Gambler's Ruin

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Consider two gamblers A, B with initial integer fortunes a, b. Let $m = a + b$ denote the initial sum of fortunes. In each round of a fair game, one player wins and is paid 1 by the other player:

$$(a, b) \mapsto \begin{cases} (a+1, b-1) & \text{with probability } 1/2, \\ (a-1, b+1) & \text{"} \end{cases}$$

Assume that rounds are independent for the remainder of this essay. The **ruin probability** p_E for a gambler E is the probability that E's fortune reaches 0 before it reaches m. For the symmetric 2-player problem,

$$p_A = \frac{b}{a+b}, \qquad p_B = \frac{a}{a+b}$$

and this can be proved using either discrete-time (1D random walk) methods or by continuous-time (1D Brownian motion) methods [1].

Before discussing the symmetric 3-player problem (which constitutes the most natural generalization of the preceding), let us examine the following **3-player C-centric game** [2, 3]:

$$(a, b, c) \mapsto \begin{cases} (a+1, b, c-1) & \text{with probability } 1/4, \\ (a-1, b, c+1) & \text{"} \\ (a, b+1, c-1) & \text{"} \\ (a, b-1, c+1) & \text{"} \end{cases}$$

In each round, C plays against either A or B (with equal probability) and wins 1 or loses 1 (again with equal probability). Let $m = a + b + c$ denote the initial sum of fortunes. By discrete-time methods, it is known that [3]

$$p_A = f(b, a, m) - f(a, a + c, m)$$

where

$$f(a, b, m) = \frac{2}{m} \sum_{\substack{1 \le j < m \\ j \text{ odd}}} \sin\left(\frac{a\, j\, \pi}{m}\right) \cot\left(\frac{j\, \pi}{2m}\right) \frac{\sinh\left((m-b)\varphi_{j,m}\right)}{\sinh\left(m\, \varphi_{j,m}\right)},$$

$$\varphi_{j,m} = \operatorname{arccosh}\left(2 - \cos(j\,\pi/m)\right).$$

For example,

$$p_A = \begin{cases} \frac{295476041655}{716708481082} = 0.4122... & \text{if } a = 3, b = 3, c = 9; \\ \frac{2964404261421089}{8592617979692098} = 0.3449... & \text{if } a = 4, b = 4, c = 7; \\ \frac{93962873}{360352742} = 0.2607... & \text{if } a = 5, b = 5, c = 5 \end{cases}$$

and these numerical results are consistent with [2] (obtained by recurrences). From

$$p_A = \begin{cases} \frac{1}{4} = 0.25 & \text{if } a = b = c = 1; \\ \frac{17}{66} = 0.2575... & \text{if } a = b = c = 2; \\ \frac{365}{1406} = 0.2596... & \text{if } a = b = c = 3; \\ \frac{223655}{858958} = 0.2603... & \text{if } a = b = c = 4 \end{cases}$$

it is clear that 3-player problems differ from 2-player problems (because scaling is not invariant) and hence 2D Brownian motion methods will only approximate (but not exactly solve) 2D random walk probabilities. If we allow $m \to \infty$ in such a way that $a/m \to \alpha > 0$ and $b/m \to \beta > 0$, then [3]

$$p_A = g(\beta, \alpha) - g(\alpha, 1 - \beta)$$

where

$$g(\alpha, \beta) = 4 \sum_{\substack{1 \le j < \infty \\ j \text{ odd}}} \frac{\sin(\alpha\, j\, \pi)}{j\, \pi} \frac{\sinh((1-\beta) j\, \pi)}{\sinh(j\, \pi)}.$$

For example,

$$p_A = \begin{cases} 0.2614366507... & \text{if } \alpha = 1/3, \beta = 1/3; \\ 0.4126822642... & \text{if } \alpha = 1/5, \beta = 1/5 \end{cases}$$

in this limiting case. If instead we allow $c \to \infty$ for fixed a, b, then [2]

$$p_A = \frac{1}{\pi} \int_0^{\pi} \frac{\sin(x)\sin(b\,x)}{1 - \cos(y)} e^{-a\,y} dx$$

where

$$\cos(x) + \cosh(y) = 2.$$

For example,

$$p_A = \begin{cases} 1/2 & \text{if } a = b; \\ 0.6976527263... & \text{if } a = 1, b = 2; \\ 0.6232861831... & \text{if } a = 2, b = 3; \\ 0.7906109052... & \text{if } a = 1, b = 3. \end{cases}$$

Let us turn attention to the **symmetric 3-player game**:

$$(a,b,c)\mapsto\begin{cases}(a+2,b-1,c-1) & \text{with probability } 1/3,\\ (a-1,b+2,c-1) & \text{''}\\ (a-1,b-1,c+2) & \text{''}\end{cases}$$

One player wins and is paid 1 by each of the other players. A discrete-time solution was outlined in [4], but it is conceptually very different from C-centric game results. For small values of m, some results are known [5, 6]:

$$p_C=\begin{cases}\frac{2}{3}=0.6666... & \text{if } a=b=c=1;\\ \frac{4}{9}=0.4444... & \text{if } a=b=c=2;\\ \frac{8}{21}=0.3809... & \text{if } a=b=c=3;\\ \frac{16}{45}=0.3555... & \text{if } a=b=c=4;\\ \frac{848}{2457}=0.3451... & \text{if } a=b=c=5;\\ \frac{49}{144}=0.3402... & \text{if } a=b=c=6.\end{cases}$$

Asymptotic numerical evaluation is feasible when modeling the game as Brownian motion in the plane of the equilateral triangle given by

$$\left\{x\begin{pmatrix}1\\0\end{pmatrix}+y\begin{pmatrix}-1\\0\end{pmatrix}+z\begin{pmatrix}0\\\sqrt{3}\end{pmatrix}:x+y+z=m,x\geq 0,y\geq 0,z\geq 0\right\}.$$

Computing p_C corresponds to finding the probability that Brownian motion first exits the triangle along the edge $z=0$, starting from $(x,y,z)=(a,b,c)$. In the event $a=b$, we determine $\eta>0$ so that

$$\frac{c}{m}=\frac{I\left(\frac{\eta^2}{1+\eta^2},\frac{1}{2},\frac{1}{6}\right)}{I\left(1,\frac{1}{2},\frac{1}{6}\right)}$$

where

$$I(\xi,\alpha,\beta)=\int_0^{\xi}t^{\alpha-1}(1-t)^{\beta-1}dt$$

is the incomplete beta function; it follows that [7, 8, 9]

$$p_C=\frac{1}{\pi}\left(\frac{\pi}{2}-\arctan\left(\frac{\eta^2-1}{2\eta}\right)\right).$$

For example,

$$p_C=\begin{cases}1/3 & \text{if } a=b=c, \text{ that is, } c/m=1/3;\\ 0.1421549761... & \text{if } 2a=2b=c, \text{ that is, } c/m=1/2;\\ 0.5617334934... & \text{if } a=b=2c, \text{ that is, } c/m=1/5.\end{cases}$$

In the event $a \neq b$, no such explicit formulas apply. A purely numerical approach [8, 9, 10, 11, 12, 13] gives, for example,

$$p_A = 0.6542207068..., \qquad p_B = 0.2923400189..., \qquad p_C = 0.0534392741...$$

when $10a = 5b = 2c$.

The final game we mention, usually referred to as the 3**-tower problem** (or **Hanoi tower problem**), is [8]:

$$(a,b,c) \mapsto \begin{cases} (a-1,b+1,c) & \text{with probability } 1/6, \\ (a-1,b,c+1) & \text{"} \\ (a+1,b-1,c) & \text{"} \\ (a,b-1,c+1) & \text{"} \\ (a+1,b,c-1) & \text{"} \\ (a,b+1,c-1) & \text{"} \end{cases}$$

In each round, one player is randomly chosen as the loser and one player (distinct from the first) is randomly chosen as the winner. A study of corresponding ruin probabilities has evidently not been done.

Another quantity of interest is the **game duration** d, which is the expected number of rounds until one of the gamblers is ruined. For the symmetric 2-player and 3-player problems, we have [14, 15, 16]

$$d = a\,b, \qquad d = \frac{a\,b\,c}{a+b+c-2}$$

respectively. For the 3-tower problem, we have [15, 16, 17, 18, 19]

$$d = \frac{3a\,b\,c}{a+b+c};$$

in fact, corresponding variance and probability distribution are also known. No one has apparently calculated d for the 3-player C-centric game. No simple formulas for d can be anticipated when the number of players exceeds three [17, 20, 21].

Here is an interesting variation on the symmetric 2-player problem:

$$(a_1,a_2,b_1,b_2) \mapsto \begin{cases} (a_1+1,a_2,b_1-1,b_2) & \text{with probability } 1/4, \\ (a_1-1,a_2,b_1+1,b_2) & \text{"} \\ (a_1,a_2+1,b_1,b_2-1) & \text{"} \\ (a_1,a_2-1,b_1,b_2+1) & \text{"} \end{cases}$$

The gamblers use two different currencies, say dollars and euros. In each round, a currency and a winner are randomly chosen. When one of the players runs out of either currency, the game is over. Ruin probabilities p are not known; if $a_1 = a_2 = b_1 = b_2 = n$, then game durations d are $O(n^2)$ and, more precisely, [22]

$$\delta = \lim_{n\to\infty} \frac{d}{n^2} = \frac{256}{\pi^4} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{(-1)^{k+\ell}}{(2k+1)(2\ell+1)\left[(2k+1)^2 + (2\ell+1)^2\right]}.$$

Another representation

$$\delta = 2\left(1 - \frac{32}{\pi^3} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3 \cosh\left[\frac{\pi}{2}(2k+1)\right]}\right) = 1.1787416525...$$

is rapidly convergent and possesses a straightforward generalization to an arbitrary number of different currencies.

0.1. Addendum. The following question is similar to our asymptotic analysis of the symmetric 3-player game. Let $a \le b$. A particle at the center of an $a \times b$ rectangle undergoes Brownian motion until it hits the rectangular boundary. What is the probability that it hits an edge of length a (rather than an edge of length b)? The answer [23, 24]

$$P(b/a) = \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} \operatorname{sech}\left(\frac{(2j+1)\pi}{2} \frac{b}{a}\right)$$

is found via solution of a steady-state heat PDE problem. This has a closed-form expression in certain cases: [25, 26, 27]

$$P(r) = \begin{cases} \frac{1}{2} & \text{if } r = 1, \\ \frac{2}{\pi} \arcsin\left[(\sqrt{2}-1)^2\right] & \text{if } r = 2, \\ \frac{2}{\pi} \arcsin\left[(\sqrt{2}-3^{1/4})(\sqrt{3}-1)/2\right] & \text{if } r = 3, \\ \frac{2}{\pi} \arcsin\left[(\sqrt{2}+1)^2(2^{1/4}-1)^4\right] & \text{if } r = 4, \\ \frac{2}{\pi} \arcsin\left[(\sqrt{5}-2)(3-2\cdot 5^{1/4})/\sqrt{2}\right] & \text{if } r = 5, \\ \frac{2}{\pi} \arcsin\left[(3-2\sqrt{2})^2(2+\sqrt{5})^2(\sqrt{10}-3)^2(5^{1/4}-\sqrt{2})^4\right] & \text{if } r = 10 \end{cases}$$

which are based on singular moduli k_1, k_4, k_9, k_{16}, k_{25}, k_{100} appearing in the theory of elliptic functions. We wonder whether heat PDE-type analysis might assist in the asymptotic study of some 4-player games.

REFERENCES

[1] H. M. Taylor and S. Karlin, *An Introduction to Stochastic Modeling*, 3rd ed., Academic Press, 1998, pp. 141–145, 509–514; MR1627763 (99c:60001).

[2] V. D. Barnett, A three-player extension of the gambler's ruin problem, *J. Appl. Probab.* 1 (1964) 321–334; MR0171330 (30 #1561).

[3] Y. Itoh and H. Maehara, A variation to the ruin problem, *Math. Japon.* 47 (1998) 97–102; MR1606328 (98m:60068).

[4] Y. C. Swan and F. T. Bruss, A matrix-analytic approach to the N-player ruin problem, *J. Appl. Probab.* 43 (2006) 755–766; MR2274798 (2007m:60221).

[5] A. L. Rocha and F. Stern, The gambler's ruin problem with n players and asymmetric play, *Statist. Probab. Lett.* 44 (1999) 87–95; MR1706327 (2000f:60063).

[6] A. L. Rocha and F. Stern, The asymmetric n-player gambler's ruin problem with equal initial fortunes, *Adv. Appl. Math.* 33 (2004) 512–530; MR2081041 (2005d:60067).

[7] T. Ferguson, Gambler's ruin in three dimensions, unpublished note (1995), http://www.math.ucla.edu/~tom/papers/unpublished.html.

[8] Y. C. Swan, On Two Unsolved Problems in Probability, Ph.D. thesis, Université Libre de Bruxelles, 2007; http://orbi.ulg.ac.be/handle/2268/188656.

[9] Y. C. Swan and F. T. Bruss, The Schwarz-Christoffel transformation as a tool in applied probability, *Math. Sci.* 29 (2004) 21–32; MR2073566.

[10] L. N. Trefethen, Numerical computation of the Schwarz-Christoffel transformation, *SIAM J. Sci. Statist. Comput.* 1 (1980) 82–102; erratum 1 (1980) 302; MR0572542 (81g:30012a) and MR0594762 (81g:30012b).

[11] T. A. Driscoll, Algorithm 756: A MATLAB toolbox for Schwarz-Christoffel mapping, *ACM Trans. Math. Software* 22 (1996) 168–186.

[12] T. A. Driscoll, Algorithm 843: Improvements to the Schwarz-Christoffel toolbox for MATLAB, *ACM Trans. Math. Software* 31 (2005) 239–251; MR2266791 (2007f:30001).

[13] T. A. Driscoll, Schwarz-Christoffel Toolbox for MATLAB, http://www.math.udel.edu/~driscoll/SC/.

[14] D. Sandell, A game with three players, *Statist. Probab. Lett.* 7 (1988) 61–63; MR0996854 (90g:60047).

[15] A. Engel, The computer solves the three tower problem, *Amer. Math. Monthly* 100 (1993) 62–64.

[16] D. Stirzaker, Tower problems and martingales, *Math. Sci.* 19 (1994) 52–59; MR1294785 (95h:60068).

[17] F. T. Bruss, G. Louchard and J. W. Turner, On the N-tower problem and related problems, *Adv. Appl. Probab.* 35 (2003) 278–294; MR1975514 (2004d:60181).

[18] A. Alabert, M. Farré and R. Roy, Exit times from equilateral triangles, *Appl. Math. Optim.* 49 (2004) 43–53; MR2023644 (2004j:60175).

[19] D. Stirzaker, Three-handed gambler's ruin, *Adv. Appl. Probab.* 38 (2006) 284–286; MR2213975 (2006k:60073).

[20] D. K. Chang, A game with four players, *Statist. Probab. Lett.* 23 (1995) 111–115; MR1341352 (96g:60055).

[21] D. Cho, A game with N players, *J. Korean Statist. Soc.* 25 (1996) 185–193; MR1423937 (98a:60051).

[22] A. Kmet and M. Petkovšek, Gambler's ruin problem in several dimensions, *Adv. Appl. Math.* 28 (2002) 107–118; MR1888839 (2003a:60012).

[23] G. R. Grimmett and D. R. Stirzaker, *Probability and Random Processes*, 3rd ed., Oxford Univ. Press, 2001, pp. 554–563; MR2059709 (2004m:60002).

[24] F. Bornemann, Short remarks on the solution of the SIAM 100-digit challenge, https://www-m3.ma.tum.de/Allgemeines/FolkmarBornemann.

[25] J. M. Borwein and P. B. Borwein, *Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity*, Wiley, 1987, pp. 69, 139, 162; MR1641658 (99h:11147).

[26] B. C. Berndt, H. H. Chan and L.-C. Zhang, Ramanujan's singular moduli, *Ramanujan J.* 1 (1997) 53–74; MR1607528 (2001b:11033).

[27] J. Boersma, Solution of Trefethen's problem 10, http://www.win.tue.nl/casa/meetings/special/siamcontest/.