Random Triangles IV

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We step back momentarily to gain perspective. By **parabolic geometry** is meant the study of distances, angles, etc. in a Riemannian manifold having zero scalar curvature; for example, geometry in two-dimensional Euclidean space $\mathbb{R}^2$ (the planar model).

By **elliptic geometry** is meant the study of such properties in a Riemannian manifold having positive scalar curvature. Given a line (geodesic) L and a point P not on L, there is no line parallel to L passing through P. The sum of the three angles of a triangle is greater than π; the quantity $(\alpha+\beta+\gamma)-\pi$ is called angular excess. The simplest example of this geometry is the spherical model S embedded in three-dimensional Euclidean space $\mathbb{R}^3$. Geodesics are great circles, that is, intersections of S with two-dimensional subspaces of $\mathbb{R}^3$.

By **hyperbolic geometry** is meant the study of such properties in a Riemannian manifold having negative scalar curvature. Given a line (geodesic) L and a point P not on L, there are at least two distinct lines parallel to L passing through P. The sum of the three angles of a triangle is less than π; the quantity $\pi-(\alpha+\beta+\gamma)$ is called angular defect. The simplest example of this geometry is the hyperboloidal model H embedded in three-dimensional Minkowski space $\mathbb{M}^3$. Geodesics are great hyperbolas, that is, *nonempty* intersections of H with two-dimensional subspaces of $\mathbb{M}^3$.

With regard to the latter, $\mathbb{M}^3$ is the vector space of ordered real triples (just like $\mathbb{R}^3$) equipped with the symmetric bilinear form [1, 2, 3]

$$q\left[(x,y,z),(u,v,w)\right]=-zw+xu+yv$$

instead of the usual (positive definite) inner product

$$p\left[(x,y,z),(u,v,w)\right]=xu+yv+zw.$$

Define the unit hyperboloid H to be the positive sheet ($z>0$) of points satisfying $q[(x,y,z),(x,y,z)]=-1$; equivalently,

$$H=\left\{(x,y,z)\in\mathbb{M}^3:z=\sqrt{1+x^2+y^2}\right\}.$$

This is analogous to the unit sphere S of points satisfying $p[(x,y,z),(x,y,z)] = 1$; equivalently,

$$S = \left\{(x,y,z) \in \mathbb{R}^3 : z = \pm\sqrt{1-x^2-y^2}\right\}.$$

Distance between two points in H:

$$\operatorname{arccosh}\left(-q\left[(x,y,z),(u,v,w)\right]\right)$$

is analogous to distance between two points in S:

$$\arccos\left(p\left[(x,y,z),(u,v,w)\right]\right)$$

(the latter is the angle at the origin determined by the two vectors).

A hyperbolic triangle T is a region enclosed by three geodesics on H. The sides of T are arcs of great hyperbolas and have length a, b, c. Since H is non-compact, there is no upper bound on these. To define a uniform distribution, we will need to introduce some restrictions. The angle α opposite side a is the dihedral angle between the two planes passing through the origin and determined by arcs b, c. The angles β, γ opposite sides b, c are similarly defined. Each of these is $\leq \pi$. By the Law of Cosines for Sides:

$$\cosh(c) = \cosh(a)\cosh(b) - \sinh(a)\sinh(b)\cos(\gamma)$$

we obtain

$$\cos(\gamma) = -\frac{\cosh(c) - \cosh(a)\cosh(b)}{\sinh(a)\sinh(b)}$$

analogous to an expression for $\cos(\gamma)$ in spherical trigonometry [4].

The disk of radius $R > 0$ on H is

$$\begin{aligned}\Delta_R &= \left\{(x,y,z) \in H : \operatorname{arccosh}\left(-q\left[(x,y,z),(0,0,1)\right]\right) \leq R\right\} \\ &= \left\{(x,y,z) \in H : z \leq \cosh(R)\right\}.\end{aligned}$$

This is analogous to the disk of radius $0 < R < \pi$ on S:

$$\left\{(x,y,z) \in S : \arccos\left(p\left[(x,y,z),(0,0,1)\right]\right) \leq R\right\} = \left\{(x,y,z) \in S : z \geq \cos(R)\right\};$$

the special case when $R = \pi/2$ is a hemisphere on S.

The orthogonal projection of Δ_R ($\subset H$) into the xy-plane gives simply the disk $x^2 + y^2 \leq \sinh(R)^2$ because

$$\sqrt{1+x^2+y^2} = z \leq \cosh(R) \qquad \text{implies} \qquad x^2+y^2 \leq \cosh(R)^2 - 1 = \sinh(R)^2.$$

It is hence apparent [5] that circular circumference is proportional to $\sinh(R)$.

An alternative mapping from Δ_R into the xy-plane is nonlinear:

$$\left(\begin{array}{c}\sqrt{z^2-1}\cos(\theta)\\ \sqrt{z^2-1}\sin(\theta)\\ z\end{array}\right)\mapsto\left(\begin{array}{c}\operatorname{arccosh}(z)\cos(\theta)\\ \operatorname{arccosh}(z)\sin(\theta)\end{array}\right)$$

but has the advantage that Δ_R is mapped onto the (even simpler) disk $x^2+y^2\leq R^2$. The inverse mapping

$$\left(\begin{array}{c}r\cos(\theta)\\ r\sin(\theta)\end{array}\right)\mapsto\left(\begin{array}{c}\sinh(r)\cos(\theta)\\ \sinh(r)\sin(\theta)\\ \cosh(r)\end{array}\right)$$

will be helpful soon; call this Φ for convenience.

We now discuss the random generation of uniform points in Δ_R. Here it is useful to first review the generation of points in the Euclidean planar disk of radius R. We want distance ξ between a random point and the center $(0,0)$ to possess density function

$$f(\xi)=\frac{2}{R^2}\xi,\qquad 0<\xi<R$$

(proportional to circular circumference, radius ξ). The cumulative distribution is

$$\eta=F(\xi)=\int_0^{\xi}\frac{2}{R^2}t\,dt=\frac{1}{R^2}\xi^2,\qquad 0<\eta<1$$

hence $\xi=R\sqrt{\eta}$. By the inverse CDF method, the point

$$\left(\begin{array}{c}R\sqrt{\eta}\cos(\theta)\\ R\sqrt{\eta}\sin(\theta)\end{array}\right)\qquad\text{where}\qquad \eta\sim\text{Unif}[0,1],\quad \theta\sim\text{Unif}[0,2\pi]$$

satisfies the desired uniformity condition.

Returning now to Δ_R, we want distance ξ between a random point and the center $(0,0,1)$ to possess density function [6, 7]

$$f(\xi)=\frac{\sinh(\xi)}{\cosh(R)-1},\qquad 0<\xi<R$$

(again by proportionality). The cumulative distribution is

$$\eta=F(\xi)=\int_0^{\xi}\frac{\sinh(t)}{\cosh(R)-1}\,dt=\frac{\cosh(\xi)-1}{\cosh(R)-1},\qquad 0<\eta<1$$

hence $\xi = \operatorname{arccosh}(1 + (\cosh(R) - 1)\eta)$. In the planar disk of radius R, the point

$$\begin{pmatrix} \operatorname{arccosh}(1 + (\cosh(R) - 1)\eta)\cos(\theta) \\ \operatorname{arccosh}(1 + (\cosh(R) - 1)\eta)\sin(\theta) \end{pmatrix} \qquad \text{where} \qquad \eta \sim \mathrm{Unif}[0,1], \quad \theta \sim \mathrm{Unif}[0,2\pi]$$

is more likely to appear near the circular boundary than near the center. Applying the transformation Φ, we obtain that

$$\begin{pmatrix} \sqrt{(1 + (\cosh(R) - 1)\eta)^2 - 1}\cos(\theta) \\ \sqrt{(1 + (\cosh(R) - 1)\eta)^2 - 1}\sin(\theta) \\ 1 + (\cosh(R) - 1)\eta \end{pmatrix} \qquad \text{where} \qquad \eta \sim \mathrm{Unif}[0,1], \quad \theta \sim \mathrm{Unif}[0,2\pi]$$

satisfies the desired uniformity condition in Δ_R.

0.1. Sides. We do not know the trivariate density $f(x, y, z)$ for sides a, b, c of a uniform random triangle in Δ_R. Let

$$X = \frac{\cosh(a)}{L^2}, \qquad Y = \frac{\cosh(b)}{L^2}, \qquad Z = \frac{\cosh(c)}{L^2}$$

denote normalized sides, where $L = \cosh(R) - 1$. The trivariate characteristic function

$$\mathrm{E}\left(\exp\left(irX + isY + itZ\right)\right)$$

has a complicated quintuple integral expression [6, 7] that we choose not to reproduce here. Setting $r = s = 0$, the following expression for the univariate characteristic function for Z emerges:

$$\frac{1}{2\pi}\int_0^{2\pi}\int_{1/L}^{1+1/L}\int_{1/L}^{1+1/L} \exp\left[it\left(uv - \cos(\varphi)\sqrt{u^2 - \frac{1}{L^2}}\sqrt{v^2 - \frac{1}{L^2}}\right)\right] du\, dv\, d\varphi$$

$$= \int_{1/L}^{1+1/L}\int_{1/L}^{1+1/L} J_0\left(t\sqrt{u^2 - \frac{1}{L^2}}\sqrt{v^2 - \frac{1}{L^2}}\right) \exp\left(ituv\right) du\, dv$$

where $J_0(\theta)$ is the zeroth Bessel function of the first kind. It follows that

$$\mathrm{E}(Z) = \left(\frac{L+2}{2L}\right)^2, \qquad \mathrm{E}(Z^2) = \frac{L^4 + 6L^3 + 13L^2 + 12L + 6}{6L^4}$$

and, in the limit as $R \to \infty$, the univariate density of Z tends to

$$-1 + \frac{2}{\pi}\sqrt{\frac{2}{\zeta} - 1} + \frac{1}{\pi}\arccos(1 - \zeta), \qquad 0 < \zeta < 2.$$

It also follows that

$$\mathrm{E}(Y\,Z) = \frac{(L+2)^2(L^2+3L+3)}{12L^4}$$

from the biivariate characteristic function for Y, Z:

$$\int_{1/L}^{1+1/L}\int_{1/L}^{1+1/L}\int_{1/L}^{1+1/L} J_0\left(s\sqrt{u^2-\frac{1}{L^2}}\sqrt{w^2-\frac{1}{L^2}}\right) J_0\left(t\sqrt{u^2-\frac{1}{L^2}}\sqrt{v^2-\frac{1}{L^2}}\right)\exp\left(isuw+ituv\right)du\,dv\,dw.$$

A complicated expression for the limiting trivariate density of X, Y, Z exists [6] in terms of a certain elliptic integral, but again we omit this.

0.2. Angles. We know even less about the density for angles α, β, γ of a uniform random triangle in Δ_R. This is unfortunate since the angular defect $\pi-(\alpha+\beta+\gamma)$ is equal to the area of the triangle and this is an important quantity to understand.

By the Law of Cosines for Sides, a triangle is acute if and only if the three inequalities

$$\cosh(a)\cosh(b) > \cosh(c),$$
$$\cosh(a)\cosh(c) > \cosh(b),$$
$$\cosh(b)\cosh(c) > \cosh(a)$$

hold, which permits a proof of [7]

$$\lim_{R\to\infty} \mathrm{P}(\text{a uniform triangle in } \Delta_R \text{ is acute}) = 1.$$

We close with an interesting variation. The **circumscribed circle** of a triangle is a circle that goes through the three vertices of the triangle. If such a circle exists, its center is called the **circumcenter** (which coincides with the intersection of the three perpendicular bisectors of the sides). We say, under such a condition, that the triangle possesses a circumcenter. This is true if and only if the three inequalities

$$\sinh\left(\frac{a}{2}\right) < \sinh\left(\frac{b}{2}\right) + \sinh\left(\frac{c}{2}\right),$$
$$\sinh\left(\frac{b}{2}\right) < \sinh\left(\frac{a}{2}\right) + \sinh\left(\frac{c}{2}\right),$$
$$\sinh\left(\frac{c}{2}\right) < \sinh\left(\frac{a}{2}\right) + \sinh\left(\frac{b}{2}\right)$$

hold, which inspires a numerical computation [7]

$$\lim_{R\to\infty} \mathrm{P}(\text{a uniform triangle in } \Delta_R \text{ possesses a circumcenter}) = 0.4596203....$$

No exact expression for this constant is known. See [8] for experimental confirmations of the preceding.

References

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