# Zero Crossings

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May 19, 2004

In this essay, we presuppose basic knowledge of stochastic processes [1]. Let $\{X_t : t \geq 0\}$ be a zero mean, unit variance, stationary Gaussian process with twice differentiable correlation function $r(|s-t|) = \mathrm{Cov}(X_s, X_t)$. We wish to study the distribution of lengths of intervals between zeroes of $X_t$. There are two cases: the first in which $r(\tau)$ is analytic (implying differentiability up to all orders) and the second in which the third derivative of $r(\tau)$ possesses a jump discontinuity at $\tau = 0$.

Define $f_m(\tau)$ to be the probability density associated with the interval length $\tau$ between an arbitrary zero $t_0$ and the $(m+1)^{\text{st}}$ later zero $t_{m+1}$. In particular, $f_0(\tau)$ is the probability density of differences between successive zeroes $t_0$ and $t_1$. We will focus on the limiting behavior of $f_m(\tau)$ as $\tau \to 0^+$.

When $r(\tau)$ is analytic, it is clear that

$$r(\tau) = 1 + \frac{r''(0)}{2!}\tau^2 + \frac{r^{(4)}(0)}{4!}\tau^4 + O(\tau^6)$$

since $r(\tau)$ must be an even function. It is known, in this case, that [2]

$$f_m(\tau) = O\left(\tau^{\frac{1}{2}(m+2)(m+3)-2}\right)$$

as $\tau \to 0^+$. Further, the big $O$ coefficient is known. We merely give an example: If $r(\tau) = \exp(-\alpha\tau^2)$ for $\alpha > 0$, then

$$\lim_{\tau\to 0^+} \frac{f_0(\tau)}{\tau} = \frac{1}{2}\alpha, \qquad \lim_{\tau\to 0^+} \frac{f_1(\tau)}{\tau^4} = \frac{\sqrt{6}}{27\pi}\alpha^{5/2}.$$

The more interesting case is when $r(\tau)$ has a singularity at the origin. If

$$r(\tau) = 1 - \frac{1}{2}\tau^2 + \alpha|\tau|^3 + o(|\tau|^3),$$

then $f_m(\tau) \to C_m\alpha$ as $\tau \to 0^+$, where $C_m > 0$ is a constant (independent of $\alpha$). Longuet-Higgins [3] determined the following bounds

$$1.1556 < C_0 < 1.158, \qquad 0.1971 < C_1 < 0.198, \qquad 0.0491 < C_2 < 0.0556,$$

but it remained for someone else to find a specific process $\{X_t\}$, and its corresponding $\alpha$, for which $f_m(\tau)$ could be computed.

Wong [4, 5, 6, 7], building upon McKean [8], examined the process

$$X_t = \sqrt{3}\exp\left(-\sqrt{3}t\right)\int\limits_0^{\exp\left(2t/\sqrt{3}\right)} W_s\,ds$$

where $W_s$ is standard Brownian motion ("standard" meaning that its variance parameter is 1). The correlation function for Wong's process is

$$r(\tau) = \frac{3}{2}\exp\left(-\frac{|\tau|}{\sqrt{3}}\right)\left(1-\frac{1}{3}\exp\left(-\frac{2|\tau|}{\sqrt{3}}\right)\right)$$

and hence $\alpha = 2\sqrt{3}/9$. It turns out that $f_0(\tau)$ can be written in terms of complete elliptic integrals, and a more complicated integral expression applies for $f_m(\tau)$, $m \geq 1$. This is sufficient to deduce that

$$C_0 = \frac{37}{32} = 1.15625, \qquad C_1 = \frac{47}{64} - \frac{108}{64\pi} = 0.1972270670...,$$

$$C_2 = \frac{121}{128} - \frac{81}{32\pi} - \frac{27}{32\pi^2} = 0.0541008518....$$

In fact,

$$C_m = \frac{27}{4\pi^2}\int\limits_0^\infty \frac{x^3-1}{x^3+1}\frac{x^m\ln(x)}{(x^2+1)^{m+1}}dx,$$

which can be evaluated exactly via residue calculus. The limiting behavior of $f_m(\tau)$ as $\tau \to 0^+$ is thus solved for all $m$. No one has found another stationary Gaussian process that permits exact analysis as this. Wong [4] also proved that $f_0(\tau) \to 0$ as $\tau \to \infty$ and, moreover,

$$\lim_{\tau\to\infty}\exp\left(\frac{\tau}{2\sqrt{3}}\right)f_0(\tau) = \frac{L}{\sqrt{2}} = K\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{4\sqrt{\pi}}\Gamma\left(\frac{1}{4}\right)^2 = 1.8540746773...$$

where $L$ is Gauss' lemniscate constant [9] and $K(x)$ denotes the complete elliptic integral of the first kind. For $m \geq 1$, such precise asymptotics for $f_m(\tau)$ as $\tau \to \infty$ remain open. See [10, 11, 12, 13, 14] as well.

We shift attention to counting zeroes in an interval of prescribed length 1. Again, $\{X_t\}$ is assumed to be a zero mean, unit variance, stationary Gaussian process with twice differentiable correlation function $r(\tau)$. Let $N$ denote the number of zeroes of $X_t$ per unit time. The expected value of $N$ is [15, 16, 17, 18]

$$\mathrm{E}(N) = \frac{1}{\pi}\sqrt{-r''(0)}$$

and the variance of $N$ is [19, 20, 21, 22, 23, 24, 25]

$$\mathrm{Var}(N) = \mathrm{E}(N) - \mathrm{E}(N)^2 + \frac{2}{\pi^2}\int\limits_0^1 (1-\tau)F(\tau)\,d\tau$$

where

$$F(\tau) = \left(1 - r(\tau)^2\right)^{-1} G(\tau)\left(1 + H(\tau)\arctan(H(\tau))\right),$$

$$G(\tau) = \sqrt{k_1(\tau)k_2(\tau)}, \qquad H(\tau) = \frac{k_3(\tau)}{\sqrt{\left(1 - r(\tau)^2\right)k_1(\tau)k_2(\tau)}},$$

$$k_1(\tau) = (1 + r(\tau))\left(r''(0) - r''(\tau)\right) + r'(\tau)^2, \qquad k_2(\tau) = (1 - r(\tau))\left(r''(0) + r''(\tau)\right) + r'(\tau)^2,$$

$$k_3(\tau) = \left(1 - r(\tau)^2\right)r''(\tau) + r(\tau)r'(\tau)^2.$$

Needless to say, an exact evaluation of $\mathrm{Var}(N)$ is generally impossible. In the case when $r(\tau)$ is analytic, we have [26]

$$\lim_{\tau\to 0^+}\frac{2}{\pi}\left(\frac{1}{H(\tau)} + \arctan\left(H(\tau)\right)\right) = 1.$$

By contrast, in the case when $r(\tau)$ has a singularity at the origin (as before),

$$\lim_{\tau\to 0^+}\frac{2}{\pi}\left(\frac{1}{H(\tau)} + \arctan\left(H(\tau)\right)\right) = \frac{2\sqrt{3}}{\pi} + \frac{1}{3} = 1.4359911241...$$

which is an interesting (coincidental?) occurrence of the first Lebesgue constant [27]. For Wong's process, $\mathrm{E}(N) = 1/\pi$ and [25]

$$\mathrm{Var}(N) = \frac{4}{3\pi} - \frac{1}{12} + \frac{3}{\pi^2}\left\{\arcsin\left(\frac{1}{2}\exp\left(-\frac{1}{\sqrt{3}}\right)\right)\right\}^2.$$

No other stationary Gaussian process is known to possess a closed-form expression for this variance. See also [28, 29, 30, 31, 32, 33, 34].

**0.1. Integrated Brownian Motion.** Wong's process involves an integral of standard Brownian motion. We briefly examine a simpler integral [35]:

$$Z_t = \int_0^t W_s \, ds,$$

which is zero mean Gaussian with covariance function

$$\operatorname{Cov}(Z_u, Z_v) = \int_0^u \int_0^v \min\{x, y\} \, dx \, dy = \begin{cases} \frac{1}{6}u^2(3v - u) & \text{if } v \geq u \geq 0 \\ \frac{1}{6}v^2(3u - v) & \text{if } u \geq v \geq 0. \end{cases}$$

One unsolved problem is concerned with the asymptotics of the maximum of $|Z_t|$ over the unit interval [36, 37, 38, 39]:

$$\lim_{\varepsilon \to 0^+} \varepsilon^{2/3} \ln \left\{ \mathrm{P}\left( \max_{0 \leq t \leq 1} |Z_t| < \varepsilon \right) \right\} = \kappa,$$

where the constant $\kappa$ is known to satisfy

$$\frac{3}{8} \leq \kappa \leq (2\pi)^{2/3} \frac{3}{8}.$$

These are the sharpest known bounds. Another unsolved problem is concerned with the probability that the integrated Wiener process is currently at its maximum value [40, 41]:

$$\lambda = \mathrm{P}\left( Z_t = \max_{0 \leq s \leq t} Z_s \right),$$

which is known to be independent of $t$. Since integration has the effect of smoothing $W_s$, it is reasonable to conjecture for $Z_t$ that $\lambda$ is positive. Two terms of a complicated infinite series were used in [40] to give an approximation $\lambda = 0.372...$, but a more accurate estimation procedure apparently has not been attempted.

**0.2. Random Polynomials.** Let $q(x)$ be a random polynomial of degree $n$, with real coefficients independently chosen from a standard Gaussian distribution. Asymptotic properties of the expected number of real zeroes of $q(x)$ were summarized in [42]; associated probabilities are more difficult to study. The probability that $q(x)$ does *not* have any zeroes in $\mathbb{R}$ is $n^{-b+o(1)}$ as $n \to \infty$ through even integers, where [43]

$$b = -4 \lim_{T \to \infty} \frac{1}{T} \ln \left( \mathrm{P}\left( \sup_{0 \leq t \leq T} Y(t) \leq 0 \right) \right)$$

and $Y(t)$ is a zero mean, unit variance, stationary Gaussian process with correlation function $r(\tau) = \operatorname{sech}(\tau/2)$. It is known [44, 45, 46] that $0.5 < b < 1.0$ and, via simulation, $b \approx 0.76$. An exact value for $b$ would be sensational! The statistics of real zeroes of $q(x)$ turn out to be identical in the four subintervals $(-\infty, -1)$, $[-1, 0]$, $[0, 1]$, $(1, \infty)$ of $\mathbb{R}$; hence the probability that $q(x)$ does not have zeroes in $[0, 1]$ is $n^{-b/4+o(1)} \approx n^{-0.19}$ [47, 48]. A related topic is the capture time in the random pursuit problem for fractional Brownian particles [44, 45, 46].

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