Power Series with Restricted Coefficients

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Define a family of functions

$$\mathcal{F}=\left\{1+\sum_{n=1}^{\infty}a_n x^n : a_n\in\{-1,0,1\}\right\}$$

and three closed subsets of the open interval $(0,1)$:

$$\Omega_2=\{x:\exists f\in\mathcal{F}\text{ for which }f(x)=f'(x)=0\},$$

$$\Omega_3=\{x:\exists f\in\mathcal{F}\text{ for which }f(x)=f'(x)=f''(x)=0\},$$

$$\Omega_4=\{x:\exists f\in\mathcal{F}\text{ for which }f(x)=f'(x)=f''(x)=f'''(x)=0\}.$$

Elements of Ω_2 are called **double zeroes**, those of Ω_3 **triple zeroes** and those of Ω_4 **quadruple zeroes**. For each $k=2,3,4$, define [1]

$$\alpha_k=\min\Omega_k,\qquad \widetilde{\alpha}_k=\sup\Omega_k^c$$

where Ω_k^c is the complement of Ω_k in $(0,1)$. The structure of Ω_k is very complicated – it appears to possess infinitely many connected components – but provably $\alpha_2=0.6684756...$ and conjecturally

$$\widetilde{\alpha}_2=0.669...,\qquad \alpha_3=0.743...,\qquad \widetilde{\alpha}_3\approx 0.75....$$

No one has yet examined α_4 or $\widetilde{\alpha}_4$ numerically, as far as is known. Elements of Ω_2^c are said to satisfy a certain **tranversality condition**, in the sense that $y\in\Omega_2^c$ and $f(y)=0$ imply that $f'(y)\neq 0$ for all $f\in\mathcal{F}$. Such a property is useful in [2] for a seemingly unrelated analysis of fractals.

Define instead

$$\hat{\mathcal{F}}=\left\{1+\sum_{n=1}^{\infty}a_n x^n : a_n\in\{-2,-1,0,1,2\}\right\}$$

and $\hat{\Omega}_2$ to be the corresponding set of double zeroes in $(0,1)$. In this case, $\min\hat{\Omega}_2$ is precisely $1/2$ and is an isolated point of $\hat{\Omega}_2$. Removing $1/2$ from $\hat{\Omega}_2$ appears to give a connected set (that is, an interval) and the minimum of this set is conjectured to be ≈ 0.5437. The fact that Ω_2 and $\hat{\Omega}_2$ are so distinct topologically is very striking [1].

A different family of functions, studied earlier in [3, 4], is

$$\mathcal{G}=\left\{1+\sum_{n=1}^{\infty}b_n x^n : b_n\in[-1,1]\right\}.$$

Let β_k denote the associated minimum zero of order k (at least) of g, taken over all $g\in\mathcal{G}$. It turns out that β_k is always algebraic: $\beta_2=0.6491378608...$ has minimal polynomial

$$2z^5-8z^2+11z-4,$$

$\beta_3=0.7278832326...$ has minimal polynomial

$$10z^{12}-14z^{11}+14z^6-10z^5-80z^3+185z^2-147z+40,$$

and $\beta_4=0.7773295434...$ has minimal polynomial

$$\begin{gathered}126z^{22}-296z^{21}+176z^{20}+44z^{12}-104z^{11}+54z^{10}+96z^7\\-146z^6+56z^5-684z^4+2236z^3-2797z^2+1584z-342.\end{gathered}$$

Of course, $\beta_1=1/2$, which corresponds to $g(x)=1-\sum_{n=1}^{\infty}x^n$. The following least squares approximation

$$\beta_k\approx 1-\frac{1}{(1.23909318...)+(0.81255949...)k}$$

was obtained in [4] and is based on data up to $k=27$. We wonder if more precise asymptotics are feasible. Additional relevant references include [5, 6, 7].

References

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