Cyclotomic Polynomials

Steven Finch

March 31, 2008

Let

$$\prod_{\gcd(j,n)=1}\left(x-e^{2\pi j\,i/n}\right)=\sum_{k=0}^{\varphi(n)}a_n(k)x^k$$

denote the n^{th} **cyclotomic polynomial**, where $\varphi(n)$ is Euler's totient function [1], i denotes the imaginary unit, and the first sum is taken over all integers $1\leq j\leq n$ coprime with n. The coefficients $a_n(k)$ are always integers. Define

$$A(n)=\max_k|a_n(k)|,$$

the largest coefficient of the n^{th} polynomial in absolute value; and

$$B(k)=\max_n|a_n(k)|,$$

the largest k^{th} coefficient in absolute value (taken over all polynomials). An simple argument gives $B(k)\leq p(k)$, where $p(k)$ is the number of integer partitions of k, hence $B(k)$ is finite.

Vaughan [2, 3, 4] proved that

$$\limsup_{n\to\infty}\frac{\ln(\ln(A(n)))}{\ln(n)/\ln(\ln(n))}=\ln(2)$$

(a maximal order) and Bachman [5] proved that

$$\lim_{k\to\infty}\frac{\ln(k)^{1/4}}{\sqrt{k}}\ln(B(k))=C=1.5394450081...$$

(an asymptotic result). The constant C is related to the solution of an interesting optimization problem involving L-series [6]. Define $\iota(D)=2$ if $D>0$ and $\iota(D)=1$ if $D<0$. Over all fundamental discriminants D, it can be proved that $D=12$ maximizes the quantity

$$\sqrt{\frac{\iota(D)}{\pi\,\varphi(D)}\frac{L_D(2)}{\sqrt{L_D(1)}}\prod_{(D/p)=-1}\left(1-\frac{1}{p^2}\right)^{1/2}}$$

where the product is taken over all primes p satisfying a negativity condition on the Legendre symbol. When $D = 12$, this quantity simplifies to

$$\begin{aligned}\sqrt{\frac{1}{2\pi}}\frac{\pi^2/(6\sqrt{3})}{\sqrt{\ln(2+\sqrt{3})/\sqrt{3}}}\prod_{\substack{p\equiv 5 \text{ or } 7 \\ \bmod 12}}\left(1-\frac{1}{p^2}\right)^{1/2} &= 0.4189414873... \\ &= 2^{-5/2}C^2.\end{aligned}$$

We hope to report later on other statistics (for example, means and variances) summarizing the coefficient array $a_n(k)$. See [7, 8] for work in this area. The precise estimate of C is due to Sebah [9].

REFERENCES

[1] S. R. Finch, Euler totient constants, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 115–118.

[2] R. C. Vaughan, Bounds for the coefficients of cyclotomic polynomials, *Michigan Math. J.* 21 (1974) 289–295; MR0364141 (51 #396).

[3] P. T. Bateman, C. Pomerance and R. C. Vaughan, On the size of the coefficients of the cyclotomic polynomial, *Topics in Classical Number Theory*, v. I, Proc. 1981 Budapest conf., Colloq. Math. Soc. János Bolyai 34, North-Holland, 1984, pp. 171–202; MR0781138 (86e:11089).

[4] C. Pomerance and N. C. Ryan, Maximal height of divisors of $x^n - 1$, *Illinois J. Math.* 51 (2007) 597–604; MR2342677.

[5] G. Bachman, *On the Coefficients of Cyclotomic Polynomials*, Amer. Math. Soc., 1993, pp. 1–10, 74–80; MR1172916 (94d:11068).

[6] S. R. Finch, Quadratic Dirichlet L-series, unpublished note (2005).

[7] H. Möller, Über die i-ten Koeffizienten der Kreisteilungspolynome, *Math. Annalen* 188 (1970) 26–38; MR0266899 (42 #1801).

[8] Y. Gallot, P. Moree and H. Hommersom, Value distribution of cyclotomic polynomial coefficients, *Unif. Distrib. Theory* 6 (2011) 177–206; arXiv:0803.2483; MR2904047.

[9] P. Sebah, An infinite product, unpublished note (2008).