Strong Triangle Inequality

Steven Finch

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Let a, b, c denote the sides of a triangle, h denote the altitude to side c, and γ denote the angle opposite c. It is known that the inequality [1, 2]

$$a+b>c+h$$

is true for all triangles with $\gamma<\pi-4\arctan(1/2)=1.2870022175...\approx 73.74^\circ$ but is false for all triangles with $\gamma\geq\pi/2$. For the intermediate range of angles, there are several ways to express the percentage of triangles satisfying the inequality. Certain authors [3] assumed that the angles α, β opposite sides a, b are uniformly distributed on the region

$$0<\alpha<\pi, \qquad 0<\beta<\pi, \qquad \alpha+\beta<\pi.$$

Let

$$K=\int\limits_0^{\pi/2}\left[2\arctan\left(1-\tan\left(\frac{x}{2}\right)\right)-\left(\frac{\pi}{2}-x\right)\right]dx=0.2922839193...$$

for convenience. Supposing $0<\gamma<\pi$, the probability that a random triangle satisfies the inequality is

$$1-\frac{2}{\pi^2}\left(\frac{\pi^2}{8}+K\right)=1-\frac{1}{4}-\frac{2K}{\pi^2}=0.690770....$$

Supposing instead $0<\gamma<\pi/2$, the probability is

$$1-\frac{8K}{3\pi^2}=0.921027....$$

(This is why $a+b>c+h$ is said to hold for "most" triangles with acute γ.) Supposing instead $\pi-4\arctan(1/2)<\gamma<\pi/2$, the probability is

$$1-\frac{8K}{64\arctan(1/2)^2-\pi^2}=0.398657....$$

We wonder about the odds corresponding to a *fixed* angle γ in the intermediate range. This is found by integrating the joint (α,β)-density

$$\begin{cases} \dfrac{2}{\pi^2} & \text{if } 0<x<\pi,\ 0<y<\pi \text{ and } x+y<\pi, \\ 0 & \text{otherwise} \end{cases}$$

to obtain a marginal density

$$f(x)=\int_0^{\pi-x}\frac{2}{\pi^2}dy=\frac{2}{\pi^2}(\pi-x)\,;$$

the desired probability is hence

$$\begin{aligned} 1-\frac{1}{f(\gamma)}\int_z^w\frac{2}{\pi^2}dx &= \frac{2z}{\pi-\gamma} \\ &= \begin{cases} 1 & \text{if } \gamma=\pi-4\arctan(1/2), \\ 0.770368... & \text{if } \gamma=5\pi/12=75^\circ, \\ 0.335397... & \text{if } \gamma=11\pi/24=82.5^\circ, \\ 0.166040... & \text{if } \gamma=23\pi/48=86.25^\circ, \\ 0 & \text{if } \gamma=\pi/2 \end{cases} \end{aligned}$$

where z is the smallest positive solution of the equation

$$\tan\left(\frac{z}{2}\right)+\cot\left(\frac{\gamma+z}{2}\right)=1$$

and $w=\pi-\gamma-z$.

We additionally wonder about the odds corresponding to a more natural choice of distribution for α, β. One difficulty with the preceding is that $\gamma=\pi-\alpha-\beta$ is not uniform; further, while α and β are independent, the same is not true for α and γ or for β and γ. If the triangle vertices are independent random Gaussian points in two dimensions, all of which have mean vector zero and covariance matrix identity, then we have joint (α,β)-density [4, 5]

$$\begin{cases} \dfrac{6}{\pi}\dfrac{\sin(x)\sin(y)\sin(x+y)}{\left(\sin(x)^2+\sin(y)^2+\sin(x+y)^2\right)^2} & \text{if } 0<x<\pi,\ 0<y<\pi \text{ and } x+y<\pi, \\ 0 & \text{otherwise.} \end{cases}$$

Integrating with respect to y over $[0,\pi-x]$, a marginal density [4, 6]

$$g(x)=\frac{3}{\pi}\frac{\cos(x)}{\left(4-\cos(x)^2\right)^{3/2}}\left(\frac{\pi}{2}+\arcsin\left(\frac{\cos(x)}{2}\right)\right)+\frac{3}{\pi}\frac{1}{4-\cos(x)^2}$$

emerges. The desired probability becomes

$$1-\frac{1}{g(\gamma)}\int\limits_{z}^{w}\frac{6}{\pi}\frac{\sin(\gamma)\sin(x)\sin(\gamma+x)}{\left(\sin(\gamma)^2+\sin(x)^2+\sin(\gamma+x)^2\right)^2}dx$$

$$=\begin{cases} 1 & \text{if } \gamma=\pi-4\arctan(1/2), \\ 0.662855... & \text{if } \gamma=5\pi/12=75^\circ, \\ 0.141612... & \text{if } \gamma=11\pi/24=82.5^\circ, \\ 0.034758... & \text{if } \gamma=23\pi/48=86.25^\circ, \\ 0 & \text{if } \gamma=\pi/2 \end{cases}$$

where z, w are exactly as before.

Another benefit of working with 2D Gaussian triangles is that the joint density for sides a, b, c is available [4, 7]:

$$\begin{cases} \dfrac{2}{3\pi}\dfrac{a\,b\,c}{\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}}\exp\left(-\dfrac{1}{6}\left(a^2+b^2+c^2\right)\right) \\ \qquad\qquad \text{if } |a-b|<c<a+b, \\ 0 \qquad\qquad \text{otherwise.} \end{cases}$$

The (ordinary) triangle inequality gives rise to an expected difference

$$\mathrm{E}(a+b-c)=\sqrt{\pi}=1.7724538509...$$

and an expected ratio

$$\mathrm{E}\left(\frac{a+b}{c}\right)\approx 2.94.$$

For the strong triangle inequality, we utilize a variation of the density function

$$\frac{1}{3\pi}a\,b\exp\left[-\frac{1}{3}\left(a^2-a\,b\cos(\gamma)+b^2\right)\right]$$

over $a>0$, $b>0$, $0<\gamma<\pi$ to compute the expected difference

$$\begin{aligned} \mathrm{E}(a+b-c-h) &= \mathrm{E}\left(a+b-\sqrt{a^2-2a\,b\cos(\gamma)+b^2}-\frac{a\,b\sin(\gamma)}{\sqrt{a^2-2a\,b\cos(\gamma)+b^2}}\right) \\ &\approx 0.79 \end{aligned}$$

and the expected ratio

$$\begin{aligned} \mathrm{E}\left(\frac{a+b}{c+h}\right) &= \mathrm{E}\left(\frac{a+b}{\sqrt{a^2-2a\,b\cos(\gamma)+b^2}+\frac{a\,b\sin(\gamma)}{\sqrt{a^2-2a\,b\cos(\gamma)+b^2}}}\right) \\ &\approx 1.44. \end{aligned}$$

Other natural models to consider are 3D Gaussian triangles [4] and broken L triangles of unit perimeter [8].

Let us turn attention away from a Euclidean setting and toward the hyperbolic plane. The strong triangle inequality holds for any hyperbolic triangle if $\gamma < \xi$ where $\xi = 1.1496525950... \approx 65.87^\circ$ is the smallest positive solution of the equation [9]

$$-1 - \cos(\xi) + \sin(\xi) + \sin\left(\frac{\xi}{2}\right)\sin(\xi) = 0.$$

Analogous probabilistic results for uniform angles are uncovered in [10]. An unusual feature of the latter paper is its careful analysis – numerical results here can be computed to arbitrary precision and the error can be bounded – we wonder if such rigor can be feasibly carried over to the Gaussian case.

References

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