Constant of Theodorus

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April 9, 2005

In the complex plane, consider the recursive sequence

$$z_n = \left(1 + \frac{i}{\sqrt{n}}\right) z_{n-1}, \qquad n \geq 1,$$

with starting point $z_0 = 1$. The points z_{n-1} and z_n determine a right triangle relative to the origin 0, with legs 1 and $\sqrt{n}$. Clearly the polar coordinates (r_n, θ_n) of z_n are given by

$$r_n = \sqrt{n+1}, \qquad \theta_n = \begin{cases} \displaystyle\sum_{j=0}^{n-1} \arctan\left(\frac{1}{\sqrt{j+1}}\right) & \text{if } n \geq 1, \\ 0 & \text{if } n = 0. \end{cases}$$

A closed-form expression for z_n is

$$z_n = \prod_{k=1}^{n} \left(1 + \frac{i}{\sqrt{k}}\right) \qquad n \geq 1,$$

and determines what is called the **discrete spiral of Theodorus**.

Davis [1, 2] and Heuvers, Moak & Boursaw [3] independently constructed the continuous analog of this spiral. A parametric representation is [1, 2]

$$\begin{aligned} f(t) &= \prod_{k=1}^{\infty} \frac{1 + \frac{i}{\sqrt{k}}}{1 + \frac{i}{\sqrt{k+t}}}, \qquad -1 < t < \infty, \\ &= \sqrt{1+t}\, \exp\left(i \sum_{k=1}^{\infty} \left(\arctan\left(\sqrt{k+t}\right) - \arctan\left(\sqrt{k}\right)\right)\right) \end{aligned}$$

and a polar representation is [3]

$$\theta(r) = \sum_{j=0}^{\infty} \left(\arctan\left(\frac{1}{\sqrt{j+1}}\right) - \arctan\left(\frac{1}{\sqrt{j+r^2}}\right)\right), \qquad r > 0.$$

Gronau [2] proved that $f(t)$ is the unique solution of the functional equation

$$f(t)=\left(1+\frac{i}{\sqrt{t}}\right)f(t-1),\qquad f(0)=1,\qquad 0<t<\infty$$

such that $|f(t)|$ is increasing and $\arg(f(t))$ is both increasing and continuous.

Among many possible questions, Davis [1] asked: What is the slope of the spiral at the point 1? Clearly

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1,0)}=\left.\frac{d\theta}{dr}\right|_{(r,\theta)=(1,0)}=\sum_{k=1}^{\infty}\frac{1}{k^{3/2}+k^{1/2}}$$

which Gautschi [4] evaluated to be 1.8600250792.... This is called the **constant of Theodorus**.

Also, what can be said about the growth of θ_n as $n\to\infty$? For convenience, given a real number ξ, let $\{\xi\}=\xi \bmod 1$ denote the fractional part of ξ. Hlawka [5] proved that

$$\theta_n=2\sqrt{n+1}+K+\frac{1}{6\sqrt{n+1}}+O\left(n^{-3/2}\right),$$

where the **square root spiral constant** $K=K_0-1-3\pi/8=-2.1577829966...$ and

$$K_0=\frac{1}{8}\int_2^{\infty}\{x\}\left(1-\{x\}\right)(3x-2)\frac{1}{x^2(x-1)^{3/2}}dx=0.0203142484....$$

The numerical estimate of K was obtained by Grünberg [6], correcting an apparent error in [5].

0.1. Addendum. The series

$$K=\frac{\pi}{4}+\sum_{m=0}^{\infty}(-1)^m\frac{\zeta\left(m+\frac{1}{2}\right)-1}{2m+1}$$

converges quickly [7], as does

$$K'=\sum_{m=0}^{\infty}\frac{\zeta\left(m+\frac{1}{2}\right)-1}{2m+1}=-1.8265078108...$$

(associated with the growth of θ_n', obtained by replacing arctan by arctanh in the definition of θ_n). Similarly, the series

$$\sum_{k=1}^{\infty}\frac{1}{(k+1)\sqrt{k}}=\frac{1}{2}+\sum_{m=1}^{\infty}(-1)^{m+1}\left\{\zeta\left(m+\frac{1}{2}\right)-1\right\}$$

converges quickly (to Theodorus' constant), as does

$$\sum_{k=2}^{\infty} \frac{1}{(k-1)\sqrt{k}} = \sum_{m=1}^{\infty} \left\{ \zeta\left(m + \frac{1}{2}\right) - 1 \right\} = 2.1840094702...$$

(obtained by simply replacing $+$ by $-$ and removing the term for $k = 1$).

References

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