Random Triangles VI

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January 7, 2011

As a conclusion of our survey, we gather various results for random triangles in the plane subject to constraints. If we break a line segment L in two places at random, the three pieces can be configured as a triangle with probability $1/4$ [1, 2, 3, 4]. If we instead select three points on a circle Γ at random, a triangle can almost surely be formed by connecting each pair of points with a line. Assuming L has length 1 and Γ has radius 1, what can be said about sides and angles of such triangles?

0.1. Unit Perimeter. Consider the broken L model, with the condition that triangle inequalities are satisfied. The bivariate density for two arbitrary sides a, b is [5, 6]

$$\begin{cases} 8 & \text{if } 0 < x < 1/2,\ 0 < y < 1/2 \text{ and } x + y > 1/2, \\ 0 & \text{otherwise.} \end{cases}$$

Integrating on y from $1/2 - x$ to $1/2$, the univariate density for a is

$$\begin{cases} 8\,x & \text{if } 0 < x < 1/2, \\ 0 & \text{otherwise} \end{cases}$$

and corresponding moments are

$$\mathrm{E}(a) = 1/3 = 0.3333333333...,\qquad \mathrm{E}(a^2) = 1/8 = 0.125.$$

As in [7], the cross-correlation coefficient $\rho(a, b) = -1/2$, hence

$$\mathrm{E}(a\,b) = 5/48 = 0.1041666666....$$

The Law of Cosines (with third side $c = 1 - a - b$) and a Jacobian determinant calculation imply that the bivariate density for two angles α, β is

$$\begin{cases} 8\dfrac{\sin(x)\sin(y)\sin(x+y)}{(\sin(x)+\sin(y)+\sin(x+y))^3} & \text{if } 0 < x < \pi,\ 0 < y < \pi \text{ and } x + y < \pi, \\ 0 & \text{otherwise.} \end{cases}$$

This is a new result, as far as is known, although it bears resemblance to formulas in [7]. Integrating on y from 0 to $\pi - x$, the univariate density for α is

$$\begin{cases} -8\dfrac{(3-\cos(x))\sin(x)}{(1+\cos(x))^3}\ln\left(\sin\left(\dfrac{x}{2}\right)\right) - 8\dfrac{\sin(x)}{(1+\cos(x))^2} & \text{if } 0 < x < \pi, \\ 0 & \text{otherwise} \end{cases}$$

and corresponding moments are

$$\mathrm{E}(\alpha) = \pi/3 = 1.0471975511\ldots, \qquad \mathrm{E}(\alpha^2) = 8/3 - \pi^2/9 = 1.5700439554....$$

Because $\rho(\alpha,\beta) = -1/2$, we have

$$\mathrm{E}(\alpha\,\beta) = -4/3 + 2\pi^2/9 = 0.8599120891....$$

It is feasible to calculate the density for the maximum angle (omitted). The probability that a broken L triangle is obtuse can be shown to be [8, 9, 10]

$$9 - 12\ln(2) = 0.6822338332... = 1 - 0.3177661667....$$

For area $\sqrt{(1/2)(1/2-a)(1/2-b)(a+b-1/2)}$, it is surprising that exact moment formulas can be found [6]:

$$\mathrm{E}(\text{area}) = \frac{\pi}{105} = 0.0299199300..., \qquad \mathrm{E}(\text{area}^2) = \frac{1}{960} = 0.0010416666....$$

A similar set of computations for triangles of unit area has not yet been undertaken.

0.2. Unit Circumradius. Consider the selection Γ model, equivalently, all triangles inscribing the unit circle. The bivariate density for two arbitrary angles α, β is [11, 12, 13]

$$\begin{cases} 2/\pi^2 & \text{if } 0 < x < \pi,\ 0 < y < \pi \text{ and } x + y < \pi, \\ 0 & \text{otherwise.} \end{cases}$$

To prove this, use the fact that an inscribed angle is one-half the length of its intercepted circular arc [14, 15]. Integrating on y from 0 to $\pi - x$, the univariate density for α is

$$\begin{cases} 2(\pi - x)/\pi^2 & \text{if } 0 < x < \pi, \\ 0 & \text{otherwise} \end{cases}$$

and corresponding moments are

$$\mathrm{E}(\alpha) = \pi/3 = 1.0471975511\ldots, \qquad \mathrm{E}(\alpha^2) = \pi^2/6 = 1.6449340668....$$

As before, the cross-correlation coefficient $\rho(\alpha,\beta)=-1/2$, hence

$$\mathrm{E}(\alpha\,\beta)=\pi^2/12=0.8224670334....$$

The angle α is maximum if $\alpha>\beta$ and $\alpha>\pi-\alpha-\beta$ [7]. Hence the density for the maximum angle is

$$\left\{\begin{array}{ll} 3\displaystyle\int\limits_{\pi-2x}^{x} 2/\pi^2\,dy & \text{if } \pi/3<x<\pi/2, \\ 3\displaystyle\int\limits_{0}^{\pi-x} 2/\pi^2\,dy & \text{if } \pi/2<x<\pi \end{array}\right. = \left\{\begin{array}{ll} 6(3\,x-\pi)/\pi^2 & \text{if } \pi/3<x<\pi/2, \\ 6(\pi-x)/\pi^2 & \text{if } \pi/2<x<\pi \end{array}\right.$$

and the probability that a selection Γ triangle is obtuse [8, 9, 13] is $3/4=0.75$.

The univariate density for a is [16, 17]

$$\left\{\begin{array}{ll} \dfrac{2}{\pi}\dfrac{1}{\sqrt{4-x^2}} & \text{if } 0<x<2, \\ 0 & \text{otherwise} \end{array}\right.$$

and corresponding moments are

$$\mathrm{E}(a)=4/\pi=1.2732395447..., \qquad \mathrm{E}(a^2)=2.$$

It can be shown that sides a, b are independent, which is delightfully paradoxical since angles α, β are *dependent* and

$$a=2\sin(\alpha), \qquad b=2\sin(\beta).$$

The remaining side c satisfies

$$c=\left\{\begin{array}{ll} \frac{1}{2}\left(a\sqrt{4-b^2}+b\sqrt{4-a^2}\right) & \text{with probability } 1/2, \\ \frac{1}{2}\left|a\sqrt{4-b^2}-b\sqrt{4-a^2}\right| & \text{with probability } 1/2 \end{array}\right.$$

but a simple expression for the trivariate density of all three sides a, b, c seems unlikely.

For area $(1/4)\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}$, it is again surprising that exact moment formulas can be found [18, 19, 20]:

$$\mathrm{E}(\text{area})=\frac{3}{2\pi}=0.4774648292..., \qquad \mathrm{E}(\text{area}^2)=\frac{3}{8}=0.375.$$

We mention that analogous results for random tetrahedra inscribing the unit sphere [19, 21, 22] are $\mathrm{E}(\text{volume})=4\pi/105\approx 0.11968$ and $\mathrm{E}(\text{volume}^2)=2/81\approx 0.02469$.

A similar set of computations for triangles circumscribing the unit circle Γ has not yet been undertaken. Caution is needed, since Γ is an incircle if and only if there is no semicircle containing all three contact points [9, 13]. Otherwise Γ is an excircle.

0.3. Side-Angle-Side Example. Thus far we have examined cases when three sides are given or three angles are given. Portnoy [23] studied an example in which two sides $a = \cos(\theta)$, $b = \sin(\theta)$ are given, where θ is Uniform $[0, \pi/2]$, as well as the included angle γ, which is independent and Uniform $[0, \pi]$. Let us focus solely on the obtuseness probability. By the Law of Cosines,

$$b^2 = a^2 + c^2 - 2a\,c\cos(\beta),$$

$$c^2 = a^2 + b^2 - 2a\,b\cos(\gamma).$$

If $\beta \geq \pi/2$, then $\cos(\beta) \leq 0$ and $b^2 \geq a^2 + c^2$, hence

$$b^2 - a^2 \geq c^2 = a^2 + b^2 - 2a\,b\cos(\gamma)$$

hence

$$2a\,b\cos(\gamma) \geq 2\,a^2$$

hence

$$\cos(\gamma) \geq a/b = \cot(\theta)$$

and conversely. The probability that $\beta \geq \pi/2$ is thus

$$\mathrm{P}\left\{\cos(\gamma) - \cot(\theta) \geq 0\right\} = 1 - \mathrm{P}\left\{\cos(\gamma) + \cot(\theta) \geq 0\right\}$$

by symmetry, and the latter probability (of a sum) is a convolution integral:

$$\frac{2}{\pi^2}\int_0^{\infty}\int_{\xi(x)}^{x+1}\frac{1}{\sqrt{1-(x-y)^2}}\frac{1}{1+y^2}dy\,dx$$

where $\xi(x) = \max\{x-1, 0\}$. Reversing the order of integration, we obtain

$$\frac{3}{4} + \frac{1}{\pi^2}\ln\left(1+\sqrt{2}\right)^2 = 1 - 0.1712917389...$$

as the value of the integral. Finally, the obtuseness probabilty for the triangle is

$$\mathrm{P}\left\{\theta \geq \pi/2\right\} + \mathrm{P}\left\{\alpha \geq \pi/2\right\} + \mathrm{P}\left\{\beta \geq \pi/2\right\}$$

which becomes

$$1 - \frac{2}{\pi^2}\ln\left(1+\sqrt{2}\right)^2 = 0.8425834778....$$

This exact evaluation is new, as far as is known, improving on [23].

Experimental confirmation of the predictions in this essay is available [24].

0.4. Addendum. The density for area of a random triangle inscribing the unit circle is $8x\Psi\left(4x^2\right)$, where

$$\begin{aligned}\Psi(y) &= \frac{1}{4\pi^3}\frac{1}{\sqrt{y}}\left\{\Gamma\left(\frac{1}{3}\right)^3\left(\frac{4y}{27}\right)^{-1/6}{}_2F_1\left(\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{4y}{27}\right)-\right.\\ &\left. 3\Gamma\left(\frac{2}{3}\right)^3\left(\frac{4y}{27}\right)^{1/6}{}_2F_1\left(\frac{2}{3},\frac{2}{3},\frac{4}{3},\frac{4y}{27}\right)\right\},\end{aligned}$$

${}_2F_1$ is the Gauss hypergeometric function [25] and $0 < y < 27/4$. This formula corrects that which appears in Case III of [26]. Random tetrahedra inscribing the unit sphere are the subject of [27]; the motivation is not a volume density but rather a coverage probability.

Random triangles of unit inradius are studied in [28]. The bivariate density for angles is the same as in the unit circumradius scenario; the univariate density for a side is

$$\frac{16}{\pi^2}\frac{x\arctan\left(\frac{x+\sqrt{x^2-4}}{2}\right)-x\arctan\left(\frac{x-\sqrt{x^2-4}}{2}\right)+\ln\left(\frac{x+\sqrt{x^2-4}}{x-\sqrt{x^2-4}}\right)}{\left(x^2+4\right)x}$$

for $x > 2$. A side has infinite mean and median 5.5482039188.... The perimeter also has infinite mean, but nothing else is known precisely.

Further analysis encompassing both unit perimeter triangles/Portnoy's SAS triangles and unit area triangles (à la "throwing paint") appears in [29, 30] with many more constants.

References

[1] E. Seneta and F. Jongmans, The problem of the broken rod and Ernesto Cesàro's early work in probability, *Math. Sci.* 30 (2005) 67–76; MR2185363 (2006h:60006).

[2] G. S. Goodman, The problem of the broken stick reconsidered, *Math. Intelligencer* 30 (2008) 43–49; MR2437199 (2010a:60026).

[3] J. J. Hunter, Random triangles, order statistics and singular distributions, *Internat. J. Math. Ed. Sci. Tech.* 17 (1986) 253–257; MR0827358.

[4] J. Whittaker, Random triangles, *Amer. Math. Monthly* 97 (1990) 228–230; MR1048437 (91i:60038).

[5] R. Pyke, Spacings, *J. Royal Statist. Soc. Ser. B* 27 (1965) 395–449; MR0216622 (35 #7451).

[6] D. E. Dobbs, The average area of a triangle, *Mathematics and Computer Education* 21 (1987) 178–181.

[7] S. R. Finch, Random triangles, unpublished note (2010).

[8] F. Budden, Some connected problems on probability, *Math. Gazette*, v. 66 (1982) n. 435, 4–10.

[9] R. K. Guy, There are three times as many obtuse-angled triangles as there are acute-angled ones, *Math. Mag.* 66 (1993) 175–179; MR1572957.

[10] D. E. Dobbs and J. L Zarestky, Acute random triangles, *Nieuw Arch. Wisk.* (4) 15 (1997) 141–162; MR1609502.

[11] R. E. Miles, The various aggregates of random polygons determined by random lines in a plane, *Adv. Math.* 10 (1973) 256–290; MR0319232 (47 #7777).

[12] J. S. Rao, Some tests based on arc-lengths for the circle, *Sankhya Ser. B* 38 (1976) 329–338; MR0652731 (58 #31571).

[13] T. Moore, RE: Random triangle problem (long summary), http://mathforum.org/kb/plaintext.jspa?messageID=86196.

[14] E. E. Moise, *Elementary Geometry from an Advanced Standpoint*, Addison-Wesley, 1963, pp. 192–196; MR0149339 (26 #6829).

[15] S. Lang and G. Murrow, *Geometry: A High School Course*, Springer-Verlag, 1983, pp. 192–197.

[16] S. S. Chiu and R. C. Larson, Bertrand's paradox revisited: more lessons about that ambiguous word, random, *J. Industrial and Systems Engineering* 3 (2009) 1–26; http://jise.ir/article_3997_0.html.

[17] E. W. Weisstein, http://mathworld.wolfram.com/CircleLinePicking.html.

[18] J. Edwards, *A Treatise on the Integral Calculus with Applications, Examples and Problems*, v. 2, MacMillan and Co., 1922, p. 772.; http://books.google.com/.

[19] R. E. Miles, Isotropic random simplices, *Adv. Appl. Probab.* 3 (1971) 353–382; MR0309164 (46 #8274).

[20] E. W. Weisstein, http://mathworld.wolfram.com/CircleTrianglePicking.html.

[21] L. Heinrich, R. Körner, N. Mehlhorn and L. Muche, Numerical and analytical computation of some second-order characteristics of spatial Poisson-Voronoi tessellations, *Statistics* 31 (1998) 235–259; MR1718316 (2000g:60016).

[22] E. W. Weisstein, http://mathworld.wolfram.com/SphereTetrahedronPicking.html.

[23] S. Portnoy, A Lewis Carroll pillow problem: probability of an obtuse triangle, *Statist. Sci.* 9 (1994) 279–284; MR1293297 (95h:60003).

[24] S. R. Finch, Simulations in R involving triangles and tetrahedra, http://www.people.fas.harvard.edu/~sfinch/csolve/rsimul.html.

[25] S. R. Finch, Riemann zeta moments, unpublished note (2007).

[26] A. M. Mathai and D. S. Tracy, On a random convex hull in an n-ball, *Comm. Statist. A - Theory Methods* 12 (1983) 1727–1736; MR0704849 (85c:60013).

[27] S. R. Finch, Covering a sphere with four random circular caps, arXiv:1502.06931.

[28] S. R. Finch, Three random tangents to a circle, arXiv:1101.3931.

[29] S. R. Finch, Uniform triangles with equality constraints, arXiv:1411.5216.

[30] S. R. Finch, Lognormality and triangles of unit area, arXiv:1412.5860.