Tauberian Constants

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A series $\sum_{k=0}^{\infty} a_k$ of complex numbers is **Abel convergent** if

$$\lim_{x\to 1^-} \sum_{k=0}^{\infty} a_k x^k \qquad \left(\text{equivalently, } \lim_{x\to 1^-} (1-x) \sum_{n=0}^{\infty} s_n x^n\right)$$

exists and **Cesàro convergent** if

$$\lim_{l\to\infty} \frac{1}{l+1} \sum_{n=0}^{l} s_n \qquad \left(\text{equivalently, } \lim_{l\to\infty} \sum_{k=0}^{l} \left(1 - \frac{k}{l+1}\right) a_k\right)$$

exists, where $s_n = \sum_{k=0}^{n} a_k$ for each $n \geq 0$. Ordinary convergence implies both Abel convergence and Cesàro convergence. Various converses of this theorem, in which ordinary convergence is deduced from a summability condition (as above) plus an additional condition (for example, $ka_k \to 0$ as $k \to \infty$), are called **Tauberian theorems** [1, 2, 3].

For notational convenience, when we use the symbol σ, we mean an arbitrary limit point of the partial sums $\{s_n\}_{n=0}^{\infty}$. By λ, we mean a limit point of the power series $\sum_{k=0}^{\infty} a_k x^k$ as $x \to 1^-$. By μ, we mean a limit point of the partial averages $\{m_l\}_{l=0}^{\infty}$, where $m_l = \sum_{n=0}^{l} s_n/(l+1)$ for each $l \geq 0$.

We start with a Tauberian theorem due to Hadwiger [4, 5] and Agnew [6, 7, 8]; it is quite general since no hypotheses are required! Constants C_1 and C_2 exist with the following properties:

- for each σ, there is a λ such that $|\lambda - \sigma| \leq C_1 \limsup_{k\to\infty} |ka_k|$,
- for each λ, there is a σ such that $|\lambda - \sigma| \leq C_2 \limsup_{k\to\infty} |ka_k|$.

The least constant C_1 is known to be

$$\gamma + \ln(\ln(2)) - 2\,\mathrm{Ei}(-\ln(2)) = 0.9680448304...$$

where Ei is the exponential integral [9]. The least constant C_2 satisfies the inequality $0.4858 \leq C_2 \leq 0.7494386$, but its exact value is unknown. Likewise [8, 10, 11], constants C_3 and C_4 exist with the following properties:

- for each σ, there is a μ such that $|\mu - \sigma| \leq C_3 \limsup_{k\to\infty} |ka_k|$,
- for each μ, there is a σ such that $|\mu - \sigma| \leq C_4 \limsup_{k\to\infty} |ka_k|$.

The least constant C_3 is known to be $\ln(2) = 0.6931471805...$ and the least constant C_4 is the unique real solution y of the equation

$$y = e^{-(\pi/2)y}, \qquad \text{that is,} \quad y = \frac{2}{\pi} W\left(\frac{\pi}{2}\right) = 0.4745409995...,$$

where W is Lambert's function [14]. See a generalization by Rajagopal [12, 13].

Different constants emerge if we are more restrictive in our choices of σ, λ and μ. For example, the best constant $\tilde{C}_1$ such that [15, 16, 17, 18]

$$\limsup_{n\to\infty} \left| \sum_{k=0}^{\infty} a_k \left(1 - \frac{1}{n}\right)^k - \sum_{k=0}^{n} a_k \right| \leq \tilde{C}_1 \limsup_{k\to\infty} |ka_k|$$

is

$$\gamma - 2\,\mathrm{Ei}(-1) = 1.0159835336... = 1.7517424160... - 2/e.$$

The constant $-\mathrm{Ei}(-1) = 0.2193839343...$ is familiar: when multiplied by e, it gives the Euler-Gompertz constant [9]. In the definition of C_4, observe that the subsequence of $\{s_n\}_{n=0}^{\infty}$ with limit point σ may depend on the sequence $\{a_k\}_{k=0}^{\infty}$. If we deny any knowledge of $\{a_k\}_{k=0}^{\infty}$, then the required constant C_4' becomes larger. More precisely, there is an increasing sequence $\{n_l\}_{l=0}^{\infty}$ independent of $\{a_k\}_{k=0}^{\infty}$ such that

$$\limsup_{l\to\infty} |m_l - s_{n_l}| \leq C_4' \limsup_{k\to\infty} |ka_k|$$

and $C_4' = \ln(2)$ is best possible; further, a simple such sequence is $n_l = \lfloor l/2 \rfloor$. Here is a variation in which we permit knowledge of $\{a_k\}_{k=0}^{\infty}$ only to make a binary decision at each step. There exist two increasing sequences $\{p_l\}_{l=0}^{\infty}$ and $\{q_l\}_{l=0}^{\infty}$ independent of $\{a_k\}_{k=0}^{\infty}$ such that

$$\limsup_{l\to\infty} |m_l - s_{n_l}| \leq C_4'' \limsup_{k\to\infty} |ka_k|$$

where n_l is, for each l, one of the two integers p_l and q_l, and the optimal C_4'' satisfies $C_4 \leq C_4'' \leq 0.56348$. The exact value of C_4'' is unknown, but it is believed to be close to its upper bound. This estimate comes from setting $p_l = \lfloor 3l/8 \rfloor$, $q_l = \lfloor 5l/8 \rfloor$ and choosing n_l appropriately.

Kotnik [19, 20] has computed certain Tauberian constants that occur in number theory [21, 22, 23]; we hope to discuss these later.

References

[1] R. L. Powell and S. M. Shah, *Summability Theory and Applications*, van Nostrand Reinhold, 1972.

[2] A. Peyerimhoff, *Lectures on Summability*, Lect. Notes in Math. 107, Springer-Verlag, 1969; MR0463744 (57 #3684).

[3] J. van de Lune, *An Introduction to Tauberian Theory: From Tauber to Wiener*, Centrum voor Wiskunde en Informatica, 1986; MR0882005 (88j:40011).

[4] H. Hadwiger, Über ein Distanz-theorem bei der A-Limitierung, *Comment. Math. Helv.* 16 (1944) 209–214; MR0010002 (5,236e).

[5] H. Hadwiger, Ueber eine Konstante Tauberscher Art, *Revista Mat. Hisp.-Amer.* 7 (1947) 65–69; MR0021602 (9,86b).

[6] R. P. Agnew, Abel transforms of Tauberian series, *Duke Math. J.* 12 (1945) 27–36; MR0012338 (7,12c).

[7] R. P. Agnew, Abel transforms and partial sums of Tauberian series, *Annals of Math.* 50 (1949) 110–117; MR0027351 (10,291i).

[8] R. P. Agnew, Tauberian relations among partial sums, Riesz transforms, and Abel transforms of series, *J. Reine Angew. Math.* 193 (1954) 94–118; MR0064159 (16,237a).

[9] S. R. Finch, Euler-Gompertz constant, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 423–428.

[10] R. P. Agnew, Arithmetic means of some Tauberian series and determination of a lower bound for a fundamental Tauberian constant, *Proc. London Math. Soc.* 2 (1952) 369–384; MR0049357 (14,160c).

[11] R. P. Agnew, Arithmetic means and the Tauberian constant .474541, *Acta Math.* 87 (1952) 347–359; MR0051331 (14,463e).

[12] C. T. Rajagopal, On an absolute constant in the theory of Tauberian series, *Proc. Indian Acad. Sci. Sect. A* 28 (1948) 537–544; postscript, 31 (1950) 60–61; MR0028443 (10,447g) and MR0035846 (12,21b).

[13] C. T. Rajagopal, On an absolute constant in the theory of Tauberian series. II, *Proc. Indian Acad. Sci. Sect. A* 39 (1954) 272–281; MR0063462 (16,125a).

[14] S. R. Finch, Iterated exponential constants, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 448–452.

[15] H. Hadwiger, Die Retardierungserscheinung bei Potenzreihen und Ermittlung zweier Konstanten Tauberscher Art, *Comment. Math. Helv.* 20 (1947) 319–332; MR0021605 (9,86e).

[16] A. Wintner, On Tauber's theorem, *Comment. Math. Helv.* 20 (1947) 216–222; MR0021604 (9,86d).

[17] P. Hartman, Tauber's theorem and absolute constants, *Amer. J. Math.* 69 (1947) 599–606; MR0021603 (9,86c).

[18] I. J. Maddox, Tauberian constants, *Bull. London Math. Soc.* 1 (1969) 193–200; MR0247328 (40 #596).

[19] T. Kotnik, Computation of eleven Tauberian constants, http://www.people.fas.harvard.edu/~sfinch/csolve/tadej.pdf.

[20] T. Kotnik, Evaluation of some integrals of sums involving the Möbius function, *Internat. J. Comput. Math.* 84 (2007) 469–475; MR2326305.

[21] K. A. Jukes, Tauberian constants connected with the prime number theorem, *Math. Z.* 145 (1975) 187–193; MR0382896 (52 #3778).

[22] H. W. Gretton and K. A. Jukes, Tauberian constants connected with the prime number theorem. II. Computational aspects, *Math. Z.* 164 (1978) 25–30; MR0514605 (80a:10064).

[23] H. W. Gretton and K. A. Jukes, Computations of Tauberian constants connected with equiconvergence and the quadratfrei numbers, *J. Comput. Appl. Math.* 5 (1979) 25–27; MR0525608 (80c:65079).