Reuleaux Triangle Constants

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Of all planar sets of constant width 1, the **Reuleaux triangle** (see Figure 1) possesses the least area [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] and is the most asymmetric [12, 13, 14, 15]. Let us examine certain key phrases in the statement of this theorem more carefully, so that we may introduce several related constants.

A compact convex set $C \subseteq \mathbb{R}^2$ is of **constant width** w if all orthogonal projections of C onto lines have the same length w. More generally, for $C \subseteq \mathbb{R}^d$, $d > 2$, the required condition becomes that every pair of parallel supporting $(d-1)$-dimensional planes are at the same distance w apart. (The word *breadth* was used in [8.4.1] for reasons of convention.) For simplicity, set $w = 1$. The first part of the theorem is that the area, $\mu(C)$, of $C \subseteq \mathbb{R}^2$ satisfies

$$\mu(C) \geq \frac{\pi - \sqrt{3}}{2} = 0.7047709230....$$

It is believed that the volume, $\mu(C)$, of $C \subseteq \mathbb{R}^3$ satisfies

$$\mu(C) \geq \left(\frac{2}{3} - \frac{\sqrt{3}}{4}\arccos\left(\frac{1}{3}\right)\right)\pi = 0.4198600459...,$$

which corresponds to Meisser's tetrahedral analog of the Reuleaux triangle [1, 16]. The best-known lower bound thus far is $(3\sqrt{6} - 7)\pi/3 = 0.3649161225...$; hence there is considerable room for improvement [8, 11].

Asymmetry is more difficult to define, primarily because there are competing notions of it! We focus on just two measures of symmetry, called the Kovner-Besicovitch (inner) and Estermann (outer) measures, respectively [14]:

$$\sigma(C) = \frac{\mu(A)}{\mu(C)}, \qquad \tau(C) = \frac{\mu(C)}{\mu(B)},$$

where A is the largest convex centrally symmetric subset of C and B is the smallest convex centrally symmetric superset of C. The second part of the theorem is that, for $C \subseteq \mathbb{R}^2$ [8, 12],

$$\sigma(C) \geq \frac{6\arccos(\frac{5+\sqrt{33}}{12}) + \sqrt{3} - \sqrt{11}}{\pi - \sqrt{3}} = 0.8403426028... = 1 - 0.1596573971...,$$

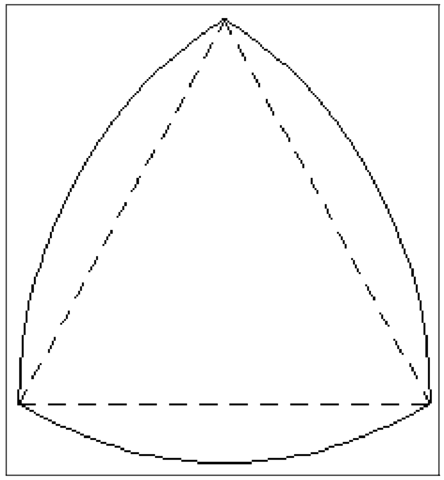

Figure 1: The Reuleaux triangle (solid curves) consists of the vertices of an equilateral triangle (dotted lines) together with three arcs of circles, each circle having a center at one of the vertices and endpoints at the other two vertices.

$$\tau(C) \geq \frac{\pi - \sqrt{3}}{\sqrt{3}} = 0.8137993642 = 1 - 0.1862006357....$$

The corresponding superset B is a regular hexagon circumscribed about the minimizing Reuleaux triangle C; the subset A is a circular hexagon obtained by reflecting C across its center, calling this new subset C', and then forming $C \cap C'$. A higher-dimensional analog of this bound is not known.

Here is one more result. What is the set $C \subseteq \mathbb{R}^2$ of maximal constant width w that avoids all vertices of the integer square lattice? The answer is a Reuleaux triangle, oriented so that one axis of symmetry lies midway between two parallel lattice edges. Its width $w = 1.5449417003...$ has minimal polynomial [9]

$$4x^6 - 12x^5 + x^4 + 22x^3 - 14x^2 - 4x + 4.$$

We mention that the Reuleaux triangle also appears in conjectures surrounding planar convex translations [8.3.1], maximal planar rendezvous constants [8.21], and exact values of the Bloch-Landau constants [7.1].

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