Riemann Zeta Moments

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The behavior of the Riemann zeta function $\zeta(z)$ on the critical line $\operatorname{Re}(z) = 1/2$ has been studied intensively for nearly 150 years. We start with a well-known asymptotic formula [1, 2, 3, 4, 5, 6]:

$$\int_0^T |\zeta(1/2 + i\,t)|^2 \, dt \sim (\ln(T) + c)\, T$$

as $T \to \infty$, where $c = 2\gamma - 1 - \ln(2\pi)$ and γ is the Euler-Mascheroni constant [7]. This is often rewritten as

$$\frac{1}{T}\int_0^T |\zeta(1/2 + i\,t)|^2 \, dt \sim \int_0^T P_1\left(\ln(\tfrac{t}{2\pi})\right) dt$$

where $P_1(x) = x + 2\gamma$ is a polynomial of degree 1. More generally,

$$\frac{1}{T}\int_0^T |\zeta(1/2 + i\,t)|^{2k} \, dt \sim \int_0^T P_k\left(\ln(\tfrac{t}{2\pi})\right) dt$$

where $P_k(x)$ is a polynomial of degree k^2. We are interested in the coefficients of $P_2(x)$, $P_3(x)$ and $P_4(x)$, but shall first assess the error term associated with $P_1(x)$. Observe that all moments examined here are of even order; the asymptotics of odd moments remain undiscovered [8].

0.1. Error for $k = 1$. Define

$$E(T) = \int_0^T |\zeta(1/2 + i\,t)|^2 \, dt - (\ln(T) + c)\, T.$$

Analogous to [9], we have a conjecture:

$$E(T) = O(T^{1/4+\varepsilon})$$

which is supported by the mean-square result [10, 11]:

$$\int_2^T E(t)^2 dt \sim C_2 T^{3/2}$$

where

$$C_2 = \frac{2}{3\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{d(n)^2}{n^{3/2}} = \frac{2\zeta(3/2)^4}{3\sqrt{2\pi}\zeta(3)}$$

and $d(n)$ is the number of divisors of n. Further supporting evidence includes [12, 13, 14, 15, 16, 17]

$$\int_2^T E(t)^m dt \sim C_m T^{1+m/4}$$

where

$$C_3 = \frac{6}{7(2\pi)^{3/4}} \sum_{\sqrt{n_1}+\sqrt{n_2}=\sqrt{n_3}} \frac{d(n_1)d(n_2)d(n_3)}{(n_1 n_2 n_3)^{3/4}},$$

$$C_4 = \frac{3}{8\pi} \sum_{\sqrt{n_1}+\sqrt{n_2}=\sqrt{n_3}+\sqrt{n_4}} \frac{d(n_1)d(n_2)d(n_3)d(n_4)}{(n_1 n_2 n_3 n_4)^{3/4}},$$

$$\begin{aligned} C_5 \;=\; & \frac{10}{9(2\pi)^{5/4}} \sum_{\sqrt{n_1}+\sqrt{n_2}+\sqrt{n_3}=\sqrt{n_4}+\sqrt{n_5}} \frac{d(n_1)d(n_2)d(n_3)d(n_4)d(n_5)}{(n_1 n_2 n_3 n_4 n_5)^{3/4}} \\ & - \frac{5}{9(2\pi)^{5/4}} \sum_{\sqrt{n_1}+\sqrt{n_2}+\sqrt{n_3}+\sqrt{n_4}=\sqrt{n_5}} \frac{d(n_1)d(n_2)d(n_3)d(n_4)d(n_5)}{(n_1 n_2 n_3 n_4 n_5)^{3/4}}. \end{aligned}$$

Numerical evaluation of such constants would be very challenging!

0.2. Coefficients for $k \geq 2$. Define

$$F(a,b,c,z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}$$

to be the **Gauss hypergeometric function** (often denoted by ${}_2F_1$). The leading coefficient $c_{k,0}$ of

$$P_k(x) = c_{k,0}x^{k^2} + c_{k,1}x^{k^2-1} + \cdots + c_{k,k^2-1}x + c_{k,k^2}$$

is conjectured to be [18]

$$c_{k,0} = \prod_p \left(\left(1 - \frac{1}{p}\right)^{k^2} F(k,k,1,1/p) \right) \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}.$$

This is provably true for the cases

$$c_{1,0} = 1, \qquad c_{2,0} = \frac{1}{12} \prod_p \left(1 - \frac{1}{p^2}\right) = \frac{1}{2\pi^2} = 0.0506605918....$$

Beyond these, the cases

$$c_{3,0} = \frac{1}{8640} \prod_p \left(1 - \frac{1}{p}\right)^4 \left(1 + \frac{4}{p} + \frac{1}{p^2}\right) = (5.708527...) \times 10^{-6}$$

$$c_{4,0} = \frac{1}{870912000} \prod_p \left(1 - \frac{1}{p}\right)^9 \left(1 + \frac{9}{p} + \frac{9}{p^2} + \frac{1}{p^3}\right) = (2.465018...) \times 10^{-13}$$

are conjectural only. For convenience, let

$$A(k) = \gamma + \sum_p \left[\frac{1}{p-1} - \frac{F(k+1,k+1,2,1/p)}{p\,F(k,k,1,1/p)} \right] \ln(p),$$

$$\begin{aligned} B(k) \;=\; & \sum_p \left[\frac{p}{(p-1)^2} + 2k^2 \frac{F(k+1,k+1,2,1/p)^2}{p^2 F(k,k,1,1/p)^2} \right. \\ & \left. - \frac{k(k+1)}{2} \frac{F(k+2,k+2,3,1/p)}{p^2 F(k,k,1,1/p)} - \frac{F(k+1,k+1,1,1/p)}{p\,F(k,k,1,1/p)} \right] \ln(p)^2. \end{aligned}$$

The next coefficient $c_{k,1}$ is conjectured to be

$$c_{k,1} = 2 c_{k,0} k^3 A(k)$$

which is provably true for $c_{1,1} = 2\gamma = 1.1544313298....$ Beyond this,

$$\begin{aligned} c_{2,1} &= \frac{8}{\pi^2} \left(\gamma + \frac{1}{2} \sum_p \frac{\ln(p)}{p^2 - 1} \right) \\ &= \frac{8}{\pi^4} \left(\gamma \pi^2 - 3\zeta'(2) \right) = 0.6988698848..., \end{aligned}$$

$$\begin{aligned} c_{3,1} &= 54c_{3,0}\left(\gamma+\frac{2}{3}\sum_p \frac{(3p+1)\ln(p)}{(p-1)(p^2+4p+1)}\right) \\ &= 0.0004050213... \end{aligned}$$

are conjectural only. The next coefficient

$$c_{k,2} = c_{k,0}k^2(k^2-1)\left(2k^2A(k)^2 - B(k) - \gamma^2 - 2\gamma_1\right)$$

gives rise to [18, 19]

$$\begin{aligned} c_{2,2} &= \frac{6}{\pi^2}\left(\frac{8}{\pi^4}(\gamma\pi^2-3\zeta'(2))^2 - 2\sum_p \frac{p^2\ln(p)^2}{(p^2-1)^2} - \gamma^2 - 2\gamma_1\right) \\ &= \frac{6}{\pi^6}\left(-48\gamma\zeta'(2)\pi^2 - 12\zeta''(2)\pi^2 + 7\gamma^2\pi^4 + 144\zeta'(2)^2 - 2\,\gamma_1\pi^4\right) \\ &= 2.4259621988..., \end{aligned}$$

$$\begin{aligned} c_{3,2} &= 72c_{3,0}\left(18A(3)^2 - \sum_p \frac{p^2(7p^2+12p+7)\ln(p)^2}{(p-1)^2(p^2+4p+1)^2} - \gamma^2 - 2\gamma_1\right) \\ &= 0.0110724552... \end{aligned}$$

where γ_m is the m^{th} Stieltjes constant [20] (for example, $\gamma_1 = -0.0728158454...$). Such values are conjectural, as well as [19]

$$\begin{aligned} c_{2,3} &= \frac{12}{\pi^8}\left(6\gamma^3\pi^6 - 84\gamma^2\zeta'(2)\pi^4 + 24\gamma_1\zeta'(2)\pi^4 - 1728\zeta'(2)^3 + 576\gamma\zeta'(2)^2\pi^2\right. \\ &\quad \left. +288\zeta'(2)\zeta''(2)\pi^2 - 8\zeta'''(2)\pi^4 - 10\gamma_1\gamma\pi^6 - \gamma_2\pi^6 - 48\gamma\zeta''(2)\pi^4\right) \\ &= 3.2279079649..., \end{aligned}$$

$$\begin{aligned} c_{2,4} &= \frac{4}{\pi^{10}}\left(-12\zeta''''(2)\pi^6 + 36\gamma_2\zeta'(2)\pi^6 + 9\gamma^4\pi^8 + 21\gamma_1^2\pi^8 + 432\zeta''(2)^2\pi^4\right. \\ &\quad +3456\gamma\zeta'(2)\zeta''(2)\pi^4 + 3024\gamma^2\zeta'(2)^2\pi^4 - 36\gamma^2\gamma_1\pi^8 - 252\gamma^2\zeta''(2)\pi^6 \\ &\quad +3\gamma\gamma_2\pi^8 + 72\gamma_1\zeta''(2)\pi^6 + 360\gamma_1\gamma\zeta'(2)\pi^6 - 216\gamma^3\zeta'(2)\pi^6 \\ &\quad -864\gamma_1\zeta'(2)^2\pi^4 + 5\gamma_3\pi^8 + 576\zeta'(2)\zeta'''(2)\pi^4 - 20736\gamma\zeta'(2)^3\pi^2 \\ &\quad \left. -15552\zeta''(2)\zeta'(2)^2\pi^2 - 96\gamma\zeta'''(2)\pi^6 + 62208\zeta'(2)^4\right) \\ &= 1.3124243859..., \end{aligned}$$

$$c_{3,3} = 0.1484007308..., \qquad c_{3,4} = 1.0459251779...,$$

$$c_{3,5} = 3.9843850948..., \qquad c_{3,6} = 8.6073191457...,$$
$$c_{3,7} = 10.2743308307..., \qquad c_{3,8} = 6.5939130206...,$$
$$c_{3,9} = 0.9165155076....$$

Why are such calculations important? Since the conjectures originate in random matrix theory and appear to agree with empirical evaluations of the zeta moments, it would follow that RMT acts as a "model" for arithmetical L-function value distributions.

0.3. Additive Divisor Problems. Estermann [21, 22, 23, 24] solved the following binary additive divisor problem:

$$\sum_{n\leq N} d_2(n)d_2(n+1) \sim \frac{6}{\pi^2}N\ln(N)^2 + \alpha\, N\ln(N) + \beta\, N$$

where $d_\ell(n)$ is the number of sequences x_1, x_2, ..., x_ℓ of positive integers such that $n = x_1 x_2 \cdots x_\ell$, and

$$\alpha = \frac{12}{\pi^4}\left(\pi^2(2\gamma-1) - 12\zeta'(2)\right) = 1.5737449203...,$$

$$\begin{aligned}\beta &= \frac{6}{\pi^6}\left(\pi^4\left[(2\gamma-1)^2+1\right] - 24\pi^2(2\gamma-1)\zeta'(2) + 288\zeta'(2)^2 - 24\pi^2\zeta''(2)\right)\\ &= -0.5243838319....\end{aligned}$$

For $\ell \geq 3$, it is conjectured that [25, 26, 27]

$$\sum_{n\leq N} d_\ell(n)d_\ell(n+1) \sim N\, Q_\ell(\ln(N))$$

where $Q_\ell(x)$ is a polynomial of degree $2(\ell-1)$, but even the leading coefficient of $Q_3(x)$ is not known. Describing the connection between ternary additive divisors as such and the sixth moment of $\zeta(1/2 + i\,t)$ would take us too far afield.

Another conjecture is [28]

$$\sum_{n\leq N} d_2(n-1)d_2(n)d_2(n+1) \sim \frac{11}{8}\kappa\, N\, \ln(N)^3$$

where

$$\kappa = \prod_p \left(1-\frac{1}{p}\right)^2\left(1+\frac{2}{p}\right) = 0.2867474284...$$

is the strongly carefree constant [29]. Discussion of generalizations and supporting evidence again would take us too far afield.

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