Planar Harmonic Mappings

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Let D denote the open unit disk. A function $f : D \to \mathbb{C}$ is **planar harmonic** if it can be written as $f(z) = u(x, y) + i\, v(x, y)$ where $z = x + i\, y$ and where $u : D \to \mathbb{R}$, $v : D \to \mathbb{R}$ are harmonic, that is, are twice continuously differentiable and obey

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

It can be shown that f is planar harmonic if and only if $f = g + \bar{h}$, where g, h are analytic on D and the overbar indicates complex conjugation ($\bar{z} = x - i\, y$).

Of course, a planar harmonic function f is analytic if and only if u and v are harmonic conjugates, that is, the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are satisfied. We are interested, in this essay, in functions f whose real and imaginary parts are not necessarily conjugate [1].

It turns out that f may be written as a twice continuously differentiable function of z and $\bar{z}$; we abuse notation and use the same letter f to represent the new function. The Cauchy-Riemann equations become a single concise equation:

$$\frac{\partial f}{\partial \bar{z}} = 0$$

and the condition that Laplacians vanish becomes

$$4 \frac{\partial^2 f}{\partial z\, \partial \bar{z}} = 0.$$

Thus the expression f is independent of $\bar{z}$ for analytic functions f, and the expression $\partial f / \partial z$ is independent of $\bar{z}$ for planar harmonic functions f.

A planar harmonic function $f : D \to \mathbb{C}$ is a **mapping** if it is one-to-one. Hence the class of planar harmonic mappings includes the subclass of univalent functions we have studied elsewhere [2, 3, 4, 5]. Define also the **dilatation** of f

$$\omega = \overline{\frac{\partial f}{\partial \bar{z}}} \Big/ \frac{\partial f}{\partial z}$$

which will be needed later.

0.1. Heinz's Inequality. We consider here planar harmonic mappings f that map D onto D, with the property that $f(0) = 0$. Heinz [6] proved that

$$\left|\frac{\partial f}{\partial z}(0,0)\right|^2 + \left|\frac{\partial f}{\partial \bar{z}}(0,0)\right|^2 \geq c$$

for some constant $c \geq 0.1788 = 0.3576/2$. The lower bound was improved to $0.32 = 0.64/2$ by Nitsche [7, 8], $0.4345 = 0.8691/2$ by de Vries [9], $0.4476 = 0.8952/2$ by Nitsche [10], $0.6411 = 1.2822/2$ by de Vries [11], and $0.6584 = 1.3168/2$ by Wegmann [12]. The conjecture that

$$c = \frac{27}{4\pi^2} = 0.6839179895... = \frac{1}{2}(1.3678359791...)$$

mentioned by Wegmann [12] seems to have been anticipated by Hopf [13]. A proof of this conjecture was first given by Hall [1, 14]; the extremal function is achieved via approximations $D \to D$ of a mapping $D \to T$, where T is an inscribed equilateral triangle, with dilatation $\omega(z) = z$.

Hall's proof involves the Fourier coeffients of homeomorphisms $C \to C$ of the unit circle C. Some related problems are given in [14]; one of these has been solved [15]. Heinz [16] also proved the inequality

$$\left|\frac{\partial f}{\partial z}(z,\bar{z})\right|^2 + \left|\frac{\partial f}{\partial \bar{z}}(z,\bar{z})\right|^2 \geq \frac{1}{\pi^2}$$

which is valid for all $z \in D$; improvements in special cases appear in [17, 18].

0.2. Minimal Surfaces. Consider a minimal surface over the unit disk D of the form

$$\left\{(x,y,z) \in \mathbb{R}^3 : z = F(x,y),\ (x,y) \in D\right\}$$

and let κ denote its Gaussian curvature at the origin. In words, the surface is locally area-minimizing: Each suitable small piece of it has the least possible area for any surface spanning the boundary of that piece. By the calculus of variations, we have the nonlinear PDE

$$\left[1 + \left(\frac{\partial F}{\partial y}\right)^2\right]\frac{\partial^2 F}{\partial x^2} - 2\frac{\partial F}{\partial x}\frac{\partial F}{\partial y}\frac{\partial^2 F}{\partial x\,\partial y} + \left[1 + \left(\frac{\partial F}{\partial x}\right)^2\right]\frac{\partial^2 F}{\partial y^2} = 0;$$

hence the mean curvature of the surface is everywhere zero. A precise determination of F is difficult – this is called **Plateau's problem** – but nature solves it effortlessly, as can be demonstrated by dipping a bent wire loop in a soap solution [19, 20].

A consequence of Heinz's inequality [6] is that

$$|\kappa| \leq \frac{4}{c} = \frac{16\pi^2}{27} = 5.8486544599...$$

by Hall's theorem [14], but this is not sharp. In fact, it is conjectured that [1]

$$|\kappa| \leq \frac{\pi^2}{2} = 4.9348022005;$$

this has however been proved only in the special case that the minimal surface has a horizontal tangent plane at the origin [21]. A general proof could be obtained utilizing the following.

Consider planar harmonic mappings f that map D onto D, with the two properties that $f(0) = 0$ and ω is the square of an analytic function. (Note that this final requirement is not met by $\omega(z) = z$.) Hall [22] recently computed that

$$\left|\frac{\partial f}{\partial z}(0,0)\right|^2 + \left|\frac{\partial f}{\partial \bar{z}}(0,0)\right|^2 \geq \tilde{c}$$

for some constant $\tilde{c} > c + 10^{-5}/2$. It is conjectured that $\tilde{c} = 8/\pi^2$ (from which $4/\tilde{c} = \pi^2/2$ would proceed immediately). The expected extremal function is a mapping $D \to S$, where S is an inscribed square, with dilatation $\omega(z) = z^2$. A proof that $\tilde{c} = 8/\pi^2$ would be a major step forward in understanding minimal surfaces. See [23] for more open questions.

0.3. Soap Films. As an aside, we give an elementary problem [24, 25]. Consider the catenoid-shaped soap film formed between two parallel rings centered at $(-\xi, 0, 0)$ and $(\xi, 0, 0)$ and of unit radius, where $\xi > 0$ is suitably small. If the rings are slowly pulled apart (that is, if ξ increases), there is a certain threshold at which the minimal surface becomes unstable and is likely to collapse to a disjoint union of two disks. More precisely, if $\xi < \xi_0 = 0.5276973969...$, then the catenoid corresponds to the global minimum for surface area while the two-disk configuration corresponds to only a local minimum. Here ξ_0 and $a = 0.8255174536...$ are solutions of the simultaneous equations

$$\begin{cases} a\cosh\left(\dfrac{\xi_0}{a}\right) = 1, \\ 2\pi a^2 \sinh\left(\dfrac{\xi_0}{a}\right)\cosh\left(\dfrac{\xi_0}{a}\right) + 2\pi a\,\xi_0 = 2\pi. \end{cases}$$

If $\xi > \xi_0$, then the two-disk configuration corresponds to the global minimum while the catenoid corresponds to only a local minimum for $\xi < \xi_1 = 0.6627434193...$; no such catenoid exists for $\xi > \xi_1$. Here ξ_1 and $b = 0.5524341245...$ are solutions of the simultaneous equations

$$\begin{cases} b\cosh\left(\dfrac{\xi_1}{b}\right) = 1, \\ \cosh\left(\dfrac{\xi_1}{b}\right) - \dfrac{\xi_1}{b}\sinh\left(\dfrac{\xi_1}{b}\right) = 0. \end{cases}$$

Interestingly, we have seen the value for ξ_1 before: In [26], it arose in a different context altogether and was called the *Laplace limit constant*.

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