Moments of Sums

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Let $X_1, X_2, \ldots, X_n$ be a sequence of independent random variables. A huge amount of work has been done on estimating the L_p-norm of the sum of the Xs:

$$\left\| \sum_{k=1}^{n} X_k \right\|_p = \left\{ \mathrm{E}\left(\left| \sum_{k=1}^{n} X_k \right|^p \right) \right\}^{1/p}, \qquad p > 0.$$

We first discuss Khintchine's inequality [1], which deals with the Rademacher sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$, where

$$\mathrm{P}(\varepsilon_k = 1) = \mathrm{P}(\varepsilon_k = -1) = 1/2 \qquad \text{(symmetric Bernoulli distribution)}$$

for each k. It is known that there exist constants A_p, B_p such that the bounds

$$A_p \left(\sum_{k=1}^{n} c_k^2 \right)^{1/2} \leq \left\| \sum_{k=1}^{n} c_k \varepsilon_k \right\|_p \leq B_p \left(\sum_{k=1}^{n} c_k^2 \right)^{1/2}$$

hold for arbitrary $c_1, c_2, \ldots, c_n \in \mathbb{R}$ and $n \geq 1$. Szarek [2] and Haagerup [3], building on [4, 5, 6, 7, 8, 9], proved that the best such constants are

$$A_p = \begin{cases} \|W\|_p & \text{if } 0 < p \leq p_0 \\ \|Z\|_p & \text{if } p_0 < p < 2 \\ 1 & \text{if } 2 \leq p < \infty \end{cases} = \begin{cases} 2^{1/2-1/p} & \text{if } 0 < p \leq p_0 \\ 2^{1/2} \left(\dfrac{\Gamma((p+1)/2)}{\sqrt{\pi}} \right)^{1/p} & \text{if } p_0 < p < 2 \\ 1 & \text{if } 2 \leq p < \infty \end{cases},$$

$$B_p = \begin{cases} 1 & \text{if } 0 < p \leq 2 \\ \|Z\|_p & \text{if } 2 < p < \infty \end{cases} = \begin{cases} 1 & \text{if } 0 < p \leq 2 \\ 2^{1/2} \left(\dfrac{\Gamma((p+1)/2)}{\sqrt{\pi}} \right)^{1/p} & \text{if } 2 < p < \infty \end{cases}$$

where $W = 2^{-1/2}(\varepsilon_1 + \varepsilon_2)$, Z is Normal$(0,1)$, and $p_0 = 1.8474163360...$ is the unique solution of the equation

$$\Gamma\left(\frac{p+1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

in the interval $0 < p < 2$. In words, if $\sum_{k=1}^{n} c_k^2 = 1$, then $A_1 = 2^{-1/2}$ and $B_1 = 1$ encompass the average of $|\pm c_1 \pm c_2 \pm \cdots \pm c_n|$ taken over all 2^n possible choices of signs. See also [10, 11, 12, 13, 14, 15].

A complex analog of Khintchine's inequality deals with the Steinhaus sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$, where ε_k is uniformly distributed on the unit circle $\{z : |z| = 1\}$ for each k. We keep notation identical to before, except that we allow $c_1, c_2, \ldots, c_n \in \mathbb{C}$. The best constants A_p, B_p in the inequality

$$A_p \left(\sum_{k=1}^{n} |c_k|^2 \right)^{1/2} \leq \left\| \sum_{k=1}^{n} c_k \varepsilon_k \right\|_p \leq B_p \left(\sum_{k=1}^{n} |c_k|^2 \right)^{1/2}$$

were conjectured by Haagerup [16] to be

$$A_p = \begin{cases} \|W\|_p & \text{if } 0 < p \leq p_0 \\ \|Z\|_p & \text{if } p_0 < p < 2 \\ 1 & \text{if } 2 \leq p < \infty \end{cases} = \begin{cases} 2^{1/2} \left(\dfrac{\Gamma((p+1)/2)}{\sqrt{\pi}\,\Gamma((p+2)/2)} \right)^{1/p} & \text{if } 0 < p \leq p_0 \\ (\Gamma((p+2)/2))^{1/p} & \text{if } p_0 < p < 2 \\ 1 & \text{if } 2 \leq p < \infty \end{cases},$$

$$B_p = \begin{cases} 1 & \text{if } 0 < p \leq 2 \\ \|Z\|_p & \text{if } 2 < p < \infty \end{cases} = \begin{cases} 1 & \text{if } 0 < p \leq 2 \\ (\Gamma((p+2)/2))^{1/p} & \text{if } 2 < p < \infty \end{cases}$$

where $W = 2^{-1/2}(\varepsilon_1 + \varepsilon_2)$, $Z = 2^{-1/2}(U + iV)$ with U, V independent and Normal$(0, 1)$, and $p_0 = 0.4756170089...$ is the unique solution of the equation

$$2^{p/2} \Gamma\left(\frac{p+1}{2}\right) = \sqrt{\pi} \left(\Gamma\left(\frac{p+2}{2}\right) \right)^2$$

in the interval $0 < p < 2$. Here, if $\sum_{k=1}^{n} |c_k|^2 = 1$, then $A_1 = \sqrt{\pi}/2$ and $B_1 = 1$ encompass an average taken over all "complex signs" rather than only "real signs" as earlier. Sawa [17] announced that he could verify significant portions of Haagerup's conjecture, but only the case $p \approx 1$ was published. See also [14, 15, 18, 19]. We mention as well the following result [20, 21] for which $p = 1$ and n is the parameter of interest:

$$\mathrm{E}\left(\left| \sum_{k=1}^{n} \varepsilon_k \right| \right) = \begin{cases} \dfrac{2}{\pi} \displaystyle\int_0^\infty \frac{1 - \cos(t)^n}{t^2} dt & \text{for the real case} \\ \displaystyle\int_0^\infty \frac{1 - J_0(t)^n}{t^2} dt & \text{for the complex case} \end{cases}$$

where $J_0(t)$ is the zeroth Bessel function of the first kind. On the one hand, we have

$$\frac{2}{\pi}\int_0^\infty \frac{1-\cos(t)^n}{t^2}dt = \frac{n!}{2^{n-1}m!(n-m-1)!} \sim \sqrt{\frac{2n}{\pi}}$$

for the real case, where $m = \lfloor (n-1)/2 \rfloor$. On the other hand, the Bessel integral takes on the values 1, $4/\pi$, 1.57459723... and 1.79909248... for $n = 1$, 2, 3 and 4. Keane [22] recently determined that the third value in this list has the following closed-form expression:

$$\frac{1}{8\pi^3}\Gamma\left(\frac{1}{6}\right)^2\Gamma\left(\frac{1}{3}\right)^2 + 48\pi\Gamma\left(\frac{1}{6}\right)^{-2}\Gamma\left(\frac{1}{3}\right)^{-2} = 1.5745972375...$$

but the fourth value still remains open.

We next discuss Rosenthal's inequalities [23]:

$$\left\|\sum_{k=1}^n X_k\right\|_p \le C_p \cdot \max\left\{\left(\sum_{k=1}^n \|X_k\|_p^p\right)^{1/p}, \left\|\sum_{k=1}^n X_k\right\|_1\right\}, \qquad p \ge 1$$

for nonnegative random variables and

$$\left\|\sum_{k=1}^n X_k\right\|_p \le D_p \cdot \max\left\{\left(\sum_{k=1}^n \|X_k\|_p^p\right)^{1/p}, \left\|\sum_{k=1}^n X_k\right\|_2\right\}, \qquad p \ge 2$$

for symmetric random variables (meaning that the distribution of $-X$ is the same as the distribution of X). A variation of the latter inequality arises if we loosen the restrictive hypothesis "symmetric" to "zero mean"; the constant is then denoted E_p rather than D_p. Johnson, Schechtman & Zinn [24] showed that the growth rate of the best constants C_p, D_p, E_p is $p/\ln(p)$ as $p \to \infty$; in contrast, the growth rate for B_p is only $\sqrt{p}$. Subsequent work [25, 26, 27, 28] yielded that

$$C_p = \begin{cases} 1 & \text{if } p = 1 \\ 2^{1/p} & \text{if } 1 < p < 2 \\ \|Q\|_p & \text{if } 2 \le p < \infty \end{cases}, \qquad D_p = \begin{cases} 1 & \text{if } p = 2 \\ \left(1 + \|Z\|_p^p\right)^{1/p} & \text{if } 2 < p < 4 \\ \|R - S\|_p & \text{if } 4 \le p < \infty \end{cases}$$

where Q is Poisson(1), Z is Normal$(0,1)$, and R, S are independent Poisson(1/2) variables. It is known that $\|Q\|_m^m = \alpha_m$ and $\|R-S\|_{2m}^{2m} = \beta_m$ for integer m, where $\{\alpha_m\}_{m=1}^\infty = \{1, 2, 5, 15, 52, 203, \ldots\}$ is the sequence of Bell numbers [29, 30]

$$\alpha_m = \frac{1}{e}\sum_{j=0}^\infty \frac{j^m}{j!} = \left.\frac{d^m}{dx^m}\exp\left(\exp(x) - 1\right)\right|_{x=0}$$

and $\{\beta_m\}_{m=1}^\infty = \{1, 4, 31, 379, \ldots\}$ is the sequence

$$\beta_m = \frac{2}{e}\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\frac{k^{2m}}{j!(j+k)!2^{2j+k}} = \left.\frac{d^{2m}}{dx^{2m}}\exp\left(\cosh(x)-1\right)\right|_{x=0}.$$

Ibragimov & Sharakhmetov [31] conjectured that

$$E_p = \begin{cases} \left(1+\|Z\|_p^p\right)^{1/p} & \text{if } 2<p<4 \\ \|Q-1\|_p & \text{if } 4\le p<\infty \end{cases}$$

and proved that this is true when $p = 2m$; further, $\|Q-1\|_{2m}^{2m} = \gamma_m$ and $\{\gamma_m\}_{m=1}^\infty = \{1, 4, 41, 715, \ldots\}$ is the sequence

$$\gamma_m = \frac{1}{e}\sum_{j=0}^{\infty}\frac{(j-1)^{2m}}{j!} = \left.\frac{d^{2m}}{dx^{2m}}\exp\left(\exp(x)-x-1\right)\right|_{x=0}.$$

Combinatorial interpretations apply for each of the three sequences: α_n is the number of partitions of an n-element set into blocks; β_n is the number of partitions of a $2n$-element set into blocks, each containing an even number of elements; and γ_n is the number of partitions of a $2n$-element set into blocks, each containing more than one element [30].

Define the following Orlicz-type norm:

$$[\Xi]_p = \inf\left\{\lambda>0 : \prod_{k=1}^{\infty}\mathrm{E}\left(\left|1+\frac{X_k}{\lambda}\right|^p\right)\le e^p\right\}$$

for an arbitrary sequence $\Xi = \{X_k\}_{k=1}^\infty$ of independent random variables, for any $p > 0$. We mention Latała's inequality [32]:

$$\frac{e-1}{2e^2}\cdot[\Xi]_p \le \left\|\sum_{k=1}^{\infty}X_k\right\|_p \le e\cdot[\Xi]_p$$

which holds either if all the Xs are nonnegative and $p \ge 1$, or if all the Xs are symmetric and $p \ge 2$. Observe here that the bounds do not depend on p, unlike the earlier inequalities. For the nonnegative case, Hitczenko & Montgomery-Smith [33] improved the left-hand constant $(e-1)/(2e^2) = 0.116272...$ to $\xi = 0.154906...$, where ξ is the unique positive solution of the equation

$$\sum_{k=0}^{\infty}\frac{(2k+1)^k}{k!}x^k = e.$$

It is not known if this improvement carries over to the symmetric case, nor whether a calculation of best constants is feasible at present.

0.1. Addendum. Assuming $\sum_{k=1}^{n} c_k^2 = 1$, it is conjectured that the Rademacher sequence satisfies [34, 35, 36, 37, 38]

$$P_n = \mathrm{P}\left(\left|\sum_{k=1}^{n} c_k \varepsilon_k \leq 1\right|\right) \geq \frac{1}{2}$$

always. This inequality is provably true if 1/2 is replaced by 3/8 [35] or if all cs are equal [37]. For the latter scenario, we deduce that

$$\lim_{n\to\infty} P_n = \operatorname{erf}\left(1/\sqrt{2}\right) = 0.6826894921...$$

by the normal approximation to the binomial distribution. This constant also appears in [39] with regard to a continued fraction expansion.

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