Stars and Watermelons

Steven Finch

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The p-**vicious walker model** of length $2n$ consists of p lattice paths W_1, W_2, …, W_p in $\mathbb{Z}^2$ where

- W_k starts at the point $(0, a_k)$ and ends at the point $(2n, b_k)$ for $k = 1, \ldots, p$
- all steps are directed northeast or southeast (that is, from (i, j) to $(i+1, j+1)$ or to $(i+1, j-1)$)
- if $k \neq \ell$, then W_k and W_ℓ never intersect (hence $a_k \neq a_\ell$ and $b_k \neq b_\ell$, for instance).

In p-**star** configurations, $a_k = 2k-2$ for each k (with no constraint on b_k); in p-**watermelon** configurations, $b_k = 2k-2$ as well [1, 2]. We often think of the horizontal axis as time and the vertical axis as space, writing $W_k(0) = a_k$ and $W_k(2n) = b_k$. A p-watermelon with a **wall** has the additional property that

- $W_k(i) \geq 0$ for all $0 \leq i \leq 2n$, for all k.

Gillet [3] demonstrated that $\lim_{n\to\infty} W_k(\lfloor 2n\, t\rfloor)/\sqrt{2n}$ tends to a family of p nonintersecting Brownian excursions, $0 \leq t \leq 1$, as an extension of a principle given in [4].

The **height** of a path W_k in a p-watermelon with wall is the maximum value of $W_k(i)$ over all i. The **area** under a path W_k is the area of the polygonal region determined by the curve $j = W_k(i)$, the horizontal line $j = 0$, and the vertical lines $i = 0$, $i = 2n$. In the case $p = 2$, we will refer to the upper height and upper area (corresponding to W_2) and the lower height and lower area (corresponding to W_1).

Counting all 1-watermelons with wall (or Dyck paths) and 2-watermelons with wall give

$$\frac{(2n)!}{n!(n+1)!}, \qquad \frac{6(2n)!(2n+2)!}{n!(n+1)!(n+2)!(n+3)!}$$

possible configurations of length $2n$, respectively. (The former is the n^{th} Catalan number.) The average height $H_1(n)$ for 1-watermelons with wall satisfies [5, 6]

$$H_1(n) \sim \sqrt{\pi n}$$

as $n \to \infty$ and the average area $A_1(n)$ satisfies [7, 8]

$$A_1(n) \sim \sqrt{\pi} n^{3/2}.$$

To go to the average L_∞-norm of Brownian excursion, divide the H_1 result by $\sqrt{2n}$ (space dimension only), yielding $\sqrt{\pi/2}$. To go to the average L_1-norm, divide the A_1 result by $(2n)^{3/2}$ (both time and space considered), yielding $\sqrt{\pi/8}$. Exact formulas for $H_1(n)$ and $A_1(n)$ are also available [9].

The average upper height $H_2(n)$ for 2-watermelons with wall satisfies

$$H_2(n) \sim (2.57758...)\sqrt{n} \sim (1.822625...)\sqrt{2n},$$

a new result due to Fulmek [6]. The coefficient can be expressed as a linear combination of several complicated integrals of theta functions; a certain double Dirichlet series also plays a role in the proof. Numerical results for $3 \leq p \leq 5$ and for higher moments were obtained by Feierl [10]. A different method was proposed in [11]. To go to the average upper L_∞-norm of Brownian excursion, divide the H_2 result by $\sqrt{2n}$. An exact formula for $H_2(n)$ is also available [12]. Similar information about the lower height is not known.

An exact formula for $A_2(n)$ seems to be an open problem. Interestingly, we have both average upper/lower L_1-norm results for Brownian excursion:

$$\tfrac{5}{8}\left(\sqrt{2}-1\right)\sqrt{\pi}, \qquad \tfrac{5}{8}\sqrt{\pi}$$

due to Tracy & Widom [13]. Multiplying each constant by $(2n)^{3/2}$ therefore provides the main asymptotic terms for average upper/lower areas under 2-watermelons with wall. Numerical results in [13] also apply for $3 \leq p \leq 9$. In a study of average upper L_1-norms as $p \to \infty$, the constant 1.7710868074... arises [14, 15] and thus random matrix theory lurks nearby.

Counting all 1-watermelons without wall (or bilateral Dyck paths) and 2-watermelons without wall give [16]

$$\frac{(2n)!}{(n!)^2}, \qquad \frac{(2n)!(2n+1)!}{(n!)^2((n+1)!)^2}$$

possible configurations of length $2n$, respectively. (The former is the n^{th} central binomial coefficient.) These tend to Brownian bridges as $n \to \infty$ [3, 17]. In the same way, p-stars with wall tend to Brownian meanders and p-stars without wall tend to Brownian motions. Corresponding questions about average heights and average areas (suitably generalized) for $p \geq 2$ seem to be unanswered.

References

[1] N. Bonichon and M. Mosbah, Watermelon uniform random generation with applications, *Theoret. Comput. Sci.* 307 (2003) 241–256; MR2022577 (2004m:05032).

[2] N. J. A. Sloane, On-Line Encyclopedia of Integer Sequences, A078920 and A103905.

[3] F. Gillet, Asymptotic behaviour of watermelons, math.PR/0307204.

[4] S. R. Finch, Variants of Brownian motion, unpublished note (2004).

[5] N. G. de Bruijn, D. E. Knuth and S. O. Rice, The average height of planted plane trees, *Graph Theory and Computing*, ed. R. C. Read, Academic Press, 1972, pp. 15–22; also in *Selected Papers on Analysis of Algorithms*, CSLI, 2000, pp. 215–223; MR0505710 (58 #21737).

[6] M. Fulmek, Asymptotics of the average height of 2-watermelons with a wall, *Elec. J. Combin.* 14 (2007) R64; math.CO/0607163; MR2350454.

[7] D. Merlini, R. Sprugnoli and M. C. Verri, The area determined by underdiagonal lattice paths, *Proc. 1996 Colloq. on Trees in Algebra and Programming (CAAP)*, Linköping, ed. H. Kirchner, Lect. Notes in Comp. Sci. 1059, Springer-Verlag, 1996, pp. 59–71; MR1415900 (97f:68140).

[8] R. Chapman, Moments of Dyck paths, *Discrete Math.* 204 (1999) 113–117; MR1691864 (2000g:05011).

[9] N. J. A. Sloane, On-Line Encyclopedia of Integer Sequences, A000108, A008549 and A136439.

[10] T. Feierl, The height of watermelons with wall, *J. Phys. A* 45 (2012) 095003; arXiv:0802.2691; MR2897031.

[11] M. Katori, M. Izumi and N. Kobayashi, Two Bessel bridges conditioned never to collide, double Dirichlet series, and Jacobi theta function, *J. Stat. Phys.* 131 (2008) 1067–1083; arXiv:0711.1710; MR2407380.

[12] N. J. A. Sloane, On-Line Encyclopedia of Integer Sequences, A005700 and A136440.

[13] C. A. Tracy and H. Widom, Nonintersecting Brownian excursions, *Annals Appl. Probab.* 17 (2007) 953–979; math.PR/0607321; MR2326237.

[14] S. R. Finch, Longest subsequence constants, *Mathematical Constants*, Cambridge Univ. Press, 2003, pp. 382–387.

[15] S. R. Finch, Hammersley's path process, unpublished note (2004).

[16] N. J. A. Sloane, On-Line Encyclopedia of Integer Sequences, A000984 and A000891.

[17] T. Feierl, The height and range of watermelons without wall, *European J. Combin.* 34 (2013) 138–154; arXiv:0806.0037; MR2974277.