Self-Convolutions

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Let f be a square-integrable probability density function supported on a subinterval of $\mathbb{R}$ of length $1/2$. Define the **self-convolution** of f to be

$$(f*f)(x)=\int\limits_{-\infty}^{\infty}f(t)f(x-t)dt.$$

Thus $f*f$ is the probability density of a sum of two independent random variables, each distributed according to f, and is supported on an interval of length 1. We are interested in the "size" of $f*f$, measured via both L_2 and L_∞ norms. Before doing this, however, let us examine f alone as a preliminary exercise.

For each integer $n\geq 1$, define

$$g_n(x)=\frac{n+1}{n}\left(\frac{1}{\sqrt{2x}}\right)^{\frac{n-1}{n}},\qquad 0<x<1/2$$

then clearly g_n is a probability density for all n,

$$\|g_n\|_2^2=\int\limits_0^{1/2}g_n(x)^2dx=\frac{(n+1)^2}{2n}\to\infty$$

as $n\to\infty$, and $\|g_n\|_\infty=\infty$ always. Consequently

$$\sup_f\|f\|_2^2=\infty=\sup_f\|f\|_\infty.$$

Also, suppose that there exists a probability density h on $[0,1/2]$ with $\|h\|_2^2<2$. By the Cauchy-Schwarz inequality,

$$2=\int\limits_0^{1/2}h(x)\cdot 2\,dx\leq\|h\|_2\cdot\|2\|_2<\sqrt{2}\cdot\sqrt{2}=2,$$

which is a contradiction. Consequently

$$\inf_f \|f\|_2^2 = 2 = \inf_f \|f\|_\infty .$$

The problem of assessing $f * f$ together is more difficult. Let us first discuss relevant infimums. Martin & O'Bryant [1, 2] conjectured that

$$\inf_f \|f * f\|_\infty = \pi/2 = 1.5707963267...$$

on the basis of their proof that the left-hand side must exceed $1.262 = (2)(0.638)$, plus their observation that $\|g * g\|_\infty = \pi/2$, where

$$g(x) = \lim_{n\to\infty} g_n(x) = 1/\sqrt{2x}.$$

Technically, g is not admissible (since it is not square-integrable). See [3, 4, 5] for discussion of a similar case.

Martin & O'Bryant [1] also proved that

$$\inf_f \|f * f\|_2^2 \geq 1.14915 = (2)(0.574575)$$

after elaborate computations. This may be nearly correct, since the probability density

$$k(x) = \frac{4}{\pi}\frac{1}{\sqrt{8x(1-2x)}}, \qquad 0 < x < 1/2$$

satisfies

$$\|k * k\|_2^2 < 1.14939.$$

Again, k is not admissible for technical reasons. No exact formula is even conjectured in this case, which renders it especially interesting!

Here is a problem involving ratios of L_p norms. Hölder's inequality gives

$$\|f\|_2^2 \leq \|f\|_\infty \cdot \|f\|_1$$

which is an equality if $f = 2$ on $[0, 1/2]$. Consequently

$$\inf_f \frac{\|f\|_\infty}{\|f\|_2^2} = 1.$$

Martin & O'Bryant [1, 2] conjectured that

$$\inf_f \frac{\|f * f\|_\infty}{\|f * f\|_2^2} = \frac{\pi}{4\ln(2)}$$

on the basis, in part, of their observation that $\|g * g\|_2^2 = 2\ln(2)$. This result gives a sense of how large $\|f * f\|_2^2$ can be, in terms of $\|f * f\|_\infty$. No other mention of relevant supremums in the literature has yet been found!

0.1. Addendum. The first conjecture is false: in fact,

$$1.2748 \le \inf_f \|f * f\|_\infty \le 1.5098.$$

The second conjecture is also false: in fact,

$$\inf_f \frac{\|f * f\|_\infty}{\|f * f\|_2^2} \le \frac{1}{0.88922...} < \frac{1}{0.88254...} = \frac{\pi}{4\ln(2)}.$$

Such adjustments open up this subject considerably since no one knows what the extremal functions f now might be [6, 7]. A sequence of lower bounds defined in [8] and numerical optimization (on a simplex in $\mathbb{R}^{2n}$) suggest an improvement 1.28 over 1.2748; the upper bound 1.5098 is believed to be close to the true value.

References

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