\,\psi + \kappa\,\pi > 0$ does not impose any additional restrictions on κ.

If $\kappa = 1/2$, then we need to consider bang-bang controls $u = \pm 1$ with two switchings. Infinitely many strategies exist by which $u = -1$ is applied for $0 < t < t_1$, $u = 1$ is applied for $t_1 < t < t_2$, $u = -1$ is applied for $t_2 < t < t_\infty$ and required initial/terminal conditions for (θ, ω) are satisfied. Of these, there is a unique strategy with minimal t_∞; see Figure 2. It is remarkable that optimality is achieved by first allowing $\omega < 0$ (clockwise motion), seemingly out of the way, before simultaneously reversing torque and exploiting gravity to push $\omega > 1.3$. Omitting the first stage would lead to the pendulum falling far short of $(\theta, \omega) = (\pi, 0)$.

If $\kappa = 3/4$, then both a one-switching strategy and a minimal two-switching strategy exist. For the former, the required time is $t_{\infty,1} = 6.5690173615...$; for the

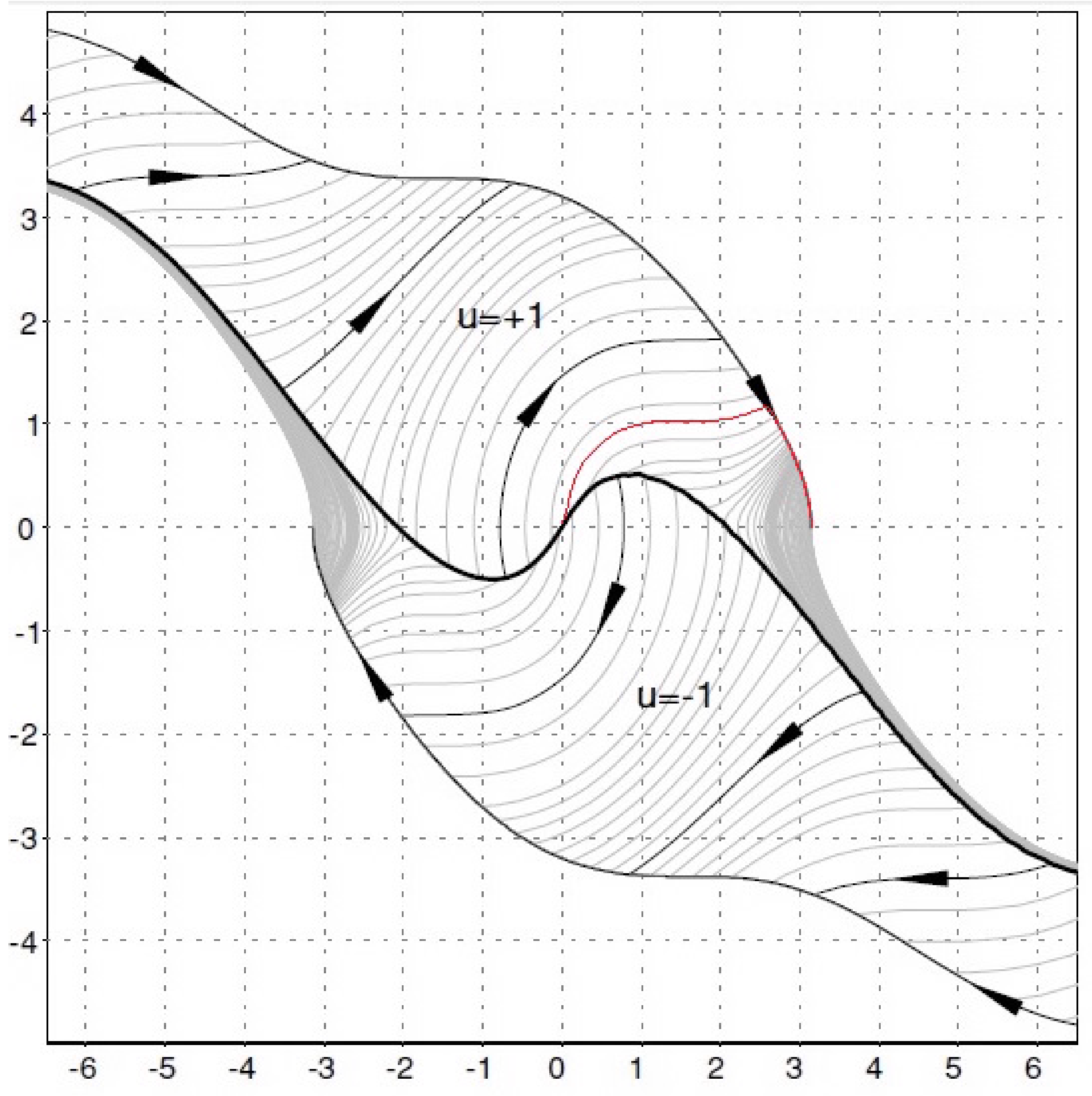

Figure 1: Phase portrait (θ on horizontal axis, ω on vertical axis) for $\kappa = 1$ from [6]; start at $(0,0)$, switching at $(2.570..., 1.207...)$, end at $(\pi, 0)$.

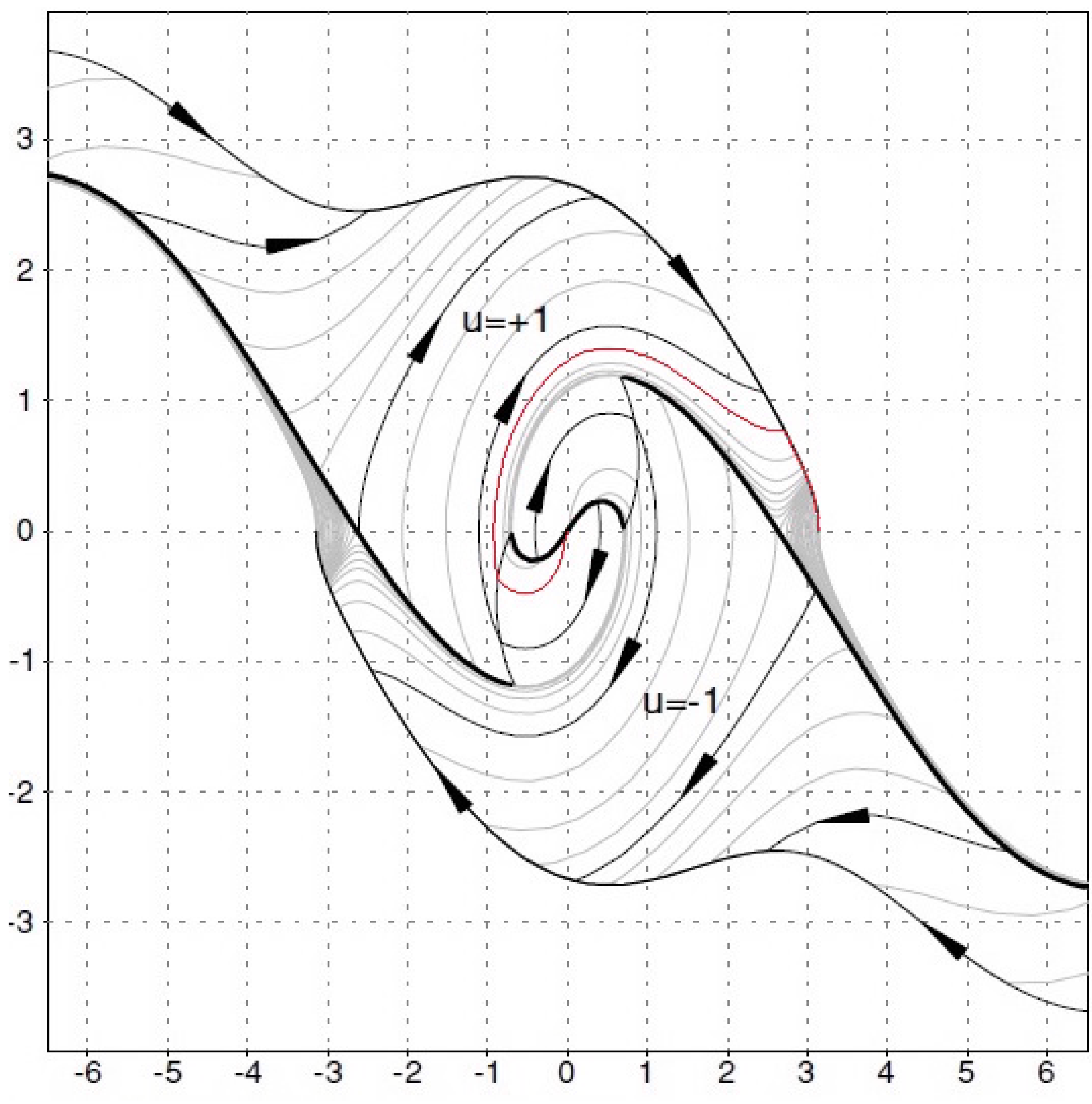

Figure 2: Phase portrait (θ on horizontal axis, ω on vertical axis) for $\kappa = 1/2$ from [6]; start at $(0, 0)$, switchings at $(-0.877..., -0.394...)$ & $(2.693..., 0.803...)$, end at $(\pi, 0)$.

latter, it is $t_{\infty,2} = 5.8397....$ The motion with two switchings is faster:

$$\frac{t_{\infty,1} - t_{\infty,2}}{t_{\infty,2}} \approx 12.5\%$$

but a motion with three (or more) switchings cannot improve upon $t_{\infty,2}$. We write $N_{3/4}(0,0) = 2$, where $(0,0)$ is the initial point and it is understood that the terminal point is $(\pi, 0)$. In the same way, $N_1(0,0) = 1$ and $N_{1/2}(0,0) = 2$. Define N_κ to be the supremum of $N_\kappa(\theta_0, \omega_0)$ over all θ_0 and ω_0 in the phase space.

Pontryagin's principle guarantees that the optimal control, for any choice of κ, must be of bang-bang type. The complexity of such a control can be characterized by the optimal switching number $N_\varkappa$. Greater knowledge of the function $\kappa \mapsto N_\varkappa$ is therefore desirable. Numerical computations suggest that [10]

$$\inf_{N_\kappa=1} \kappa \approx 0.80, \qquad \inf_{N_\kappa=2} \kappa \approx 0.44$$

which are bifurcation values of the parameter κ (analogous to bifurcation values of the parameter a discussed in [11] with regard to quadratic iterates and period doubling). More precise estimates of these values would be good to see someday.

0.1. Damping Control. This scenario is dual to that for swing-up [8, 9]. Given $(\theta_0, \omega_0) = (\pi, 0)$, what is the unique strategy to drive the pendulum to $(\theta, \omega) = (0,0)$ via a bang-bang control $u = \pm 1$ with one switching? The solution is to initially apply $u = -1$ until the precise time $\tilde{t}_1$ when

$$(\theta, \omega) = \left(\frac{\pi}{2} + \frac{1}{\kappa}, -\sqrt{\kappa\,\pi - 2\sin\left(\frac{1}{\kappa}\right)} \right)$$

and subsequently apply $u = 1$ until the precise time $\tilde{t}_\infty$ when $(\theta, \omega) = (0,0)$. For example, if $\kappa = 1$, then

$$\tilde{t}_1 = T_- \left(\pi, 0, 0, \kappa; \frac{\pi}{2} + \frac{1}{\kappa} \right) = 1.0236648603... = t_\infty - t_1,$$

$$\tilde{t}_\infty = T_+ \left(\frac{\pi}{2} + \frac{1}{\kappa}, -\sqrt{\kappa\,\pi - 2\sin\left(\frac{1}{\kappa}\right)}, \tilde{t}_1, \kappa; 0 \right) = 4.0300186879... = t_\infty$$

but again this is valid only since $\cos(\varphi) - 1 + \kappa\,\varphi > 0$ for all $0 < \varphi < \pi$.

We write $\tilde{N}_1(\pi, 0) = 1$, where $(\pi, 0)$ is the initial point and it is understood that the terminal point is $(0,0)$. One might expect that $\tilde{N}_1(\theta_0, \omega_0)$ to be 1 always, but this is false. By an example given in [12], $\tilde{N}_1(-100, 14.16) = 2$ and the time improvement is 0.27% (less dramatic than before). The principal bifurcation value here is [13]

$$\inf_{\tilde{N}_\kappa=1} \kappa \approx 1.04$$

and we have asymptotics [14]

$$\inf_{\tilde{N}_\kappa=n} \kappa \sim \frac{1}{n}\frac{G}{2} = \frac{0.9259685259...}{n}$$

as $n \to \infty$, where the constant

$$G = \int_0^\pi \frac{\sin(z)}{z}\,dz = \sum_{j=0}^\infty \frac{(-1)^j \pi^{2j+1}}{(2j+1)(2j+1)!} = 1.8519370519...$$

is well-known from approximation theory [15]. Although the theory in [14] is devoted to damping, which differs substantially from swing-up, the asymptotic constant $G/2$ evidently remains the same.

More references appear in [16], including mention of a double pendulum and chaos. Time optimal control of such appears to be difficult [17].

0.2. Addendum. The formula $\theta_1 = \pi/2 + 1/\kappa$ corresponding to one switching has a complicated analog for two switchings [18]. Define

$$\xi(\rho) = \frac{\rho}{2} + \frac{\cos(\rho) - 1}{2\kappa}, \qquad \eta(\rho) = \xi(\rho) + \frac{\pi}{2} + \frac{1}{\kappa},$$

$$F(u, \rho) = \sqrt{-2\kappa(u - \rho) - 2\left[\cos(u) - \cos(\rho)\right]}$$

and solve for ρ via the following equation:

$$\frac{1}{F(\xi(\rho), \rho)} + \frac{1}{F(\eta(\rho), \rho)} + \int_{\xi(\rho)}^{\rho} \frac{-\sin(u) + \sin(\rho)}{F(u, \rho)^3} du + \int_{\eta(\rho)}^{\rho} \frac{-\sin(v) + \sin(\rho)}{F(v, \rho)^3} dv = 0.$$

In the event $\kappa = 1/2$, we obtain $\rho = -0.937739...$ and hence $\theta_1 = \xi = -0.877...$, $\theta_2 = \eta = 2.693....$ In the event $\kappa = 3/4$, we obtain $\rho = -0.521237...$ and hence $\theta_1 = -0.349...$, $\theta_2 = 2.554....$ To compute ω_1 and ω_2 involves $F(\xi, \rho)$ and $F(\eta, \rho)$ respectively.

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