Random Triangles III

Steven Finch

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Let Ω be a compact convex set in Euclidean n-space with nonempty interior. Random triangles are defined here by selecting three independent uniformly distributed points in Ω to be vertices. Generating such points for $(n,\Omega) = (2,$unit square$)$ or $(n,\Omega) = (3,$unit cube$)$ is straightforward. For $(n,\Omega) = (2,$unit disk$)$ or $(n,\Omega) = (3,$unit ball$)$, we use the following result [1]. Let X_1, X_2, X_3, Y_1, Y_2, Y_3, Z_1, Z_2, Z_3 be independent normally distributed random variables with mean 0 and variance 1/2. Let W_1, W_2, W_3 be exponential random variables, independent of the others, with mean 1. Then the points

$$\frac{(X_1,Y_1)}{\sqrt{X_1^2+Y_1^2+W_1}}, \qquad \frac{(X_2,Y_2)}{\sqrt{X_2^2+Y_2^2+W_2}}, \qquad \frac{(X_3,Y_3)}{\sqrt{X_3^2+Y_3^2+W_3}}$$

are uniform in the disk, and the points

$$\frac{(X_1,Y_1,Z_1)}{\sqrt{X_1^2+Y_1^2+Z_1^2+W_1}}, \qquad \frac{(X_2,Y_2,Z_2)}{\sqrt{X_2^2+Y_2^2+Z_2^2+W_2}}, \qquad \frac{(X_3,Y_3,Z_3)}{\sqrt{X_3^2+Y_3^2+Z_3^2+W_3}}$$

are uniform in the ball. Compared with the intricate joint distributions of sides and angles for Gaussian triangles [2] and for spherical triangles [3], little is known for uniform triangles in Ω.

0.1. Disk. The density $f(x)$ for an arbitrary side a of a random uniform triangle in the unit disk is [4, 5, 6, 7, 8, 9, 10, 11]

$$\frac{4x}{\pi}\arccos\left(\frac{x}{2}\right) - \frac{x^2}{\pi}\sqrt{4-x^2}, \qquad 0 < x < 2$$

and

$$\mathrm{E}(a) = \frac{128}{45\pi} = 0.9054147873..., \qquad \mathrm{E}(a^2) = 1.$$

No one has attempted to extend this univariate result to a bivariate or trivariate density, as far as is known.

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The density $g(x)$ for an arbitrary angle α is [12, 13]

$$\frac{\cos(5x)-(1-12\pi x+12x^2)\cos(x)}{12\pi^2\sin(x)^3}-\frac{(\pi-x)\cos(4x)+10(\pi-x)\cos(2x)+(\pi-13x)}{12\pi^2\sin(x)^2}$$

when $0<x<\pi$ and

$$\mathrm{E}(\alpha)=\frac{\pi}{3}=1.0471975511...,\qquad \mathrm{E}(\alpha^2)=\frac{\pi^2}{6}+\frac{1}{12}=1.7282674001....$$

We can also give partial results for the maximum angle (analogous to the Gaussian case [2]). Corresponding to the density of $\max\{\alpha,\beta,\gamma\}$, the expression $3g(x)$ holds when $\pi/2<x<\pi$; an expression when $\pi/3<x<\pi/2$ remains open, although a numerical approach is employed in [14]. It can also be shown that [15, 16, 17]

$$\mathrm{P}(\text{a uniform triangle in the disk is acute})=\frac{4}{\pi^2}-\frac{1}{8}=0.2802847345....$$

Moments of area are known [18, 19, 20, 21, 22, 23, 24, 25]:

$$\mathrm{E}(\text{area})=\frac{35}{48\pi}=0.2321009586...=(0.0738800297...)\pi,$$

$$\mathrm{E}(\text{area}^2)=\frac{3}{32}=0.09375$$

but only a complicated integral representation [22] is available for the density $h(x)$ of area. Conceivably the integrals might someday be evaluated in closed-form. It can be shown that $\lim_{x\to 0^+}h(x)=12/\pi$ and $\lim_{x\to 3\sqrt{3}/4^-}h(x)=0$; it seems that $h(x)$ is strictly decreasing (but a proof is lacking). We do not even have an expression for the second moment of perimeter.

0.2. Ball. The density $f(x)$ for an arbitrary side a of a random uniform triangle in the unit ball is [6, 10, 11]

$$\frac{3}{16}x^5-\frac{9}{4}x^3+3x^2,\qquad 0<x<2$$

and

$$\mathrm{E}(a)=\frac{36}{35}=1.0285714285...,\qquad \mathrm{E}(a^2)=\frac{6}{5}=1.2.$$

A recent extraordinary calculation [26, 27] gives a trivariate density for the sides (a,b,c). For reasons of space, we report only the bivariate density $f(x,y)$ for (a,b):

$$f(x,y)=\begin{cases}\varphi(x,y) & \text{if } x+y\leq 2,\\ \psi(x,y) & \text{if } x+y>2 \text{ and } x\leq 2\end{cases}$$

when $0 \leq y \leq x$ (use symmetry otherwise) where

$$\varphi(x,y) = \frac{9}{16}x^5y^2 - \frac{27}{4}x^3y^2 + \frac{27}{16}x^3y^3 + \frac{9}{16}x^3y^4 + 9x^2y^2 - \frac{27}{8}x^2y^3 + \frac{9}{32}x^2y^5 - \frac{9}{8}xy^4 + \frac{9}{160}xy^6,$$

$$\psi(x,y) = -\frac{9}{160}x^6y + \frac{9}{32}x^5y^2 + \frac{9}{8}x^4y - \frac{9}{16}x^4y^3 - \frac{9}{4}x^3y - \frac{27}{8}x^3y^2 + \frac{27}{16}x^3y^3 + \frac{9}{2}x^2y^2 + \frac{9}{5}xy - \frac{9}{4}xy^3.$$

It follows that

$$\mathrm{E}(a\,b) = \frac{884}{825} = 1.0715..., \qquad \rho(a,b) = \frac{884/825 - (36/35)^2}{6/5 - (36/35)^2} = \frac{274}{2871} = 0.0954...$$

and the cross-correlation coefficient is somewhat smaller than that found in the Gaussian case [2].

With regard to angles, apart from $\mathrm{E}(\alpha) = \pi/3$, all we know is that [16, 17]

$$\mathrm{P}(\text{a uniform triangle in the ball is acute}) = \frac{33}{70} = 0.4714285714....$$

We also have [28]

$$\mathrm{E}(\text{area}) = \frac{9\pi}{77} = 0.3671991413...,$$

$$\mathrm{E}(\text{perimeter}) = \frac{108}{35}, \qquad \mathrm{E}(\text{perimeter}^2) = \frac{2758}{275}.$$

Evaluating $\mathrm{E}(\text{area}^2)$ remains open; it is surprising that $\mathrm{E}(\text{perimeter}^2)$ is presently known in three dimensions but not in two dimensions.

0.3. Square. The density $f(x)$ for an arbitrary side a of a random uniform triangle in the unit square is [10, 29]

$$\begin{cases} 2x^3 - 8x^2 + 2\pi x & \text{if } 0 \leq x \leq 1, \\ 8x\sqrt{x^2-1} - 2x^3 + 2(\pi-2)x - 8x\arctan\left(\sqrt{x^2-1}\right) & \text{if } 1 < x \leq \sqrt{2} \end{cases}$$

and

$$\mathrm{E}(a) = \frac{1}{15}\left(2 + \sqrt{2} + 5\ln(1+\sqrt{2})\right) = 0.5214054331..., \qquad \mathrm{E}(a^2) = \frac{1}{3}.$$

Nothing comparable is known for an arbitrary angle α; we have only [30, 31]

$$\begin{aligned} \mathrm{P}(\text{a uniform triangle in the square is acute}) &= 1 - \left(\frac{97}{150} + \frac{\pi}{40}\right) \\ &= 1 - 0.7252064830... \\ &= 0.2747935169.... \end{aligned}$$

A remarkable formula holds for the density $h(x)$ for area [22, 32, 33, 34, 35]:

$$\begin{aligned}&(-16\pi^2x^2 - 16\pi^2x - 24x + 12) + (240x^2 - 96x - 12)\ln(1-2x)\\&-240x^2\ln(2x) + 48x^2\ln(2x)^2 + (96x^2 + 96x)\operatorname{Li}_2(2x)\end{aligned}$$

where $\operatorname{Li}_2(\xi)$ is the dilogarithm function [2]. As a consequence,

$$\mathrm{E}(\text{area}) = \frac{11}{144}, \qquad \mathrm{E}(\text{area}^2) = \frac{1}{96}.$$

Evaluating $\mathrm{E}(\text{perimeter}^2)$ remains open.

0.4. Cube. The density $f(x)$ for an arbitrary side a of a random uniform triangle in the unit cube is [36, 37, 38]

$$\begin{cases} -x^5 + 8x^4 - 6\pi x^3 + 4\pi x^2 & \text{if } 0 \le x \le 1, \\ \begin{aligned}&-8x(2x^2+1)\sqrt{x^2-1} + 2x^5 + 6x^3 - 8\pi x^2\\&+(6\pi - 1)x + 24x^3\arctan\left(\sqrt{x^2-1}\right)\end{aligned} & \text{if } 1 < x \le \sqrt{2} \\ \begin{aligned}&8x(x^2+1)\sqrt{x^2-2} - x^5 + 6(\pi-1)x^3 - 8\pi x^2 + (6\pi-5)x\\&-24x(x^2+1)\arctan\left(\sqrt{x^2-2}\right) + 24x^2\arctan\left(x\sqrt{x^2-2}\right)\end{aligned} & \text{if } \sqrt{2} < x \le \sqrt{3} \end{cases}$$

and

$$\mathrm{E}(a) = \frac{1}{105}\left(4 + 17\sqrt{2} - 6\sqrt{3} + 21\ln(1+\sqrt{2}) + 42\ln(2+\sqrt{3}) - 7\pi\right) = 0.6617071822...,$$

$$\mathrm{E}(a^2) = \frac{1}{2}.$$

Essentially nothing else is known: it would be good someday to learn more about the associated acuteness probability and $\mathrm{E}(\text{area})$. See [39] for experimental confirmations of the preceding; see [40] for related discussion.

0.5. Addendum. Mathai [41] expressed the area densities for both disk and ball in terms of the generalized hypergeometric function

$$ {}_pF_q(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; z) = \frac{\Gamma(b_1)\Gamma(b_2)\cdots\Gamma(b_q)}{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_p)} \sum_{k=0}^{\infty} \frac{\Gamma(a_1+k)\Gamma(a_2+k)\cdots\Gamma(a_p+k)}{\Gamma(b_1+k)\Gamma(b_2+k)\cdots\Gamma(b_q+k)} \frac{z^k}{k!}$$

(we saw the special case ${}_2F_1$ in [42]). Define

$$\Phi_n(y) = \frac{9}{16\pi}\frac{2^{n/2}}{(n-2)!}\Gamma\left(1+\frac{n}{2}\right)^3 y^{(n-3)/2}\left\{-\frac{4\pi^{3/2}}{\sqrt{3}}\frac{1}{\Gamma(\frac{1}{2}+\frac{n}{4})\Gamma(1+\frac{n}{4})}\left(\frac{4y}{27}\right)^{1/2}-\right.$$
$$\frac{3\Gamma(-\frac{5}{6})\Gamma(\frac{1}{6})}{2^{2/3}\sqrt{\pi}}\frac{1}{\Gamma(\frac{1}{6}+\frac{n}{4})\Gamma(\frac{2}{3}+\frac{n}{4})}\left(\frac{4y}{27}\right)^{5/6}{}_4F_3\left(\frac{1}{3},\frac{1}{3},\frac{5}{6}-\frac{n}{4},\frac{1}{3}-\frac{n}{4};\frac{2}{3},\frac{11}{6},\frac{4}{3};\frac{4y}{27}\right)+$$
$$\frac{9\sqrt{\pi}\Gamma(-\frac{7}{6})}{2^{1/3}\Gamma(\frac{1}{6})}\frac{1}{\Gamma(-\frac{1}{6}+\frac{n}{4})\Gamma(\frac{1}{3}+\frac{n}{4})}\left(\frac{4y}{27}\right)^{7/6}{}_4F_3\left(\frac{2}{3},\frac{2}{3},\frac{7}{6}-\frac{n}{4},\frac{2}{3}-\frac{n}{4};\frac{4}{3},\frac{13}{6},\frac{5}{3};\frac{4y}{27}\right)+$$
$$\left.\frac{4\sqrt{\pi}}{3}\frac{1}{\Gamma(1+\frac{n}{4})\Gamma(\frac{3}{2}+\frac{n}{4})}{}_4F_3\left(-\frac{1}{2},-\frac{1}{2},-\frac{n}{4},-\frac{1}{2}-\frac{n}{4};\frac{1}{6},-\frac{1}{6},\frac{1}{2};\frac{4y}{27}\right)\right\}$$

for $n \geq 2$ and $0 < y < 27/4$. Then the density $h(x)$ for area of a random uniform triangle in the unit disk is $8x\Phi_2\left(4x^2\right)$, which satisfies the limiting properties mentioned earlier. Also, using $8x\Phi_3\left(4x^2\right)$, we deduce that $\mathrm{E}(\text{area}^2) = 9/50$ for the unit ball.

The bivariate density for (a, b) in the unit disk can be found, imitating Parry's [26] analysis. Finch [43] concluded that

$$\mathrm{E}(a\,b) = 0.8378520652..., \qquad \rho(a, b) = 0.1002980835...,$$

$$\mathrm{E}(\text{perimeter}^2) = 8.0271123917...$$

but exact evaluation of these constants remains open.

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