Uncertainty Inequalities

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If an integrable function $f : \mathbb{R}^n \to \mathbb{R}$ is thought of as the amplitude of a time signal or space image, then the Fourier transform $\hat{f}$ of f:

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} f(x)\, dx$$

conveys information on how $f(x)$ is built from sine waves of different frequencies. Assume that $f \in L_r(\mathbb{R}^n)$ for some $r \geq 1$; equivalently, $|f(x)|^r$ is integrable and decays rapidly enough as $|x| \to \infty$ so that

$$||f||_r = \left(\int_{\mathbb{R}^n} |f(x)|^r\, dx \right)^{\frac{1}{r}} < \infty.$$

Define $P_p f$ and $Q_q f$ to be the functions

$$(P_p f)(x) = |x|^p f(x), \qquad (Q_q f)(\xi) = |\xi|^q \hat{f}(\xi).$$

Heisenberg's famous inequality arises from the case when $p = q = 1$ and $r = 2$ [1]:

$$||P_1 f||_2 \cdot ||Q_1 f||_2 \geq \frac{n}{4\pi} ||f||_2^2.$$

In words, if $f(x)$ is concentrated close to 0 (having a small variance), then $\hat{f}(\xi)$ must be relatively spread out (having a large variance) unless $f(x)$ is zero almost everywhere. The constant $n/(4\pi)$ is best possible if $n = 1$: consider functions of the form $a \exp(-bx^2)$ for some $b > 0$ [2].

When f is smooth, it follows that $||\nabla f||_2 = 2\pi ||Q_1 f||_2$ where ∇f is the gradient of f and $|\nabla f|$ is its Euclidean norm. Therefore Heisenberg's inequality is an uncertainty principle in the same sense as expressed in [3].

Here are two sample variations [4, 5]. Let $f \in L_1(\mathbb{R}^n) \cap L_2(\mathbb{R}^n)$ and recall that J_ν is the Bessel function of the first kind [6]. For $r > 0$, define

$$J(r) = r^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(r)$$

and, for $y > 0$,

$$g_y(x) = \begin{cases} J(|x|) - J(y) + \dfrac{J'(y)}{2y}(y^2 - |x|^2) & \text{if } |x| < y, \\ 0 & \text{if } |x| \geq y. \end{cases}$$

The best constant μ_n in the inequality

$$||P_2 f||_1 \cdot ||Q_1 f||_2^2 \geq \frac{\mu_n}{4\pi^2} ||f||_1 ||f||_2^2$$

is achieved when $f = g_c$, where c is the smallest positive root of the equation

$$||g_y||_2 = ||\nabla g_y||_2.$$

In particular, if $n = 1$, the equation simplifies to

$$y(5 - 2y^2)\tan(y)^2 + 5(3 - 2y^2)\tan(y) - 15y = 0$$

and hence $c = 1.7502456171...$ and $\mu_n = 0.4283683675... = \frac{1}{2}(0.8567367350...) = \frac{M}{2}$. The constant M will be useful to us later.

Also, the best constant μ_n in the inequality

$$||P_2 f||_1^{\frac{2}{n+6}} \cdot ||Q_1 f||_2^{\frac{n+4}{n+6}} \geq \mu_n ||f||_2$$

is achieved when $f = g_c$, where c is the smallest positive root of the equation

$$\sqrt{n+4}\,||g_y||_2 = \sqrt{n+6}\,||\nabla g_y||_2, \qquad \text{that is,} \qquad (y^2 - 2n)J'(y) = 2yJ(y).$$

In particular, if $n = 1$ (and thus the two exponents are $2/7$ and $5/7$), the equation simplifies to

$$(2 - y^2)\tan(y) = 2y$$

and hence $c = 2.0815759778$ and $\mu_n^{-1} = 4.1731026567....$ Closed-form expressions do not seem to be possible here! This formulation is, in fact, only a special case of a considerably broader theorem [5].

0.1. Positive Definite Probability Densities. A probability density function $f : \mathbb{R}^n \to \mathbb{R}$ is **positive definite** if [7, 8]

$$\sum_{j=1}^{m} \sum_{k=1}^{m} f(x_k - x_j) z_j \bar{z}_k \geq 0$$

for all $x_j \in \mathbb{R}^n$, for all $z_j \in \mathbb{C}$ $(j = 1, \ldots, n)$ and for each $m \geq 1$, where $\bar{z}$ denotes the complex conjugate of z. Clearly $f(-x) = f(x) < f(0)$ for all $x \neq 0$. Let F_n

denote the class of all continuous, positive definite probability density functions on $\mathbb{R}^n$. If $f \in F_n$, then $\hat{f}$ is nonnegative and integrable over $\mathbb{R}^n$; in fact, $\hat{f}/f(0)$ is itself a probability density.

Fix, for now, $n = 1$. Among the well-known members of F_1 are the normal, t, and logistic densities. Define a product of variances

$$\lambda(f) = 4\pi^2 \frac{||P_2 f||_1 \cdot ||Q_2 f||_1}{\hat{f}(0) \cdot f(0)}$$

and a greatest lower bound, called **Laue's constant** [8]:

$$\Lambda = \inf_{f \in F_1} \lambda(f).$$

An immediate consequence of Laeng & Morpurgo's work [4], for example, is that $\Lambda \leq M < 0.85674$. Estimating Λ has occupied several researchers over several years [9, 10, 11, 12]:

$$0.543 < \Lambda < 0.85024$$

yet a determination of its exact value still seems faraway.

For $n \geq 1$, choose an arbitrary unit vector $u \in \mathbb{R}^n$. If X is a random n-vector with density $f \in F_n$, let $f_u \in F_1$ denote the density of the one-dimensional projection $u \cdot X$ of X onto u. Then define [11]

$$\Lambda_n = \inf_{f \in F_n} \sup_{||u||=1} \lambda(f_u).$$

Clearly $\Lambda_1 = \Lambda$ and $\Lambda_{n+1} \geq \Lambda_n$ for all n. We have the following estimates [12]:

$$\begin{aligned} \Lambda_n \leq \frac{1}{2} \frac{9 + 4\sqrt{5}}{(1+\sqrt{5})^2} < 0.856763... \qquad & \text{if } n \leq 7, \\ 1 - \frac{3}{n} \leq \Lambda_n \leq 1 - \frac{n-5}{2(n-4)} \frac{3}{n} \qquad & \text{if } n \geq 8. \end{aligned}$$

which demonstrate that $\lim_{n\to\infty} \Lambda_n = 1$. These constants deserve to be better known!

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