Quinn-Rand-Strogatz Constant

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We present two problems: one is easy (for the sake of comparison) and the other is difficult. The unique solution $s > 0$ of the equation

$$0 = \sum_{j=1}^{n} \left[1 - 3s^2 \left(1 - 2\tfrac{j-1}{n-1} \right)^2 \right]$$

is

$$s = \sqrt{\tfrac{n-1}{n+1}} \sim 1 - \tfrac{1}{n} + \tfrac{1}{2}\tfrac{1}{n^2} - \tfrac{1}{2}\tfrac{1}{n^3} + \tfrac{3}{8}\tfrac{1}{n^4} - \tfrac{3}{8}\tfrac{1}{n^5} + - \cdots$$

as $n \to \infty$. Define $s_n = 1 - 1/n$, the first-order approximation, and a partial sum

$$f_n(x) = \sum_{j=1}^{n} \left[1 - s_n^2 \left(1 - 2\tfrac{j-1}{n-1} \right)^2 \right]^{-x}$$

for $x > 0$. It follows that

$$\lim_{n\to\infty} \frac{f_n(1)}{n \ln(n)} = \frac{1}{2}, \qquad \lim_{n\to\infty} \frac{f_n(2)}{n^2} = \frac{\pi^2}{16}$$

and such formulas for other values of x are possible.

The unique solution $s > 0$ of the equation

$$0 = \sum_{j=1}^{n} \left[2\sqrt{1 - s^2 \left(1 - 2\tfrac{j-1}{n-1} \right)^2} - \frac{1}{\sqrt{1 - s^2 \left(1 - 2\tfrac{j-1}{n-1} \right)^2}} \right]$$

satisfies [1, 2, 3]

$$s \sim 1 - \tfrac{c_1}{n} - \tfrac{c_2}{n^2} - \tfrac{c_3}{n^3} - \tfrac{c_4}{n^4} - \cdots$$

as $n \to \infty$, where

$$c_1 = 0.6054436571..., \qquad c_2 = -0.1046854594...,$$

$$c_3 = 0.1263143361..., \qquad c_4 = -0.0159376251....$$

Bailey, Borwein & Crandall [2] proved the c_1 is the unique solution $y \in (0,2)$ of the equation

$$\zeta(1/2, y/2) = 0$$

where

$$\zeta(z,a) = \sum_{\substack{k=0 \\ k+a\neq 0}}^{\infty} \frac{1}{(k+a)^z}$$

is the Hurwitz zeta function (with analytic continuation). Further,

$$c_2 = c_1 - c_1^2 - 30\frac{\zeta(-1/2, c_1/2)}{\zeta(3/2, c_1/2)}$$

but exact expressions for c_3, c_4 remain open [3]. Define $s_n = 1 - 1/n$, the first-order approximation, and a partial sum $f_n(x)$ exactly as before. It follows that

$$\lim_{n\to\infty} \frac{f_n(3/2)}{n^{3/2}} = \frac{1}{4}\zeta\left(\frac{3}{2}, \frac{c_1}{2}\right) = 2.0381693797....$$

It is believed that analogous formulas involving Hurwitz zeta function values should exist for other choices of x.

We refer to [1, 4] for discussion of the theory of self-synchronizing systems, and hope to motivate the second (difficult) equation more clearly at a later time.

References

[1] D. D. Quinn, R. H. Rand and S. H. Strogatz, Singular unlocking transition in the Winfree model of coupled oscillators, *Phys. Rev. E* 75 (2007) 036218; MR2358543.

[2] D. H. Bailey, R. E. Crandall and J. M. Borwein, Resolution of the Quinn-Rand-Strogatz constant of nonlinear physics, *Experim. Math.* 18 (2009) 107–116; http://www.davidhbailey.com/dhbpapers/.

[3] N. J. Durgin, S. M. Garcia, T. Flournoy and D. H. Bailey, 'Syncing' up with the Quinn-Rand-Strogatz constant: Hurwitz-zeta functions in non-linear physics, unpublished note (2007); http://www.davidhbailey.com/dhbpapers/.

[4] A. Pikovsky, M. Rosenblum and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Science,* Cambridge Univ. Press, 2001.