Random Triangles II

Steven Finch

February 9, 2010

Let S denote the unit sphere in Euclidean 3-space. A spherical triangle T is a region enclosed by three great circles on S; a great circle is a circle whose center is at the origin. The sides of T are arcs of great circles and have length a, b, c. Each of these is $\leq \pi$. The angle α opposite side a is the dihedral angle between the two planes passing through the origin and determined by arcs b, c. The angles β, γ opposite sides b, c are similarly defined. Each of these is $\leq \pi$ too [1].

The sum of the angles is $\leq 3\pi$ yet $\geq \pi$. In particular, the sum need not be the constant π. Define the spherical excess $E = \alpha + \beta + \gamma - \pi$. The sum of the sides is ≥ 0 yet $\leq 2\pi$. Define the spherical defect $D = 2\pi - (a + b + c)$. It can be shown that the area of T is E and a calculus-based proof appears in [2]; see also [3]. Clearly the perimeter of T is $2\pi - D$.

The probability density functions for sides, angles, excess and defect on S will occupy us in this essay. Random triangles are defined here by selecting three independent uniformly distributed points on the sphere to be vertices. One way to do this is to let X_1, X_2, X_3, Y_1, Y_2, Y_3, Z_1, Z_2, Z_3 be independent normally distributed random variables with mean 0 and variance 1; then the points

$$\frac{(X_1, Y_1, Z_1)}{\sqrt{X_1^2 + Y_1^2 + Z_1^2}}, \qquad \frac{(X_2, Y_2, Z_2)}{\sqrt{X_2^2 + Y_2^2 + Z_2^2}}, \qquad \frac{(X_3, Y_3, Z_3)}{\sqrt{X_3^2 + Y_3^2 + Z_3^2}}$$

satisfy our requirements. Any spherically-symmetric underlying distribution will do, in fact, but we shall refer to the normal variables X_i, Y_j, Z_k again at a later time.

0.1. Sides. The trivariate density $f(x, y, z)$ for sides a, b, c is [4]

$$\begin{cases} \dfrac{1}{4\pi} \dfrac{\sin(x)\sin(y)\sin(z)}{\sqrt{1 - \cos(x)^2 - \cos(y)^2 - \cos(z)^2 + 2\cos(x)\cos(y)\cos(z)}} & \\ \qquad\qquad \text{if } x + y + z < 2\pi,\ x + y > z,\ y + z > x \text{ and } z + x > y, & \\ 0 \qquad\qquad \text{otherwise.} & \end{cases}$$

As a consequence, the univariate density for a is

$$\frac{1}{2}\sin(x), \qquad 0 < x < \pi$$

and

$$\mathrm{E}(a) = \frac{\pi}{2} = 1.5707963267..., \qquad \mathrm{E}(a^2) = \frac{\pi^2}{2} - 2 = 2.9348022005....$$

Sides a, b, c are uncorrelated and, moreover, pairwise independent. They are, however, mutually dependent, since [5, 6, 7, 8]

$$\mathrm{P}\left(a < \frac{\pi}{2}, b < \frac{\pi}{2}, c < \frac{\pi}{2}\right) = \frac{1}{4}\left(1 - \frac{1}{\pi}\right) > \frac{1}{8},$$

$$\mathrm{P}\left(a > \frac{\pi}{2}, b > \frac{\pi}{2}, c > \frac{\pi}{2}\right) = \frac{1}{4\pi} < \frac{1}{8}$$

and since $\mathrm{E}(a\,b\,c) = 3.694... < \pi^3/8$.

0.2. Angles. The trivariate density $g(x, y, z)$ for angles α, β, γ is [4]

$$\begin{cases} -\dfrac{1}{\pi}\dfrac{\cos\left(\frac{x+y+z}{2}\right)\cos\left(\frac{-x+y+z}{2}\right)\cos\left(\frac{x-y+z}{2}\right)\cos\left(\frac{x+y-z}{2}\right)}{\sin(x)^2\sin(y)^2\sin(z)^2} & \\ \qquad\qquad \text{if } x+y+z > \pi,\ x+y < \pi+z,\ y+z < \pi+x \text{ and } z+x < \pi+y, & \\ 0 \qquad\qquad \text{otherwise.} & \end{cases}$$

As a consequence, α is uniformly distributed on $[0, \pi]$ and

$$\mathrm{E}(\alpha) = \frac{\pi}{2} = 1.5707963267..., \qquad \mathrm{E}(\alpha^2) = \frac{\pi^2}{3} = 3.8757845850....$$

Angles α, β, γ are uncorrelated but, unlike before, pairwise *dependent*. Integrating out z, the bivariate density for α, β is

$$\frac{1}{2\pi}\frac{1}{\sin(x)^2\sin(y)^2}\cdot\begin{cases} -\cos(y)\sin(y) + y & \text{if } x - y > 0 \text{ and } x + y < \pi, \\ \pi + \cos(y)\sin(y) - y & \text{if } x - y < 0 \text{ and } x + y > \pi, \\ -\cos(x)\sin(x) + x & \text{if } x - y < 0 \text{ and } x + y < \pi, \\ \pi + \cos(x)\sin(x) - x & \text{if } x - y > 0 \text{ and } x + y > \pi \end{cases}$$

which is not uniform on $[0, \pi] \times [0, \pi]$. The mutual dependence can also be seen from [5, 6, 7, 8]

$$\mathrm{P}\left(\alpha < \frac{\pi}{2}, \beta < \frac{\pi}{2}, \gamma < \frac{\pi}{2}\right) = \frac{1}{2}\left(\frac{1}{\pi} - \frac{1}{4}\right) < \frac{1}{8},$$

$$\mathrm{P}\left(\alpha > \frac{\pi}{2}, \beta > \frac{\pi}{2}, \gamma > \frac{\pi}{2}\right) = \frac{1}{2}\left(\frac{3}{4} - \frac{1}{\pi}\right) > \frac{1}{8}$$

and from $\mathrm{E}(\alpha\,\beta\,\gamma) = 4.688... > \pi^3/8$.

0.3. Excess and Defect. In this section, we gather several results which seem to defy easy analysis. A proof that angle α is uncorrelated with either adjacent side b or c is known, hence $\mathrm{E}(\alpha\, b) = \pi^2/4 = \mathrm{E}(\alpha\, c)$ immediately. The joint moment of α with its opposite side a is obviously a triple integral:

$$\mathrm{E}(\alpha\, a) = \frac{1}{4\pi}\int_0^\pi\int_0^\pi\int_0^\pi \sin(x)\sin(y) z \arccos\left[\cos(x)\cos(y) + \sin(x)\sin(y)\cos(z)\right] dx\, dy\, dz$$

whose exact evaluation seems difficult. Miles [4] proved, via stochastic geometry, that $\mathrm{E}(\alpha\, a) = \pi^2/2 - 2$ as a special case of a more general theorem. As a consequence, the correlation coefficient between α and a is

$$\rho(\alpha, a) = \frac{\sqrt{3(\pi^2 - 8)}}{\pi} = 0.7538511740....$$

Recall from [9] that analogous results for Gaussian triangles in the plane remain open.

Clearly

$$\mathrm{E}(\alpha + \beta + \gamma - \pi) = \frac{\pi}{2}, \qquad \mathrm{E}((\alpha + \beta + \gamma - \pi)^2) = \frac{\pi^2}{2},$$
$$\mathrm{E}(2\pi - a - b - c) = \frac{\pi}{2}, \qquad \mathrm{E}((2\pi - a - b - c)^2) = \pi^2 - 6$$

however the verification of

$$\mathrm{E}((\alpha + \beta + \gamma - \pi)(2\pi - a - b - c)) = 6 - \frac{\pi^2}{2},$$

$$\rho(E, D) = -\frac{\sqrt{3(\pi^2 - 8)}}{\pi} = -0.7538511740...$$

rests on the aforementioned nontrivial result.

A proposed density $h(x)$ for excess E was published in 1867 [10]:

$$-\frac{(x^2 - 4\pi x + 3\pi^2 - 6)\cos(x) - 6(x - 2\pi)\sin(x) - 2(x^2 - 4\pi x + 3\pi^2 + 3)}{16\pi\cos(x/2)^4}$$

for $0 < x < 2\pi$ and remained obscure until it was cited in a recent paper [11]. Details of the supporting geometric proof need to be carefully examined. No analytic proof using our trivariate density for α, β, γ has yet been found. In some relevant 1928 calculations, Burnside [12] remarked that, "in a similar way", the probability that the area of T should lie between x and $x + dx$ "may be determined". Miles [4] confessed in 1971 that the functional form of $h(x)$ has "so far eluded the author", but then mentioned (in a footnote) work of J. N. Boots evidently leading to $h(x)$. No additional information is known, nor has anyone conjectured a density formula for defect D.

0.4. Proof for (a, b, γ)**.** We will demonstrate that a, b, γ are independent random variables; the sides a, b each have the sine density on $[0, \pi]$ and the angle γ is uniformly distributed on $[0, \pi]$. Our starting point is the fact that a is an angle between two vectors (X_1, Y_1, Z_1) and (X_3, Y_3, Z_3), where X_i, Y_j, Z_k were defined earlier, and b likewise for the vectors (X_2, Y_2, Z_2) and (X_3, Y_3, Z_3). The formulas [13, 14, 15]

$$\cos(a) = \frac{X_1 X_3 + Y_1 Y_3 + Z_1 Z_3}{\sqrt{X_1^2 + Y_1^2 + Z_1^2}\sqrt{X_3^2 + Y_3^2 + Z_3^2}}, \qquad \cos(b) = \frac{X_2 X_3 + Y_2 Y_3 + Z_2 Z_3}{\sqrt{X_2^2 + Y_2^2 + Z_2^2}\sqrt{X_3^2 + Y_3^2 + Z_3^2}}$$

are familiar: $\cos(a)$ is the sample correlation coefficient r_{13} between two samples of size three (each sample coming from a population of known mean $= 0$) and $\cos(b)$ is likewise the sample correlation coefficient r_{23}. Also, by the Law of Cosines for Sides:

$$\cos(c) = \cos(a)\cos(b) + \sin(a)\sin(b)\cos(\gamma)$$

we obtain

$$\cos(\gamma) = \frac{\cos(c) - \cos(a)\cos(b)}{\sin(a)\sin(b)} = \frac{r_{12} - r_{13} r_{23}}{\sqrt{1 - r_{13}^2}\sqrt{1 - r_{23}^2}}$$

and recognize this as the sample partial correlation coefficient $r_{12 \cdot 3}$ between samples 1 and 2, holding variable 3 fixed. An exercise in [16] states that r_{13}, r_{23}, $r_{12 \cdot 3}$ are independent because X_i, Y_j, Z_k are independent and normally distributed. Hence a, b, γ are independent as well.

The sample correlation coefficient r_{13} is uniformly distributed on $[-1, 1]$, as a special case of results given in [17, 18, 19, 20], hence

$$\begin{aligned} \mathrm{P}(a < \xi) &= \mathrm{P}(\cos(a) > \cos(\xi)) = \mathrm{P}(r_{13} > \cos(\xi)) = \frac{1}{2}\int_{\cos(\xi)}^{1} d\eta \\ &= \frac{1 - \cos(\xi)}{2} \end{aligned}$$

and $d\,\mathrm{P}(a < \xi)/d\xi = \sin(\xi)/2$. The sample partial correlation coefficient $r_{12 \cdot 3}$ has the arcsine distribution on $[-1, 1]$, hence

$$\begin{aligned} \mathrm{P}(\gamma < \xi) &= \mathrm{P}(\cos(\gamma) > \cos(\xi)) = \mathrm{P}(r_{12 \cdot 3} > \cos(\xi)) = \frac{1}{\pi}\int_{\cos(\xi)}^{1} \frac{d\eta}{\sqrt{1 - \eta^2}} \\ &= \frac{1}{2} - \frac{1}{\pi}\arcsin(\cos(\xi)) = \frac{1}{\pi}\xi \end{aligned}$$

and $d\,\mathrm{P}(\gamma < \xi)/d\xi = 1/\pi$, as was to be shown.

Geisser & Mantel [21] were the first to notice that the correlations r_{13}, r_{23}, r_{12} are pairwise but not mutually independent (for samples of arbitrary size). This "natural" example has been justly celebrated and is of "valuable pedagogical use" [22]. Recasting the example in terms of spherical triangle sides a, b, c makes it even more remarkable, in our opinion. No one seems to have linked Miles' paper [4] in geometric probability to ongoing research in theoretical statistics.

0.5. Proof for (a,β,γ)**.** We bring β into the trivariate density $\sin(a)\sin(b)/(4\pi)$, removing b. From the Law of Cosines for Angles:

$$-\cos(\alpha) = \cos(\beta)\cos(\gamma) - \sin(\beta)\sin(\gamma)\cos(a)$$

we have

$$\begin{aligned} \sin(\alpha)^3 &= \left(1-\cos(\alpha)^2\right)^{3/2} \\ &= \left(1-(\cos(\beta)\cos(\gamma)-\sin(\beta)\sin(\gamma)\cos(a))^2\right)^{3/2} \end{aligned}$$

since $0<\alpha<\pi$. Differentiating the identity [1, 4]

$$\sin(a)\cot(b) = \cot(\beta)\sin(\gamma) + \cos(\gamma)\cos(a)$$

with respect to b, we obtain

$$-\sin(a)\csc(b)^2 db = -\csc(\beta)^2\sin(\gamma)d\beta$$

hence

$$db = \frac{\sin(b)^2\sin(\gamma)}{\sin(a)\sin(\beta)^2}d\beta.$$

Via the Law of Sines:

$$\frac{\sin(a)}{\sin(\alpha)} = \frac{\sin(b)}{\sin(\beta)} = \frac{\sin(c)}{\sin(\gamma)}$$

the density $\sin(a)\sin(b)/(4\pi)$ becomes

$$\begin{aligned} \frac{1}{4\pi}\sin(a)\frac{\sin(b)^3\sin(\gamma)}{\sin(a)\sin(\beta)^2} &= \frac{1}{4\pi}\frac{\sin(b)^3}{\sin(\beta)^3}\sin(\beta)\sin(\gamma) \\ &= \frac{1}{4\pi}\frac{\sin(a)^3}{\sin(\alpha)^3}\sin(\beta)\sin(\gamma) \\ &= \frac{1}{4\pi}\frac{\sin(\beta)\sin(\gamma)\sin(a)^3}{(1-(\cos(\beta)\cos(\gamma)-\sin(\beta)\sin(\gamma)\cos(a))^2)^{3/2}}. \end{aligned}$$

More elaborate arguments lead to the trivariate densities of (a,b,c) and (α,β,γ).

This preceding expression is helpful for computing the bivariate density of (β,γ). Integrating out a gives

$$\frac{|\sin(\beta-\gamma)|\cos(\beta+\gamma)-|\sin(\beta+\gamma)|\cos(\beta-\gamma)+\arcsin(\cos(\beta-\gamma))-\arcsin(\cos(\beta+\gamma))}{4\pi\sin(\beta)^2\sin(\gamma)^2}$$

which seems complicated at first glance. Everything simplifies if we partition the square $[0,\pi]\times[0,\pi]$ into four isosceles right triangles according to the diagonal lines $\beta-\gamma=0$, $\beta+\gamma=\pi$. For example, if $\beta-\gamma>0$ and $\beta+\gamma<\pi$, then the numerator becomes $-2\cos(\gamma)\sin(\gamma)+2\gamma$. As another example, if $\beta-\gamma<0$ and $\beta+\gamma>\pi$, then the numerator becomes $2\pi+2\cos(\gamma)\sin(\gamma)-2\gamma$. For the remaining two triangles, γ is merely replaced by β, by symmetry. Such formulas can be used to confirm directly that β, γ are each uniformly distributed on $[0,\pi]$ and $\mathrm{E}(\beta\,\gamma)=\pi^2/4$.

A joint density for (a,α) might assist in evaluating the triple integral mentioned earlier, but finding this (and the joint density of $(r_{13},r_{13.2})$) seems to be hard.

0.6. Addendum. Jones & Benyon-Tinker [23] expressed the perimeter density in terms of elliptic integrals [9]:

$$\frac{1}{4\pi}\int\limits_0^{x/2}\frac{E\left(\sin\left(\frac{t}{2}\right)\right)-\cos\left(\frac{x-t}{2}\right)^2K\left(\sin\left(\frac{t}{2}\right)\right)}{\sqrt{\cos\left(\frac{t}{2}\right)^2-\cos\left(\frac{x-t}{2}\right)^2}}\sin(t)\,dt$$

which contrasts with the 1867 area density (for which an elementary formula is available). Finch & Jones [24] found that the perimeter density has value $3\sqrt{2}/32$ at $x=\pi$, and also revisited the proof of the area density formula (valid at all x).

Miles' [4] proof that $\mathrm{E}(\alpha\,a)=\pi^2/2-2$ is clarified in [25]. Let ${}_pF_q$ denote the generalized hypergeometric function and G denote Catalan's constant [26]. It is interesting that the conditional moment

$$\begin{aligned}\mathrm{E}(\alpha\,a\,|\,b &= \pi/2)=3.0538319164...\\ &= \frac{1}{4}\int\limits_0^{\pi}\left\{2-\frac{4}{\pi}\left[\frac{1}{\cos(s)^2}E\left(\cos\left(s\right)\right)+\left(1-\frac{1}{\cos(s)^2}\right)K\left(\cos\left(s\right)\right)\right]\cos(s)\right\}s\,ds\\ &= \frac{\pi^2}{2}-\frac{4G}{\pi}-\frac{2}{\pi}\,{}_4F_3\left(\frac{1}{2},\frac{1}{2},1,1;\frac{3}{2},\frac{3}{2},\frac{3}{2};1\right)\end{aligned}$$

remains complicated whereas

$$\mathrm{E}(\alpha\,a\,|\,\beta=\pi/2)=2.8708787614...=\frac{\pi}{4}\left[2+(1+\ln(2))\,\pi-4G\right]$$

is simple.

Any spherical triangle T determines a unique chordal triangle T' (with sides as straight lines through the interior of S) and vice versa. Let r' denote the radius of the unique circle passing through the three vertices of T'. The density of two T' angles is given in [27], as well as the trivariate density of two T' sides coupled with r'. Such results lead to progress in answering an open question: What is the exact probability that four random circular caps of angular radius 88° completely cover S? The progress is, however, insignificant if 88° is replaced by, say, 71°. We hope to see resolution of this issue someday.

REFERENCES

[1] I. Todhunter, *Spherical Trigonometry for the Use of Colleges and Schools*, 5^{th} ed., Macmillan, 1886, pp. 6–9, 16–32, 67–74; http://digital.library.cornell.edu/m/math/browse/title/s.php.

[2] J. Makesh Pravin, Symmetry on a sphere, *Resonance* 7 (2002) 84–88.

[3] J. C. Polking, The area of a spherical triangle. Girard's theorem, http://math.rice.edu/~pcmi/sphere/.

[4] R. E. Miles, Random points, sets and tessellations on the surface of a sphere, *Sankhya Ser. A* 33 (1971) 145–174; MR0321150 (47 #9683).

[5] Y. Maeda, Seven types of random spherical triangle in S^n and their probabilities, *Computational Geometry and Graph Theory*, Proc. 2007 Kyoto conf., Lect. Notes in Comp. Sci. 4535, Springer-Verlag, 2008, pp. 119–126.

[6] E. J. Watson and E. H. Lockwood, Probability of an acute-angled spherical triangle, *Math. Gazette* 58 (1974) 217–218.

[7] M. W. Crofton and Exhumatus, Problem 2326, *Mathematical Questions and Solutions from the "Educational Times"*, v. 7, ed. W. J. Miller, Hodgson & Son, Jan.-Jul. 1867, pp. 105–106; http://books.google.com/.

[8] S. R. Finch, Simulations in R involving triangles and tetrahedra, http://www.people.fas.harvard.edu/~sfinch/csolve/rsimul.html.

[9] S. R. Finch, Random triangles, unpublished note (2010).

[10] M. W. Crofton and Exhumatus, Problem 2370, *Mathematical Questions and Solutions from the "Educational Times"*, v. 8, ed. W. J. Miller, Hodgson & Son, Jul.-Dec. 1867, pp. 21–23; http://books.google.com/.

[11] M. Hindmarsh, A.-C. Davis and R. Brandenberger, Formation of topological defects in first order phase transitions, *Phys. Rev. D* 49 (1994) 1944–1950; arXiv:hep-ph/9307203.

[12] W. Burnside, *Theory of Probability*, Cambridge Univ. Press, 1928, pp. 74–79.

[13] D. Jackson, The trigonometry of correlation, *Amer. Math. Monthly* 31 (1924) 275–280.

[14] J. O. Irwin, Note on the addition formula for the Jacobian elliptic functions and their connection with the theory of correlation in statistics, *Math. Gazette* 49 (1965) 425–427.

[15] G. Thomas, G., and J. O'Quigley, A geometric interpretation of partial correlation using spherical triangles, *Amer. Statist.* 47 (1993) 30–32.

[16] T. W. Anderson, *An Introduction to Multivariate Statistical Analysis*, 2nd ed., Wiley, 1984, pp. 102–109, 130–133, 251, 266–268, 282–283 (esp. exercises 21 & 22 in sect. 7.6); MR0771294 (86b:62079).

[17] N. L. Johnson, S. Kotz and N. Balakrishnan, *Continuous Univariate Distributions*, v. II, 2nd ed., Wiley, 1995, pp. 545–638; MR1326603 (96j:62029).

[18] W. A. Chance, A geometric derivation of the distribution of the correlation coefficient $|r|$ when $\rho = 0$, *Amer. Math. Monthly* 93 (1986) 94–98; MR0827582 (87d:62102).

[19] M. S. Srivastava and C. G. Khatri, *An Introduction to Multivariate Statistics*, North-Holland, 1979, pp. 87–92, 100–101; MR0544670 (81j:62001).

[20] T. W. Anderson, R. A. Fisher and multivariate analysis, *Statist. Sci.* 11 (1996) 20–34; MR1437125.

[21] S. Geisser and N. Mantel, Pairwise independence of jointly dependent variables, *Annals Math. Statist.* 33 (1962) 290–291; MR0137188 (25 #644).

[22] R. A. Schaufele, An application of pairwise independence of random variables to regression analysis, *Canad. Math. Bull.* 18 (1975) 397–404; MR0405733 (53 #9525).

[23] A. J. Jones and G. Benyon-Tinker, The jet-set travelling salesman: (1) Triangular tours, unpublished manuscript (1994), http://users.cs.cf.ac.uk/Antonia.J.Jones/.

[24] S. R. Finch and A. J. Jones, Random spherical triangles, arXiv:1009.5329.

[25] S. R. Finch, Correlation between angle and side, arXiv:1012.0781.

[26] S. R. Finch, Catalan's constant, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 53–59.

[27] S. R. Finch, Covering a sphere with four random circular caps, arXiv:1502.06931.