Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division

The Analysis of Power Series

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division

1. Now that we know that every analytic function is locally equal to a power series, we want to know more about power series themselves.

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division

- 1. Now that we know that every analytic function is locally equal to a power series, we want to know more about power series themselves.
- 2. Whereas so far we had a function and found from it the power series, now we start with a power series and want to know the properties of the function it defines.

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- 1. Now that we know that every analytic function is locally equal to a power series, we want to know more about power series themselves.
- 2. Whereas so far we had a function and found from it the power series, now we start with a power series and want to know the properties of the function it defines.
- 3. It turns out that functions defined by power series are very well behaved.

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- 1. Now that we know that every analytic function is locally equal to a power series, we want to know more about power series themselves.
- 2. Whereas so far we had a function and found from it the power series, now we start with a power series and want to know the properties of the function it defines.
- 3. It turns out that functions defined by power series are very well behaved.
- 4. But to prove the requisite results, we first must more closely investigate the convergence of power series.

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Theorem.

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converges.

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Proof.

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ConvergenceContinuityDifferentiabilityUniquenessMultiplication and Division**Proof.** Because
$$\sum_{n=0}^{\infty} a_n (z_1 - z_0)^n$$
 converges, we know that $|a_n (z_1 - z_0)^n|$ converges to zero.

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Convergence Continuity Differentiability Uniqueness Multiplication and Division **Proof.** Because $\sum_{n=0}^{\infty} a_n (z_1 - z_0)^n$ converges, we know that $|a_n (z_1 - z_0)^n|$ converges to zero. In particular, there is an M > 0 so that $|a_n (z_1 - z_0)^n| < M$ for all n.

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that $|a_n (z_1 - z_0)^n| < M$ for all n. Therefore, with $q := \left| \frac{z - z_0}{z_1 - z_0} \right| < 1$ we have

$$\sum_{n=N+1}^{\infty} \left| a_n (z-z_0)^n \right|$$

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Convergence Continuity Differentiability Uniqueness Multiplication and Division $\begin{aligned}
\mathbf{Proof.} & \text{Because } \sum_{n=0}^{\infty} a_n (z_1 - z_0)^n \text{ converges, we know that} \\
\left| a_n (z_1 - z_0)^n \right| \text{ converges to zero. In particular, there is an } M > 0 \text{ so} \\
\text{that } \left| a_n (z_1 - z_0)^n \right| < M \text{ for all } n. \text{ Therefore, with } q := \left| \frac{z - z_0}{z_1 - z_0} \right| < 1 \\
\text{we have}
\end{aligned}$

$$\sum_{n=N+1}^{\infty} \left| a_n (z-z_0)^n \right| = \sum_{n=N+1}^{\infty} \left| a_n (z_1-z_0)^n \right| \left| \frac{z-z_0}{z_1-z_0} \right|^n$$

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 Proof. Because $\sum_{n=0}^{\infty} a_n (z_1 - z_0)^n$ converges, we know that $|a_n (z_1 - z_0)^n|$ converges to zero. In particular, there is an M > 0 so that $|a_n (z_1 - z_0)^n| < M$ for all n. Therefore, with $q := \left| \frac{z - z_0}{z_1 - z_0} \right| < 1$ we have

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$$\leq M \sum_{n=N+1}^{\infty} q^n$$

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$$\leq M \sum_{n=N+1}^{\infty} q^n = M \frac{q^{N+1}}{1-q}$$

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Observat	ions From 7	The Theorem series $\sum_{n=0}^{\infty} a_n(z-z)$	$_{0})^{n}$, there must	be a number <i>c</i>
so	that the power	series converges	for $ z - z_0 < c$	and diverges
for	$ z-z_0 >c.$			

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Observat	ions From	The Theorem		

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Observations From The Theorem				
Obber value				

 $\Im(z) \blacktriangle$

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 $\Re(z)$

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Observa	tions From '	The Theorem		

 $\Im(z)$

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 $\Re(z)$

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Observatio	ons From	The Theorem		



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The Analysis of Power Series

3(z)▲

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Observati	ions From	The Theorem $_{\infty}$		



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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Observati	ons From	The Theorem $_{\infty}$		



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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division

1. For every power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, there must be a number c so that the power series converges for $|z-z_0| < c$ and diverges for $|z-z_0| > c$. Indeed, if it converges at z_1 , then it must converge on the disk $|z-z_0| < |z_1-z_0|$, which rules out any other shape for the region of convergence.



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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division

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- 2. The number c is called the **radius of convergence**

Convergence Continuity Differentiability Uniqueness Multiplication and Division

Observations From The Theorem

- 1. For every power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$, there must be a number *c* so that the power series converges for $|z-z_0| < c$ and diverges for $|z-z_0| > c$. Indeed, if it converges at z_1 , then it must converge on the disk $|z-z_0| < |z_1-z_0|$, which rules out any
 - other shape for the region of convergence.
- 2. The number *c* is called the **radius of convergence**, and the circle $|z z_0| = c$ is called the **circle of convergence**.
Convergence Continuity Differentiability Uniqueness Multiplication and Division

Observations From The Theorem

1. For every power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$, there must be a number *c* so that the power series converges for $|z-z_0| < c$ and diverges

for $|z - z_0| > c$. Indeed, if it converges at z_1 , then it must converge on the disk $|z - z_0| < |z_1 - z_0|$, which rules out any other shape for the region of convergence.

- 2. The number *c* is called the **radius of convergence**, and the circle $|z z_0| = c$ is called the **circle of convergence**.
- 3. There are no general theorems about what happens when $|z z_0| = c$.

Convergence Continuity Differentiability Uniqueness Multiplication and Division

Observations From The Theorem

1. For every power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$, there must be a number *c*

so that the power series converges for $|z - z_0| < c$ and diverges for $|z - z_0| > c$. Indeed, if it converges at z_1 , then it must converge on the disk $|z - z_0| < |z_1 - z_0|$, which rules out any other shape for the region of convergence.

- 2. The number *c* is called the **radius of convergence**, and the circle $|z z_0| = c$ is called the **circle of convergence**.
- 3. There are no general theorems about what happens when $|z z_0| = c$.
- 4. And problems in the complex plane can influence the behavior of functions on the real line.

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
$f(x) = \frac{1}{1+x}$	$\frac{1}{x^2} = \sum_{n=0}^{\infty} \left(\frac{1}{x^2} - \frac{1}{x^2} \right)$	$\left(-x^2\right)^n$ Has R	adius of Co	onvergence 1

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
$f(x) = \frac{1}{1+x}$	$\frac{1}{x^2} = \sum_{n=0}^{\infty} \left(\frac{1}{x^2} - \frac{1}{x^2} \right)$	$\left(-x^2\right)^n$ Has R	adius of Co	onvergence 1

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
f(x) =	$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}$	$\left(-x^2\right)^n$ Has	Radius of (Convergence 1
		𝔅(z) ▲		
		1		
		i I		
		1		
		1		
		۱ ــــــــــــــــــــــــــــــــــــ		►
		1		$\Re(z)$
		1		
		l I		
		1		
		1		





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Observations From The Proof

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Observat 1. N re	tions From ote that in the provided that in the provided that $\sum_{n=N+1}^{\infty}$	The Proof preceding proof the $ a_n(z-z_0)^n $ dependence	e upper bound ends only on the	for the e distance from
z^{\dagger}	to z_0 .			

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Observ	ations From '	The Proof		
1.	Note that in the p_{∞}	preceding proof th	e upper bound	for the
	remainder $\sum_{n=N+1}^{\infty}$	$ a_n(z-z_0)^n $ depe	nds only on the	e distance from
	z to z_0 . That's be	cause $q = \left \frac{z - z_0}{z_1 - z_0} \right $	-	

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Observa 1 2	ations From 7 Note that in the p remainder $\sum_{n=N+1}^{\infty}$ z to z ₀ . That's bea In particular, if 0	The Proof preceding proof the $ a_n(z-z_0)^n $ dependence cause $q = \left \frac{z-z_0}{z_1-z_0} + \frac{z-z_0}{z_1-z_0} \right $	e upper bound ands only on the $-\frac{1}{2}$.	for the e distance from be bounded
1	uniformly for all	z so that $ z-z_0 \leq$	$\leq r$.	

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Observ	ations From	The Proof		
1.	Note that in the p	preceding proof th	e upper bound f	for the
	remainder $\sum_{n=N+1}^{\infty}$	$ a_n(z-z_0)^n $ depe	nds only on the	distance from
	z to z_0 . That's be	cause $q = \left \frac{z - z_0}{z_1 - z_0} \right $	-	
2.	In particular, if 0	< r < R, then the	remainder can	be bounded
	uniformly for all	z so that $ z - z_0 \leq z_0$	$\leq r$.	

3. That, in turn, means that on such sub-disks $|z - z_0| \le r$ there is a uniform minimum speed of convergence.

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Definition.

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Theorem.

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Definition. The power series
$$\sum_{n=0}^{\infty} a_n(z-z_0)^n$$
 is said to **converge**
uniformly in the region R if and only if for every $\varepsilon > 0$ there is a
natural number N_{ε} so that for all $N > N_{\varepsilon}$ and for all z in the region R
we have $\left|\sum_{n=N}^{\infty} a_n(z-z_0)^n\right| < \varepsilon$.
Theorem. Suppose the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ has circle of
convergence $|z-z_0| = R$ and let $R_1 < R$.

Convergence Continuity Differentiability Uniqueness Multiplication and Division
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Theorem. Suppose the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ has circle of
convergence $|z-z_0| = R$ and let $R_1 < R$. Then the power series
converges uniformly for all z with $|z-z_0| < R_1$.
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Proof.

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division

Proof. Let z_R be so that $R_1 < |z_R - z_0| < R$

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof. I	Let z_R be so th	at $R_1 < z_R - z_0 <$	< R and so that	$\sum_{n=0}^{\infty} a_n (z_R - z_0)^n$
converg	es. Then $ a_n $	$z_R - z_0)^n $ converg	es to zero. In pa	rticular, there
is an M	> 0 so that $ a $	$ _n(z_R-z_0)^n < M$	for all <i>n</i> . Theref	ore, with
$q := \frac{1}{ z_R }$	$\frac{R_1}{ z-z_0 }$			

Controlgence	continuity	Differentiability	omqueness	interpretation and Division
Proof. I convergis an <i>M</i>	Let z_R be so the es. Then $ a_n(z) > 0$ so that $ a $	at $R_1 < z_R - z_0 $ $ z_R - z_0 ^n $ convergence $ z_R - z_0 ^n < M$	< <i>R</i> and so that es to zero. In pa for all <i>n</i> . Theret	$\sum_{n=0}^{\infty} a_n (z_R - z_0)^n$ articular, there fore, with
$q := \frac{1}{ z_R }$	$\frac{R_1}{ z_0 } > \left \frac{z-z_0}{ z_R-z_0 }\right $	$\left \frac{z_0}{z_0} \right $		

Differentiabilit

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Convergence

Multiplication and Division

Convergence Continuity Differentiability Uniqueness Multiplication and Division **Proof.** Let z_R be so that $R_1 < |z_R - z_0| < R$ and so that $\sum_{n=0}^{\infty} a_n (z_R - z_0)^n$ converges. Then $|a_n (z_R - z_0)^n|$ converges to zero. In particular, there is an M > 0 so that $|a_n (z_R - z_0)^n| < M$ for all n. Therefore, with $q := \frac{R_1}{|z_R - z_0|} > \left| \frac{z - z_0}{z_R - z_0} \right|$ (note that q < 1)

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof. L	Let z_R be so the	at $R_1 < z_R - z_0 <$	<i>R</i> and so that	$\sum_{n=0}^{\infty} a_n (z_R - z_0)^n$
converge	es. Then $ a_n $	$ z_R - z_0)^n $ converge	es to zero. In pa	articular, there
is an M	> 0 so that $ a $	$\left (z_R - z_0)^n \right < M$ f	for all <i>n</i> . There	fore, with
$q := \frac{1}{ z_R }$ $ z - z_0 < 0$	$\frac{R_1}{ z_R } > \left \frac{z}{ z_R } - \frac{z}{ z_R } \right $	$\left \frac{z_0}{z_0} \right $ (note that $q <$	(1) we have for	r all

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof. I converge is an M $q := \frac{1}{ z_R }$ $ z - z_0 < \sum_{n=N+1}^{\infty}$	Let z_R be so that es. Then $ a_n(z) > 0$ so that $ a_n(z) = \frac{R_1}{ z_R } > \left \frac{z}{ z_R } - \frac{z}{ z_R } - \frac{z}{ z_R } \right = \frac{z}{ z_R }$	at $R_1 < z_R - z_0 < z_R - z_0 < z_R - z_0 $ $ z_R - z_0 ^n $ convergence $ z_R - z_0 ^n < M = z_0 $ $ z_0 $ (note that $q < z_0 $	< <i>R</i> and so that es to zero. In pa for all <i>n</i> . There < 1) we have fo	$\sum_{n=0}^{\infty} a_n (z_R - z_0)^n$ articular, there fore, with r <i>all</i>
n = n + 1				

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof	Let z_R be so the	at $R_1 < z_R - z_0 $	< <i>R</i> and so that	$\sum_{n=0}^{\infty} a_n (z_R - z_0)^n$
conve	rges. Then $ a_n(z) $	$ _{R}-z_{0})^{n} $ converg	es to zero. In p	articular, there
is an <i>l</i>	$M > 0$ so that $ a_i $	$ (z_R - z_0)^n < M$	for all <i>n</i> . There	fore, with
$q := \frac{1}{ z-z_0 }$	$\frac{R_1}{z_R - z_0 } > \left \frac{z - z_0}{z_R - z_0}\right < R_1$	$\left \frac{z_0}{z_0}\right $ (note that $q < $	< 1) we have fo	r all
$\sum_{n=N}^{\infty}$	$ a_n(z-z_0)^n $	$= \sum_{n=N+1}^{\infty} a_n(z_n) ^2$	$\left \frac{z-z_0}{z_R-z}\right ^n \left \frac{z-z_0}{z_R-z}\right $	$\left \begin{array}{c} 0 \\ 0 \end{array} \right ^n$

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proo	f. Let z_R be so that	at $R_1 < z_R - z_0 $	< R and so that	$\sum_{n=0}^{\infty} a_n (z_R - z_0)^n$
conv	erges. Then $ a_n(z) $	$ _{R} - z_0)^n $ converge	ges to zero. In pa	articular, there
is an	$M > 0$ so that $ a_n $	$ (z_R - z_0)^n < M$	for all <i>n</i> . There	fore, with
q :=	$\frac{R_1}{ z_R - z_0 } > \left \frac{z - z}{z_R - z_0} \right $	$\left \frac{z_0}{z_0} \right $ (note that q -	< 1) we have fo	r all
$\sum_{n=N}^{\infty}$	$\sum_{N+1}^{\infty} a_n(z-z_0)^n $	$= \sum_{n=N+1}^{\infty} a_n(z) ^2$	$ _{R}-z_{0})^{n} \left \frac{z-z_{0}}{z_{R}-z}\right $	$\frac{1}{0} \Big ^n$
		$\leq M \sum_{n=N+1}^{\infty} q^n$		

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof.	Let z_R be so that	$R_1 < z_R - z_0 $	< R and so that	$\sum_{n=0}^{\infty} a_n (z_R - z_0)^n$
converg	ges. Then $ a_n(z_R) $	$(-z_0)^n$ converge	ges to zero. In pa	articular, there
is an M	> 0 so that $ a_n $	$ z_R - z_0)^n < M$	for all <i>n</i> . There	fore, with
$q := \frac{1}{ z }$	$\frac{R_1}{R-z_0} > \left \frac{z-z_0}{z_R-z_0} \right $	$\left \begin{array}{c} 0 \\ 0 \end{array} \right $ (note that q	< 1) we have fo	r all
$ z - z_0 $	$< R_1$			
$\sum_{n=N+1}^{\infty}$	$\left a_n(z-z_0)^n\right =$	$= \sum_{n=N+1}^{\infty} a_n(z) $	$ _{R} - z_0)^n \left \frac{z - z_0}{z_R - z_0} \right $	$\frac{1}{0}$
	-	$\leq M \sum_{n=N+1}^{\infty} q^n$	$= M \frac{q^{N+1}}{1-q}$	

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof. Converg	Let z_R be so that the set $a_n(z)$	at $R_1 < z_R - z_0 <$ $R_R - z_0)^n $ converge	$< R$ and so that $\sum_{n=1}^{\infty} \frac{1}{n}$	$\sum_{n=0}^{\infty} a_n (z_R - z_0)^n$
is an M	> 0 so that $ a_n $	$\frac{(7n-70)^n}{(7n-70)^n} < M$	for all <i>n</i> Therefo	ore with
15 dii 191	P $ -$	$(x \times x \times 0) \mid \langle m \rangle$	or an <i>n</i> . Thereix	sie, with
$q := \frac{1}{ z_K }$	$\frac{ \mathbf{x}_1 }{ \mathbf{z}_R-\mathbf{z}_0 } > \left \frac{ \mathbf{z}_R-\mathbf{z}_R }{ \mathbf{z}_R-\mathbf{z}_R }\right $	$\left \begin{array}{c} \frac{z_0}{z_0} \end{array} \right $ (note that $q < $	(1) we have for	all
$ z - z_0 $	$< R_1$			
$\sum_{n=N+1}^{\infty}$	$\left a_n(z-z_0)^n\right $	$= \sum_{n=N+1}^{\infty} a_n(z_R) ^2$	$\left -z_0\right)^n \left \left \frac{z-z_0}{z_R-z_0} \right \right $	ⁿ
		$\leq M \sum_{n=N+1}^{\infty} q^n =$	$= M \frac{q^{N+1}}{1-q} \to 0$	$(N ightarrow \infty)$

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Pro	bof. Let z_R be so that	$t R_1 < z_R - z_0 <$	< R and so that	$\sum_{n=0}^{\infty} a_n (z_R - z_0)^n$
con	verges. Then $ a_n(z_R) $	$ -z_0)^n$ converg	es to zero. In pa	rticular, there
is a	n $M > 0$ so that $ a_n $	$\left (z_R - z_0)^n\right < M$	for all <i>n</i> . Theref	ore, with
q := z -	$= \frac{R_1}{ z_R - z_0 } > \left \frac{z - z}{z_R - z} \right \\ - z_0 < R_1$	$\left \begin{array}{c} \frac{0}{z_0} \right $ (note that $q < $	< 1) we have for	all
n=	$\sum_{=N+1}^{\infty} \left a_n (z-z_0)^n \right $	$= \sum_{n=N+1}^{\infty} a_n(z_N) = \sum_{n=N+1}^{\infty} a_$	$\left \frac{z-z_0}{z_R-z_0}\right \left \frac{z-z_0}{z_R-z_0}\right $	- ⁿ
		$\leq M \sum_{n=N+1}^{\infty} q^n =$	$= M \frac{q^{N+1}}{1-q} \to 0$	$(N ightarrow \infty)$
and	hence $\sum_{n=0}^{\infty} a_n(z-z) $	$_{0})^{n}$ converges u	niformly for all	$ z-z_0 < R_1.$

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Pro	of. Let z_R be so that	$R_1 < z_R - z_0 $	$< R$ and so that $\sum_{n=1}^{\infty}$	$\int_{0}^{\infty} a_n (z_R - z_0)^n$
conv	verges. Then $ a_n(z_R - a_n) $	$(-z_0)^n$ converge	ges to zero. In part	ticular, there
is ar	$M > 0$ so that $ a_n(z) $	$ z_R - z_0)^n < M$	for all n. Therefor	re, with
q := z - z	$= \frac{R_1}{ z_R - z_0 } > \left \frac{z - z_0}{z_R - z_0} \right $ $z_0 < R_1$	$\left \text{(note that } q \right $	< 1) we have for a	ıll
n=	$\sum_{n=N+1}^{\infty} \left a_n (z-z_0)^n \right =$	$= \sum_{n=N+1}^{\infty} a_n(z) $	$ _{R}-z_{0})^{n} \left \frac{z-z_{0}}{z_{R}-z_{0}}\right $	n
	<	$\leq M \sum_{n=N+1}^{\infty} q^n$	$= M \frac{q^{N+1}}{1-q} \to 0$	$(N \to \infty)$
and	hence $\sum_{n=0}^{\infty} a_n(z-z_0) $	n converges u	niformly for all z	$ z-z_0 < R_1.$

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division

Theorem.

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convergence $|z - z_0| = R$.

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division

Proof (Visualization).

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof ((Visualization).			
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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof (Visualization).	S(z) ▲		
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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof (Visualization).		R	$-\frac{1}{\Re(z)}$

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof (Visualization).	3(z) ▲ • z R		- ¬► ℜ(z)

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof ((Visualization).			► ℜ(z)

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof (Visualization)		R	- - ► ℜ(z)

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof (Visualization)		R	- - ► ℜ(z)

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof.				

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division

Proof. Let *z* be so that $|z - z_0| < R$

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division	
Proof. Let <i>z</i> be so that $ z - z_0 < R$ (we will prove that <i>f</i> is continuous at <i>z</i>)					
Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division	
-------------	--------------------	------------------------------	-----------------	-----------------------------	
Proof.	Let z be so that	$ z - z_0 < R \text{ (we)}$	will prove that	f is continuous	

at z), let $R_1 > 0$ be so that $|z - z_0| < R$ (we will p at z), $|z - R_1| < R$

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof.	Let z be so that	$ z-z_0 < R \text{ (we)}$	will prove that	f is continuous

at z), let $R_1 > 0$ be so that $|z - z_0| < R_1 < R$ and let $\varepsilon > 0$.

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof. at z), let is a nat $\sum_{n=N+1}^{\infty} $	Let z be so that t $R_1 > 0$ be so ural number N $a_n(\tilde{z} - z_0)^n < 0$	t $ z - z_0 < R$ (we that $ z - z_0 < R_1$ so that for all \tilde{z} w $\frac{\varepsilon}{3}$.	will prove that $< R$ and let $\varepsilon >$ with $ \tilde{z} - z_0 < R$	<i>f</i> is continuous 0. Then there 1 we have that

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof.	Let z be so that	$t z - z_0 < R \text{ (we}$	will prove that	f is continuous
at <i>z</i>), le	t $R_1 > 0$ be so	that $ z - z_0 < R_1$	$< R$ and let $\varepsilon >$	• 0. Then there
is a natural number N so that for all \tilde{z} with $ \tilde{z} - z_0 < R_1$ we have that				
$\sum_{n=N+1}^{\infty} a $	$a_n(\tilde{z}-z_0)^n <\frac{1}{2}$	$\frac{\varepsilon}{3}$. Now, because	polynomials are	e continuous
n-n+1				

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof. I at z), let is a natu $\sum_{n=N+1}^{\infty} a $ there is	Let z be so that $R_1 > 0$ be so ural number N $u_n(\tilde{z} - z_0)^n < 2$	t $ z - z_0 < R$ (we that $ z - z_0 < R_1$ so that for all \tilde{z} w $\frac{\varepsilon}{3}$. Now, because	will prove that $< R$ and let $\varepsilon >$ ith $ \tilde{z} - z_0 < R$ polynomials are	<i>f</i> is continuous > 0. Then there ₁ we have that e continuous,
there is	a $\delta > 0$			

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof.	Let z be so that $P \ge 0$ be so	$t z - z_0 < R \text{ (we}$	will prove that $(B) = (B) + $	f is continuous
at z), le	$t R_1 > 0$ be so	that $ z - z_0 < R_1$	$< R$ and let $\varepsilon >$	> 0. Then there
is a natural number N so that for all \tilde{z} with $ \tilde{z} - z_0 < R_1$ we have that				
$\sum_{n=N+1}^{\infty} a $	$a_n(\tilde{z}-z_0)^n <$	$\frac{\varepsilon}{3}$. Now, because	polynomials are	e continuous,
there is	a $\delta > 0$ so that	at $\delta < R_1 - z - z_0 $		

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof.	Let z be so that	$t z - z_0 < R $ (we	will prove that	f is continuous
at <i>z</i>), le	t $R_1 > 0$ be so	that $ z - z_0 < R_1$	$< R$ and let ε >	> 0. Then there
is a nat	ural number N	so that for all \tilde{z} w	with $ \tilde{z} - z_0 < R$	1_1 we have that
$\sum_{n=N+1}^{\infty} a_n(\tilde{z}-z_0)^n < \frac{\varepsilon}{3}.$ Now, because polynomials are continuous,				
there is	a $\delta > 0$ so that	$\text{ tt } \delta < R_1 - z - z_0 $	and so that fo	r all \tilde{z} with
$ \tilde{z}-z <$	$<\delta$ we have $\left \sum_{n=1}^{\infty} \right _{n}$	$\sum_{n=0}^{N} a_n (z - z_0)^n - \sum_{n=0}^{N} a_n (z - z_0)^n - \sum_{n$	$\sum_{n=0}^{N} a_n (\tilde{z} - z_0)^n \left \right \leq \frac{1}{2}$	$< \frac{\varepsilon}{3}.$

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof.	Let z be so that	$t z - z_0 < R $ (we	will prove that	f is continuous
at <i>z</i>), le	et $R_1 > 0$ be so	that $ z - z_0 < R_1$	$< R$ and let ε >	> 0. Then there
is a nat	ural number N	so that for all \tilde{z} w	with $ \tilde{z} - z_0 < R$	R_1 we have that
$\sum_{n=N+1}^{\infty} $	$a_n(\tilde{z}-z_0)^n <$	$\frac{\varepsilon}{3}$. Now, because	polynomials ar	e continuous,
there is	a $\delta > 0$ so that	$t \delta < R_1 - z - z_0 $	and so that fo	or all \tilde{z} with
$ \tilde{z}-z $	$<\delta$ we have \int_{n}^{∞}	$\sum_{n=0}^{N} a_n (z - z_0)^n - \sum_{n=0}^{N} a_n (z - z_0)^n - \sum_{n$	$\sum_{n=0}^{N} a_n (\tilde{z} - z_0)^n \bigg \cdot$	$<\frac{\varepsilon}{3}$. Hence for
all \hat{z} wi	th $ z-z < \delta$ v	we have		

	Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
	Proof. I	Let z be so that	$t z - z_0 < R $ (we	will prove that	f is continuous
	at <i>z</i>), let	$R_1 > 0$ be so	that $ z - z_0 < R_1$	$< R$ and let ε >	> 0. Then there
	is a natu	ral number N	so that for all \tilde{z} w	$ith \tilde{z} - z_0 < R$	R_1 we have that
	$\sum_{n=N+1}^{\infty} a $	$ n(\tilde{z}-z_0)^n < 1$	$\frac{\varepsilon}{3}$. Now, because	polynomials ar	e continuous,
	there is a	a $\delta > 0$ so that	t $\delta < R_1 - z - z_0 $	and so that fo	or all \tilde{z} with
	$ \tilde{z}-z <$	δ we have \int_{n}^{∞}	$\sum_{n=0}^{N} a_n (z-z_0)^n - \sum_{n=0}^{N} a_n (z-z_0)^n - $	$\left \sum_{n=0}^{N} a_n (\tilde{z} - z_0)^n \right \cdot$	$<\frac{\varepsilon}{3}$. Hence for
	all \tilde{z} with	$\mathbf{h} \tilde{z} - z < \delta' \mathbf{v}$	ve have		
1.0					

 $|f(z) - f(\tilde{z})|$

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof.	Let z be so that	$t z - z_0 < R $ (we	will prove that	f is continuous
at <i>z</i>), le	t $R_1 > 0$ be so	that $ z - z_0 < R_1$	$< R$ and let $\varepsilon >$	> 0. Then there
is a nat	ural number N	so that for all \tilde{z} w	ith $ \tilde{z} - z_0 < R$	$_1$ we have that
$\sum_{n=N+1}^{\infty} a $	$a_n(\tilde{z}-z_0)^n <1$	$\frac{\varepsilon}{3}$. Now, because	polynomials are	e continuous,
there is	a $\delta > 0$ so that	it $\delta < R_1 - z - z_0 $	and so that for	z all \tilde{z} with
$ \tilde{z}-z <$	$<\delta$ we have $\left \sum_{n} \right $	$\sum_{n=0}^{N} a_n (z - z_0)^n - \sum_{n=0}^{N} a_n (z - z_0)^n - \sum_{n$	$\left \sum_{n=0}^{\infty} a_n (\tilde{z} - z_0)^n \right <$	$<\frac{\varepsilon}{3}$. Hence for
all ž wi	th $ \tilde{z}-z < \delta$ v	ve have		
$\left f(z)-f(\tilde{z})\right =$	$= \left \sum_{n=0}^{\infty} a_n (z - z_0) \right ^{\infty}$	$)^n - \sum_{n=0}^{\infty} a_n (\tilde{z} - z_0)$	n	

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof.	Let z be so that	$ z - z_0 < R \text{ (we)}$	will prove that	f is continuous
at z), let	t $R_1 > 0$ be so	that $ z - z_0 < R_1$	$< R$ and let $\varepsilon >$	> 0. Then there
is a natu	ural number N	so that for all \tilde{z} w	$ith \tilde{z} - z_0 < R$	$_1$ we have that
$\sum_{n=N+1}^{\infty} a $	$a_n(\tilde{z}-z_0)^n <$	$\frac{\varepsilon}{3}$. Now, because j	polynomials are	e continuous,
there is	a $\delta > 0$ so that	at $\delta < R_1 - z - z_0 $	and so that for	r all \tilde{z} with
$ \tilde{z}-z <$	$<\delta$ we have	$\sum_{n=0}^{N} a_n (z - z_0)^n - \sum_{n=0}^{N} a_n (z - z_0)^n - \sum_{n$	$\left \sum_{n=0}^{N} a_n (\tilde{z} - z_0)^n \right <$	$<\frac{\varepsilon}{3}$. Hence for
all \tilde{z} with	th $ \tilde{z}-z < \delta'$	we have		
$\left f(z)-f(\tilde{z})\right =$	$= \left \sum_{n=0}^{\infty} a_n (z - z_0) \right ^2$	$a_n)^n - \sum_{n=0}^{\infty} a_n (\tilde{z} - z_0)^n$) ⁿ	
$\leq \sum_{n=1}^{\infty}$	$\sum_{N+1}^{\infty} a_n(z-z_0)^n $	1		

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof.	Let z be so that	$ z-z_0 < R \text{ (we}$	will prove that	<i>f</i> is continuous
at <i>z</i>), le	t $R_1 > 0$ be so	that $ z - z_0 < R_1$	$< R$ and let $\varepsilon >$	> 0. Then there
is a nat	ural number N	so that for all \tilde{z} w	$ \tilde{z} - z_0 < R$	$_1$ we have that
$\sum_{n=N+1}^{\infty} a $	$a_n(\tilde{z}-z_0)^n <$	$\frac{\varepsilon}{3}$. Now, because	polynomials are	e continuous,
there is	a $\delta > 0$ so that	at $\delta < R_1 - z - z_0 $	and so that for	r all \tilde{z} with
$ \tilde{z}-z <$	$<\delta$ we have $\left \sum_{n=1}^{\infty} \right _{n}$	$\sum_{n=0}^{N} a_n (z - z_0)^n - \sum_{n=0}^{N} a_n (z - z_0)^n - \sum_{n$	$\left \sum_{n=0}^{N}a_{n}(\tilde{z}-z_{0})^{n}\right <$	$<\frac{\varepsilon}{3}$. Hence for
all \tilde{z} wi	th $ \tilde{z}-z < \delta$ v	we have		
$\left f(z)-f(\tilde{z})\right =$	$= \left \sum_{n=0}^{\infty} a_n (z - z_0) \right ^2$	$a_n)^n - \sum_{n=0}^{\infty} a_n (\tilde{z} - z_0)^n$) ⁿ	
$\leq \sum_{n=1}^{n}$	$\sum_{N+1}^{\infty} a_n(z-z_0)^n $	$ z + \sum_{n=0}^{N} a_n (z - z_0)^n$	$n - \sum_{n=0}^{N} a_n (\tilde{z} - z_0)$	$)^n$

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof.	Let z be so tha	$t z - z_0 < R $ (we	will prove that	f is continuous
at <i>z</i>), le	t $R_1 > 0$ be so	that $ z - z_0 < R_1$	$< R$ and let $\varepsilon >$	> 0. Then there
is a nati	ural number N	so that for all \tilde{z} w	with $ \tilde{z} - z_0 < R$	2_1 we have that
$\sum_{n=N+1}^{\infty} a $	$a_n(\tilde{z}-z_0)^n <$	$\frac{\varepsilon}{3}$. Now, because	polynomials ar	e continuous,
there is	a $\delta > 0$ so that	$\text{it } \delta < R_1 - z - z_0 $	and so that fo	r all \tilde{z} with
$ \tilde{z}-z <$	$<\delta$ we have \int_{n}^{∞}	$\sum_{n=0}^{N} a_n (z - z_0)^n - \sum_{n=0}^{N} a_n (z - z_0)^n - \sum_{n$	$\left \sum_{n=0}^{N}a_{n}(\tilde{z}-z_{0})^{n}\right \leq $	$< \frac{\varepsilon}{3}$. Hence for
all \tilde{z} with	th $ \tilde{z}-z < \delta$ v	we have		
$\left f(z)-f(\tilde{z})\right =$	$= \left \sum_{n=0}^{\infty} a_n (z - z_0) \right ^2$	$)^n - \sum_{n=0}^{\infty} a_n (\tilde{z} - z_0)^n$	$)^n$	
$\leq \sum_{n=1}^{\infty}$	$\sum_{N+1}^{\infty} a_n(z-z_0)^n $	$ +\left \sum_{n=0}^{N}a_n(z-z_0)\right $	$n - \sum_{n=0}^{N} a_n (\tilde{z} - z_0)$	$)^n \left + \sum_{n=N+1}^{\infty} a_n (\tilde{z} - z_0)^n \right $

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division

Theorem.

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Theor	em. Let the pow	ver series $\sum_{n=0}^{\infty} a_n(z)$	$(-z_0)^n$ have ci	rcle of
conver	gence $ z - z_0 =$	R, let C be a con	tour that is en	irely contained
in the i	interior of the ci	ircle of convergen	ce and let g be	a function that
is cont	inuous on the in	terior of the circl	e of convergen	ce.

Convergence Continuity Differentiability Uniqueness Multiplication and Division **Theorem.** Let the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ have circle of convergence $|z-z_0| = R$, let C be a contour that is entirely contained in the interior of the circle of convergence and let g be a function that is continuous on the interior of the circle of convergence. Then for the continuous function $f(z) := \sum_{n=0}^{\infty} a_n(z-z_0)^n$ the integral of g(z)f(z)over C can be computed term-by-term.

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Convergence Continuity Differentiability Uniqueness **Multiplication and Division Theorem.** Let the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ have circle of convergence $|z - z_0| = R$, let C be a contour that is entirely contained in the interior of the circle of convergence and let g be a function that is continuous on the interior of the circle of convergence. Then for the continuous function $f(z) := \sum_{n=1}^{\infty} a_n (z - z_0)^n$ the integral of g(z)f(z)over C can be computed term-by-term. That is, $\int_C g(z)f(z) dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z-z_0)^n dz.$

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Visuali	zation.			

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Visuali	zation.	$\Im(z)$		
				$\Re(z)$
		1		

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Visuali	zation.	$\Im(z)$		
		•		► ℜ(z)
				0

1

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Convergence Continuity Differentiability Uniqueness Multip	plication and Division
Visualization.	

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Visualiz	ation.			⊾ ℜ(z)

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Visuali	zation.			- - ► ℜ(z)

Convergence Continuity Differentiability Uniqueness Multip	plication and Division
Visualization.	

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof.				

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division

Proof. Let $R_1 > 0$ be so that for all z on C we have $|z - z_0| < R_1 < R$

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof. Let $R_1 > 0$ be so that for all <i>z</i> on <i>C</i> we have $ z - z_0 < R_1 < R$				
and let	$\varepsilon > 0.$			
Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
---	---	--	--	---
Proof. and let $ \tilde{z} - z_0 $	Let $R_1 > 0$ be $\varepsilon > 0$. Then the $\langle R_1$ we have	so that for all z or ere is a natural nu that $\sum_{n=N+1}^{\infty} a_n(\tilde{z} - t) ^2$	The formula for the constant of the formula formula formula for the constant of the constant	$\begin{aligned} z_0 < R_1 < R \\ z \text{ for all } \tilde{z} \text{ with } \\ \frac{\varepsilon}{ v d w }. \end{aligned}$

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof.	Let $R_1 > 0$ be	so that for all z on	C we have $ z - z = 1$	$ -z_0 < R_1 < R$
and let	$\varepsilon > 0$. Then the	ere is a natural nu	umber N so that	for all \tilde{z} with
$ \tilde{z}-z_0 $	$< R_1$ we have	that $\sum_{n=N+1}^{\infty} a_n(\tilde{z} -$	$ z_0)^n < \frac{1}{\int_C g(v) }$	$\frac{\varepsilon}{ w }$. Then
$\int_C g(z) j$	f(z) dz			

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof. 1	Let $R_1 > 0$ be	so that for all z on	C we have $ z - z = 1$	$ z_0 < R_1 < R$
and let a	$\varepsilon > 0$. Then the	nere is a natural nu	The imber N so that	for all \tilde{z} with
$ \tilde{z}-z_0 $	$< R_1$ we have	that $\sum_{n=N+1}^{\infty} a_n(\tilde{z} -$	$ z_0)^n < \frac{\varepsilon}{\int_C g(w) }$	$\frac{\varepsilon}{ v d w }$. Then
$\int_C g(z)f$	$f(z) dz = \int$	$\int_C g(z) \sum_{n=0}^\infty a_n (z - z)$	$(0)^n dz$	

Proof. Let
$$R_1 > 0$$
 be so that for all z on C we have $|z - z_0| < R_1 < R$
and let $\varepsilon > 0$. Then there is a natural number N so that for all \tilde{z} with
 $|\tilde{z} - z_0| < R_1$ we have that $\sum_{n=N+1}^{\infty} |a_n(\tilde{z} - z_0)^n| < \frac{\varepsilon}{\int_C |g(w)| \, d|w|}$. Then
 $\int_C g(z)f(z) \, dz = \int_C g(z) \sum_{n=0}^{\infty} a_n(z - z_0)^n \, dz$
 $= \int_C g(z) \left(\sum_{n=0}^N a_n(z - z_0)^n + \sum_{n=N+1}^\infty a_n(z - z_0)^n\right) \, dz$

Convergence Continuity Differentiability Uniqueness Multiplication and Division
Proof. Let
$$R_1 > 0$$
 be so that for all z on C we have $|z - z_0| < R_1 < R$
and let $\varepsilon > 0$. Then there is a natural number N so that for all \tilde{z} with
 $|\tilde{z} - z_0| < R_1$ we have that $\sum_{n=N+1}^{\infty} |a_n(\tilde{z} - z_0)^n| < \frac{\varepsilon}{\int_C |g(w)| \, d|w|}$. Then
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 $= \int_C g(z) \left(\sum_{n=0}^N a_n(z - z_0)^n + \sum_{n=N+1}^{\infty} a_n(z - z_0)^n \right) \, dz$
 $= \sum_{n=0}^N a_n \int_C g(z)(z - z_0)^n \, dz + \int_C g(z) \sum_{n=N+1}^{\infty} a_n(z - z_0)^n \, dz$

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof.				

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division

$$\left| \int_C g(z) \sum_{n=N+1}^{\infty} a_n (z-z_0)^n \, dz \right|$$

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division

$$\left| \int_{C} g(z) \sum_{n=N+1}^{\infty} a_n (z-z_0)^n \, dz \right| \leq \int_{C} |g(z)| \sum_{n=N+1}^{\infty} |a_n (z-z_0)^n| \, d|z|$$

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division

$$\begin{aligned} \left| \int_{C} g(z) \sum_{n=N+1}^{\infty} a_n (z-z_0)^n dz \right| &\leq \int_{C} \left| g(z) \right| \sum_{n=N+1}^{\infty} \left| a_n (z-z_0)^n \right| d|z| \\ &\leq \int_{C} \left| g(z) \right| \frac{\varepsilon}{\int_{C} \left| g(w) \right| d|w|} d|z| \end{aligned}$$

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division

$$\begin{aligned} \left| \int_{C} g(z) \sum_{n=N+1}^{\infty} a_n (z-z_0)^n dz \right| &\leq \int_{C} \left| g(z) \right| \sum_{n=N+1}^{\infty} \left| a_n (z-z_0)^n \right| d|z| \\ &\leq \int_{C} \left| g(z) \right| \frac{\varepsilon}{\int_{C} \left| g(w) \right| d|w|} d|z| < \varepsilon \end{aligned}$$

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division

$$\begin{aligned} \left| \int_{C} g(z) \sum_{n=N+1}^{\infty} a_n (z-z_0)^n dz \right| &\leq \int_{C} \left| g(z) \right| \sum_{n=N+1}^{\infty} |a_n (z-z_0)^n| d|z| \\ &\leq \int_{C} \left| g(z) \right| \frac{\varepsilon}{\int_{C} |g(w)| d|w|} d|z| < \varepsilon \end{aligned}$$

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division

Theorem.

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Theorem. Let the power series
$$\sum_{n=0}^{\infty} a_n(z-z_0)^n$$
 have circle of
convergence $|z-z_0| = R$ and let $f(z) := \sum_{n=0}^{\infty} a_n(z-z_0)^n$ for all z in the
circle of convergence. Then f is analytic in the circle of convergence.
Moreover, the power series can be differentiated term-by-term. That
is, for all z in the circle of convergence we have
 $f'(z) = \sum_{n=1}^{\infty} na_n(z-z_0)^{n-1}$

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof.				

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division	
Proof. Note that for any closed contour C in the circle of					

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof. converg	of			
J	$\int_{C} f(z) dz$			

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division				
Proof. Note that for any closed contour C in the circle of								
convergence, we can integrate f term-by-term. Thus								

$$\int_C f(z) dz = \int_C \sum_{n=0}^{\infty} a_n (z-z_0)^n dz$$

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division				
Proof. Note that for any closed contour C in the circle of								
convergence, we can integrate f term-by-term. Thus								

$$\int_{C} f(z) dz = \int_{C} \sum_{n=0}^{\infty} a_{n} (z - z_{0})^{n} dz = \sum_{n=0}^{\infty} a_{n} \int_{C} (z - z_{0})^{n} dz$$

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division					
Proof.	Proof. Note that for any closed contour C in the circle of								

$$\int_{C} f(z) dz = \int_{C} \sum_{n=0}^{\infty} a_{n} (z - z_{0})^{n} dz = \sum_{n=0}^{\infty} a_{n} \int_{C} (z - z_{0})^{n} dz$$
$$= \sum_{n=0}^{\infty} a_{n} \cdot 0$$

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division					
Proof.	Proof. Note that for any closed contour C in the circle of								

$$\int_{C} f(z) dz = \int_{C} \sum_{n=0}^{\infty} a_{n} (z - z_{0})^{n} dz = \sum_{n=0}^{\infty} a_{n} \int_{C} (z - z_{0})^{n} dz$$
$$= \sum_{n=0}^{\infty} a_{n} \cdot 0 = 0$$

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division					
Proof.	Proof Note that for any closed contour C in the circle of								

$$\int_{C} f(z) dz = \int_{C} \sum_{n=0}^{\infty} a_{n} (z - z_{0})^{n} dz = \sum_{n=0}^{\infty} a_{n} \int_{C} (z - z_{0})^{n} dz$$
$$= \sum_{n=0}^{\infty} a_{n} \cdot 0 = 0$$

Thus, by Morera's Theorem, f is analytic in the interior of the circle of convergence

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division				
Proof. Note that for any closed contour C in the circle of								

$$\int_{C} f(z) dz = \int_{C} \sum_{n=0}^{\infty} a_{n} (z - z_{0})^{n} dz = \sum_{n=0}^{\infty} a_{n} \int_{C} (z - z_{0})^{n} dz$$
$$= \sum_{n=0}^{\infty} a_{n} \cdot 0 = 0$$

Thus, by Morera's Theorem, f is analytic in the interior of the circle of convergence(of f).

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Converge	nce	Continuiț	Differe	ntiability	Uı	niquene	ss	N	Iultiplication a	nd Division	
_											

$$\int_{C} f(z) dz = \int_{C} \sum_{n=0}^{\infty} a_{n} (z - z_{0})^{n} dz = \sum_{n=0}^{\infty} a_{n} \int_{C} (z - z_{0})^{n} dz$$
$$= \sum_{n=0}^{\infty} a_{n} \cdot 0 = 0$$

Thus, by Morera's Theorem, f is analytic in the interior of the circle of convergence(of f). Thus for all z in the circle of convergence, $f'(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division

$$\int_{C} f(z) dz = \int_{C} \sum_{n=0}^{\infty} a_{n} (z - z_{0})^{n} dz = \sum_{n=0}^{\infty} a_{n} \int_{C} (z - z_{0})^{n} dz$$
$$= \sum_{n=0}^{\infty} a_{n} \cdot 0 = 0$$

Thus, by Morera's Theorem, f is analytic in the interior of the circle of convergence(of f). Thus for all z in the circle of convergence, $f'(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$, where $b_n = \frac{(f')^{(n)}(z_0)}{n!}$

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division

$$\int_{C} f(z) dz = \int_{C} \sum_{n=0}^{\infty} a_{n} (z - z_{0})^{n} dz = \sum_{n=0}^{\infty} a_{n} \int_{C} (z - z_{0})^{n} dz$$
$$= \sum_{n=0}^{\infty} a_{n} \cdot 0 = 0$$

Thus, by Morera's Theorem, *f* is analytic in the interior of the circle of convergence(of *f*). Thus for all *z* in the circle of convergence, $f'(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n$, where $b_n = \frac{(f')^{(n)}(z_0)}{n!} = \frac{f^{(n+1)}(z_0)}{n!}$

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division

$$\int_{C} f(z) dz = \int_{C} \sum_{n=0}^{\infty} a_{n} (z - z_{0})^{n} dz = \sum_{n=0}^{\infty} a_{n} \int_{C} (z - z_{0})^{n} dz$$
$$= \sum_{n=0}^{\infty} a_{n} \cdot 0 = 0$$

Thus, by Morera's Theorem, f is analytic in the interior of the circle of convergence(of f). Thus for all z in the circle of convergence, $f'(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$, where $b_n = \frac{(f')^{(n)}(z_0)}{n!} = \frac{f^{(n+1)}(z_0)}{n!} \frac{n+1}{n+1}$

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division

$$\int_{C} f(z) dz = \int_{C} \sum_{n=0}^{\infty} a_{n} (z - z_{0})^{n} dz = \sum_{n=0}^{\infty} a_{n} \int_{C} (z - z_{0})^{n} dz$$
$$= \sum_{n=0}^{\infty} a_{n} \cdot 0 = 0$$

Thus, by Morera's Theorem, f is analytic in the interior of the circle of convergence(of f). Thus for all z in the circle of convergence, $f'(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n$, where $b_n = \frac{(f')^{(n)}(z_0)}{n!} = \frac{f^{(n+1)}(z_0)}{n!} \frac{n+1}{n+1} = (n+1) \frac{f^{(n+1)}(z_0)}{(n+1)!}$

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division

$$\int_{C} f(z) dz = \int_{C} \sum_{n=0}^{\infty} a_{n} (z - z_{0})^{n} dz = \sum_{n=0}^{\infty} a_{n} \int_{C} (z - z_{0})^{n} dz$$
$$= \sum_{n=0}^{\infty} a_{n} \cdot 0 = 0$$

Thus, by Morera's Theorem, f is analytic in the interior of the circle of convergence(of f). Thus for all z in the circle of convergence, $f'(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$, where $b_n = \frac{(f')^{(n)}(z_0)}{n!} = \frac{f^{(n+1)}(z_0)}{n!} \frac{n+1}{n+1} = (n+1) \frac{f^{(n+1)}(z_0)}{(n+1)!} = (n+1)a_{n+1}$

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division

$$\int_{C} f(z) dz = \int_{C} \sum_{n=0}^{\infty} a_{n} (z - z_{0})^{n} dz = \sum_{n=0}^{\infty} a_{n} \int_{C} (z - z_{0})^{n} dz$$
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Thus, by Morera's Theorem, f is analytic in the interior of the circle of convergence(of f). Thus for all z in the circle of convergence, $f'(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n$, where $b_n = \frac{(f')^{(n)}(z_0)}{n!} = \frac{f^{(n+1)}(z_0)}{n!} \frac{n+1}{n+1} = (n+1)\frac{f^{(n+1)}(z_0)}{(n+1)!} = (n+1)a_{n+1}$, which is what was to be proved.

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division

$$\int_{C} f(z) dz = \int_{C} \sum_{n=0}^{\infty} a_{n} (z - z_{0})^{n} dz = \sum_{n=0}^{\infty} a_{n} \int_{C} (z - z_{0})^{n} dz$$
$$= \sum_{n=0}^{\infty} a_{n} \cdot 0 = 0$$

Thus, by Morera's Theorem, f is analytic in the interior of the circle of convergence(of f). Thus for all z in the circle of convergence, $f'(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n$, where $b_n = \frac{(f')^{(n)}(z_0)}{n!} = \frac{f^{(n+1)}(z_0)}{n!} \frac{n+1}{n+1} = (n+1)\frac{f^{(n+1)}(z_0)}{(n+1)!} = (n+1)a_{n+1}$, which is what was to be proved.

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division

Theorem.

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof.				

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof. $ z-z_0 $	The power ser $< R_1 < R$	ies converges unif	ormly inside ar	ny circle

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof.	The power ser	ies converges unif	ormly inside a	ny circle

 $|z-z_0| < R_1 < R$, so that the power series is analytic there.

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof.	The power ser	ies converges unif	formly inside an	y circle
$ z - z_0 $	$< R_1 < R$, so t	that the power ser	ies is analytic th	ere. But that
means	that the coeffic	ients of the analy	tic function	
f(z) =	$\sum_{n=0}^{\infty} b_n (z-z_0)^n$	are obtained via		
	$b_n = \frac{1}{2\pi}$	$\frac{1}{\tau i} \int_{C(z_0,R_1)} \frac{f(\xi)}{(\xi-z_0)} d\xi$	$rac{)}{)^{n+1}}d\xi$	

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Proof.	The power	series converges uni	formly inside an	iy circle
$ z - z_0 $	$< R_1 < R,$	so that the power ser	ries is analytic th	here. But that
means t	hat the \cos_{∞}	efficients of the analy	tic function	
$f(z) = \sum_{i=1}^{n}$	$\sum_{n=1}^{\infty} b_n(z-z)$	$(z_0)^n$ are obtained via		
n	n=0			
	b_n =	$-\frac{1}{2\pi i}\int_{C(z_0,R_1)}\frac{f(\xi)}{(\xi-z_0)}$	$\left({j \over 0} ight)^{n+1} d\xi$	
	=	$-\frac{1}{2\pi i}\int_{C(z_0,R_1)}\frac{\sum_{j=0}^{\infty}a_j}{(\xi-$	$rac{i}{(\xi-z_0)^{j-1}}d\xi$	

Convergence Continuity Differentiability Uniqueness **Multiplication and Division Proof.** The power series converges uniformly inside any circle $|z-z_0| < R_1 < R$, so that the power series is analytic there. But that means that the coefficients of the analytic function $f(z) = \sum_{n} b_n (z - z_0)^n$ are obtained via $b_n = \frac{1}{2\pi i} \int_{C(z_0,R_1)} \frac{f(\xi)}{(\xi-z_0)^{n+1}} d\xi$ $= \frac{1}{2\pi i} \int_{C(z_0, R_1)} \frac{\sum_{j=0}^{\infty} a_j (\xi - z_0)^j}{(\xi - z_0)^{n+1}} d\xi$ $= \frac{1}{2\pi i} \int_{C(z_0,R_1)} \sum_{i=0}^{\infty} a_i (\xi - z_0)^{j-n-1} d\xi$

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Convergence Continuity Differentiability Uniqueness **Multiplication and Division Proof.** The power series converges uniformly inside any circle $|z-z_0| < R_1 < R$, so that the power series is analytic there. But that means that the coefficients of the analytic function $f(z) = \sum_{n} b_n (z - z_0)^n$ are obtained via $b_n = \frac{1}{2\pi i} \int_{C(z_0,R_1)} \frac{f(\xi)}{(\xi-z_0)^{n+1}} d\xi$ $= \frac{1}{2\pi i} \int_{C(z_{-}, P_{-})} \frac{\sum_{j=0}^{\infty} a_{j} (\xi - z_{0})^{j}}{(\xi - z_{0})^{n+1}} d\xi$ $= \frac{1}{2\pi i} \int_{C(z_0,R_1)} \sum_{i=0}^{\infty} a_i (\xi - z_0)^{j-n-1} d\xi$ $= \frac{1}{2\pi i} \sum_{i=0}^{\infty} a_j \int_{C(z_0,R_1)} (\xi - z_0)^{j-n-1} d\xi$

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Convergence Continuity Differentiability Uniqueness **Multiplication and Division Proof.** The power series converges uniformly inside any circle $|z-z_0| < R_1 < R$, so that the power series is analytic there. But that means that the coefficients of the analytic function $f(z) = \sum b_n (z - z_0)^n$ are obtained via $b_n = \frac{1}{2\pi i} \int_{C(z_0,R_1)} \frac{f(\xi)}{(\xi-z_0)^{n+1}} d\xi$ $= \frac{1}{2\pi i} \int_{C(z_{-}, P_{-})} \frac{\sum_{j=0}^{\infty} a_{j} (\xi - z_{0})^{j}}{(\xi - z_{0})^{n+1}} d\xi$ $= \frac{1}{2\pi i} \int_{C(z_0,R_1)} \sum_{i=0}^{\infty} a_i (\xi - z_0)^{j-n-1} d\xi$ $= \frac{1}{2\pi i} \sum_{i=0}^{\infty} a_j \int_{C(z_0,R_1)} (\xi - z_0)^{j-n-1} d\xi = a_n$

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Convergence Continuity Differentiability Uniqueness **Multiplication and Division Proof.** The power series converges uniformly inside any circle $|z-z_0| < R_1 < R$, so that the power series is analytic there. But that means that the coefficients of the analytic function $f(z) = \sum_{n} b_n (z - z_0)^n$ are obtained via $b_n = \frac{1}{2\pi i} \int_{C(z_0,R_1)} \frac{f(\xi)}{(\xi-z_0)^{n+1}} d\xi$ $= \frac{1}{2\pi i} \int_{C(z_{-}, P_{-})} \frac{\sum_{j=0}^{\infty} a_{j} (\xi - z_{0})^{j}}{(\xi - z_{0})^{n+1}} d\xi$ $= \frac{1}{2\pi i} \int_{C(z_0,R_1)} \sum_{i=0}^{\infty} a_i (\xi - z_0)^{j-n-1} d\xi$ $= \frac{1}{2\pi i} \sum_{i=0}^{\infty} a_j \int_{C(z_0,R_1)} (\xi - z_0)^{j-n-1} d\xi = a_n$ Thus the original power series really is the Taylor series of f.

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The Analysis of Power Series

Convergence Continuity Differentiability Uniqueness **Multiplication and Division Proof.** The power series converges uniformly inside any circle $|z-z_0| < R_1 < R$, so that the power series is analytic there. But that means that the coefficients of the analytic function $f(z) = \sum_{n} b_n (z - z_0)^n$ are obtained via $b_n = \frac{1}{2\pi i} \int_{C(z_0,R_1)} \frac{f(\xi)}{(\xi-z_0)^{n+1}} d\xi$ $= \frac{1}{2\pi i} \int_{C(z_{-}, P_{-})} \frac{\sum_{j=0}^{\infty} a_{j} (\xi - z_{0})^{j}}{(\xi - z_{0})^{n+1}} d\xi$ $= \frac{1}{2\pi i} \int_{C(z_0,R_1)} \sum_{i=0}^{\infty} a_i (\xi - z_0)^{j-n-1} d\xi$ $= \frac{1}{2\pi i} \sum_{i=0}^{\infty} a_j \int_{C(z_0,R_1)} (\xi - z_0)^{j-n-1} d\xi = a_n$ Thus the original power series really is the Taylor series of f. ("Series

are their own Taylor series")

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Convergence Continuity Differentiability Uniqueness **Multiplication and Division Proof.** The power series converges uniformly inside any circle $|z-z_0| < R_1 < R$, so that the power series is analytic there. But that means that the coefficients of the analytic function $f(z) = \sum_{n} b_n (z - z_0)^n$ are obtained via $b_n = \frac{1}{2\pi i} \int_{C(z_0,R_1)} \frac{f(\xi)}{(\xi-z_0)^{n+1}} d\xi$ $= \frac{1}{2\pi i} \int_{C(z_{-}, P_{-})} \frac{\sum_{j=0}^{\infty} a_{j} (\xi - z_{0})^{j}}{(\xi - z_{0})^{n+1}} d\xi$ $= \frac{1}{2\pi i} \int_{C(z_0,R_1)} \sum_{i=0}^{\infty} a_i (\xi - z_0)^{j-n-1} d\xi$ $= \frac{1}{2\pi i} \sum_{i=0}^{\infty} a_j \int_{C(z_0,R_1)} (\xi - z_0)^{j-n-1} d\xi = a_n$ Thus the original power series really is the Taylor series of f. ("Series

are their own Taylor series ")

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division

Theorem.

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Proof.

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Proof. Let $r_0 < R_0$ be so that $r < r_0 < R_0 < R$.

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The Analysis of Power Series

 $0 < |z - z_0| < R_0$

ConvergenceContinuityDifferentiabilityUniquenessMultiplication and DivisionTheorem. If the doubly infinite series
$$\sum_{n=-\infty}^{\infty} c_n(z-z_0)^n$$
 converges to $f(z)$ at all points inside an annular domain $r < |z - z_0| < R$ with $R > r \ge 0$, then it is the Laurent series expansion of the function fabout z_0 .Proof. Let $r_0 < R_0$ be so that $r < r_0 < R_0 < R$. From the way serieswork, the series $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ converges uniformly for $0 \le |z-z_0| < R_0$ and the series $\sum_{n=-\infty}^{0} c_n(z-z_0)^n$ converges uniformlyfor $r_0 < |z-z_0| < \infty$.

Convergence Continuity Differentiability Uniqueness Multiplication and Division
Theorem. If the doubly infinite series
$$\sum_{n=-\infty}^{\infty} c_n(z-z_0)^n$$
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Proof. Let $r_0 < R_0$ be so that $r < r_0 < R_0 < R$. From the way series
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Convergence Continuity Differentiability Uniqueness Multiplication and Division
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ConvergenceContinuityDifferentiabilityUniquenessMultiplication and DivisionTheorem. If the doubly infinite series
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Convergence Continuity Differentiability Uniquenes Multiplication and Division
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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division

Theorem.

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division

Theorem. Leibniz' Rule.

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Convergence

Multiplication and Division

Theorem. Leibniz' Rule. If g and h are both infinitely differentiable at z, then $(gh)^{(n)}(z) = \sum_{k=0}^{n} {n \choose k} g^{(k)}(z)h^{(n-k)}(z).$

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Theorem. Leibniz' Rule. If g and h are both infinitely differentiable at z, then $(gh)^{(n)}(z) = \sum_{k=0}^{n} \binom{n}{k} g^{(k)}(z)h^{(n-k)}(z).$

Proof.

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Proof. Induction on *n*.

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Proof. Induction on *n*. **Base step, n=0.**

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division

Induction step, $n \rightarrow n+1$ **.**

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division

Induction step, $n \rightarrow n+1$ **.**

 $(gh)^{(n+1)}(z)$

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Induct	ion step, $n \rightarrow 1$	n + 1.		

$$(gh)^{(n+1)}(z) = \frac{d}{dz} \sum_{k=0}^{n} \binom{n}{k} g^{(k)}(z)h^{(n-k)}(z)$$

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Inductio	on step, $n \rightarrow r$	n + 1.		
$(gh)^{(n+1)}(z)$	$= \frac{d}{dz} \sum_{k=0}^{n} \left($	$\binom{n}{k}g^{(k)}(z)h^{(n-k)}(z)$		
	$\frac{n}{n}$ (n)		(1) (

$$\begin{aligned} (gh)^{(n+1)}(z) &= \frac{d}{dz} \sum_{k=0}^{n} \binom{n}{k} g^{(k)}(z) h^{(n-k)}(z) \\ &= \sum_{k=0}^{n} \binom{n}{k} \left(g^{(k+1)}(z) h^{(n-k)}(z) + g^{(k)}(z) h^{(n-k+1)}(z) \right) \end{aligned}$$

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Inducti	ion step, $n \rightarrow r$	n + 1.		
$(gh)^{(n+1)}(z)$	$= \frac{d}{dz} \sum_{k=0}^{n} \left($	$\binom{n}{k}g^{(k)}(z)h^{(n-k)}(z)$:)	
	$=\sum_{k=0}^{n} \binom{n}{k}$	$\Big(g^{(k+1)}(z)h^{(n-k)}($	$z) + g^{(k)}(z)h^{(n-1)}$	$^{-k+1)}(z)\Big)$

$$= \sum_{k=0}^{n} \binom{n}{k} g^{(k+1)}(z) h^{(n-k)}(z) + \sum_{k=0}^{n} \binom{n}{k} g^{(k)}(z) h^{(n-k+1)}(z)$$

Convergence	C	ontinuity	Differentiability	Uniqueness	Multiplication and Division
Induction	on st	tep, $n \rightarrow n +$	- 1.		
$(gh)^{(n+1)}(z)$	=	$\frac{d}{dz}\sum_{k=0}^{n}\binom{n}{k}$	$g^{(k)}(z)h^{(n-k)}(z)$		
	=	$\sum_{k=0}^{n} \binom{n}{k} \left(\xi \right)$	$g^{(k+1)}(z)h^{(n-k)}(z) -$	$+g^{(k)}(z)h^{(n-k+1)}$	$^{(1)}(z)\Big)$
	=	$\sum_{k=0}^{n} \binom{n}{k} g^{(k)}$	$^{(k+1)}(z)h^{(n-k)}(z) +$	$\sum_{k=0}^{n} \binom{n}{k} g^{(k)}(z)$	$h^{(n-k+1)}(z)$
	=	$\sum_{j=1}^{n+1} \binom{n}{j-1}$	$g^{(j)}(z)h^{(n+1-j)}(z)$	$+\sum_{k=0}^{n} \binom{n}{k} g^{(k)}$	$(z)h^{(n+1-k)}(z)$

Convergence	U	ontinuity	Differentiability	Uniqueness	Multiplication and Division
Inducti	on st	tep, $n \rightarrow n$	+1.		
		d n (n)	\ \		
$(gh)^{(n+1)}(z)$	=	$\frac{a}{1}\sum_{n} \binom{n}{n}$	$g^{(k)}(z)h^{(n-k)}(z)$)	
		$dz = 0 \setminus k$)		
		$\sum_{n=1}^{n} (n)$	(k+1) $(-)$ $k(n-k)$ $(-k)$	$(k) = c^{(k)} (-) \mathbf{L}^{(n-1)}$	-k+1)
	=	$\sum_{k=0} (k) (k)$	g(z)n(z)n(z)	(z) + g(z)n(z)	(z)
		n (n)		n (m)	
	=	$\sum \binom{n}{k} g$	$h^{(k+1)}(z)h^{(n-k)}(z)$	$+\sum \binom{n}{k} g^{(k)}$	$^{)}(z)h^{(n-k+1)}(z)$
		$k=0 \langle k \rangle$		$k=0 \langle k \rangle$	
		$\sum^{n+1} (n)$	(i) (1) (n+1-i)	$\sum_{n=1}^{n} (n)$	$(k) \land (n+1-k) \land)$
	=	$\sum_{i=1} (j-1)$	$\int g^{(j)}(z) n^{(\alpha+\alpha-j)}(z) dz$	$(z) + \sum_{k=0}^{\infty} {\binom{k}{k}}$	$g^{(\alpha)}(z)h^{(\alpha+z-\alpha)}(z)$
		<i>j</i> =1 ♥	'n	$k=0$ $\langle \rangle$	
	=	$\binom{n}{g^{(n+1)}}$	$h^{(1)}(z)h^{(0)}(z) + \sum_{n=1}^{n}$	$\binom{n}{1}g^{(j)}(z)$	$h^{(n+1-j)}(z)$
		$(n)^{\circ}$	j=1	$(j-1)^{\circ}$, , , , , , , , , , , , , , , , , , ,
		$\sum_{n=1}^{n} (n)$	$(k) \land (n+1-k) \land$	(n) (0)	() $(n+1)$ $()$
		$+ \sum_{k \in I} (k)$	$\int g^{(n)}(z)h^{(n+1-n)}(z)$	$z^{(0)} + (0) g^{(0)}$	$(z)h^{(n+1)}(z)$
		$\kappa = 1 \langle \rangle$		× /	

Continuity

Differentiabilit

$$(gh)^{(n+1)}(z) = \binom{n}{n} g^{(n+1)}(z)h^{(0)}(z) + \sum_{j=1}^{n} \binom{n}{j-1} g^{(j)}(z)h^{(n+1-j)}(z) + \sum_{k=1}^{n} \binom{n}{k} g^{(k)}(z)h^{(n+1-k)}(z) + \binom{n}{0} g^{(0)}(z)h^{(n+1)}(z)$$

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Continuity

Differentiabilit

$$\begin{split} (gh)^{(n+1)}(z) &= \binom{n}{n} g^{(n+1)}(z) h^{(0)}(z) + \sum_{j=1}^{n} \binom{n}{j-1} g^{(j)}(z) h^{(n+1-j)}(z) \\ &+ \sum_{k=1}^{n} \binom{n}{k} g^{(k)}(z) h^{(n+1-k)}(z) + \binom{n}{0} g^{(0)}(z) h^{(n+1)}(z) \\ &= \binom{n+1}{n+1} g^{(n+1)}(z) h^{(0)}(z) \end{split}$$

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Continuity

$$(gh)^{(n+1)}(z) = \binom{n}{n} g^{(n+1)}(z)h^{(0)}(z) + \sum_{j=1}^{n} \binom{n}{j-1} g^{(j)}(z)h^{(n+1-j)}(z) + \sum_{k=1}^{n} \binom{n}{k} g^{(k)}(z)h^{(n+1-k)}(z) + \binom{n}{0} g^{(0)}(z)h^{(n+1)}(z) = \binom{n+1}{n+1} g^{(n+1)}(z)h^{(0)}(z) + \sum_{j=1}^{n} \left(\binom{n}{j-1} + \binom{n}{j}\right) g^{(j)}(z)h^{(n+1-j)}(z)$$

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Continuity

$$gh)^{(n+1)}(z) = \binom{n}{n} g^{(n+1)}(z)h^{(0)}(z) + \sum_{j=1}^{n} \binom{n}{j-1} g^{(j)}(z)h^{(n+1-j)}(z) + \sum_{k=1}^{n} \binom{n}{k} g^{(k)}(z)h^{(n+1-k)}(z) + \binom{n}{0} g^{(0)}(z)h^{(n+1)}(z) = \binom{n+1}{n+1} g^{(n+1)}(z)h^{(0)}(z) + \sum_{j=1}^{n} \left(\binom{n}{j-1} + \binom{n}{j}\right) g^{(j)}(z)h^{(n+1-j)}(z) + \binom{n+1}{0} g^{(0)}(z)h^{(n+1)}(z)$$

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Continuity

Differentiability

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division

Theorem.

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Convergence Continuity Differentiability Uniqueness Multiplication and Division
Theorem. If
$$g(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$
 and $h(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^n$
converge at all points inside a circle $|z-z_0| < R$ of nonzero radius
 $R > 0$, then the function $f := gh$ has a power series expansion there,
too, and the coefficients of its expansion are $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

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Proof.

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Proof. By the preceding theorem, *f* is analytic for $|z - z_0| < R$.

Convergence Continuity Differentiability Uniqueness Multiplication and Division
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Convergence Continuity Differentiability Uniqueness Multiplication and Division
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 $\sum_{k=0}^{\infty} e^{(n)(z-1)}$

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$$|z - z_0| < R$$
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$$= \frac{1}{n!} \sum_{j=0}^n \frac{n!}{j!(n-j)!}$$

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, $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$ and $c_n = \frac{f^{(n)}(z_0)}{n!}$. Again by the

$$c_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{n!} \sum_{j=0}^n \binom{n}{j} g^{(j)}(z_0) h^{(n-j)}(z_0)$$
$$= \frac{1}{n!} \sum_{j=0}^n \frac{n!}{j!(n-j)!} j! a_j$$

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Theorem. If
$$g(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$
 and $h(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n$
converge at all points inside a circle $|z-z_0| < R$ of nonzero radius $R > 0$, then the function $f := gh$ has a power series expansion there, too, and the coefficients of its expansion are $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.
Proof. By the preceding theorem, f is analytic for $|z-z_0| < R$. Thus,

for
$$|z - z_0| < R$$
, $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$ and $c_n = \frac{f^{(n)}(z_0)}{n!}$. Again by the

$$c_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{n!} \sum_{j=0}^n {n \choose j} g^{(j)}(z_0) h^{(n-j)}(z_0)$$
$$= \frac{1}{n!} \sum_{j=0}^n \frac{n!}{j!(n-j)!} j! a_j(n-j)! b_{n-j}$$

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Convergence Continuity Differentiability Uniqueness Multiplication and Division **Theorem.** If $g(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ and $h(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^n$ converge at all points inside a circle $|z-z_0| < R$ of nonzero radius R > 0, then the function f := gh has a power series expansion there, too, and the coefficients of its expansion are $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. **Proof.** By the preceding theorem, f is analytic for $|z-z_0| < R$. Thus,

for
$$|z - z_0| < R$$
, $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$ and $c_n = \frac{f^{(n)}(z_0)}{n!}$. Again by the

preceding theorem,

$$c_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{n!} \sum_{j=0}^n {n \choose j} g^{(j)}(z_0) h^{(n-j)}(z_0)$$

= $\frac{1}{n!} \sum_{j=0}^n \frac{n!}{j!(n-j)!} j! a_j(n-j)! b_{n-j} = \sum_{j=0}^n a_j b_{n-j}$

Bernd Schröder

Theorem. If
$$g(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$
 and $h(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n$
converge at all points inside a circle $|z-z_0| < R$ of nonzero radius $R > 0$, then the function $f := gh$ has a power series expansion there, too, and the coefficients of its expansion are $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.
Proof. By the preceding theorem, f is analytic for $|z-z_0| < R$. Thus,

for
$$|z - z_0| < R$$
, $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$ and $c_n = \frac{f^{(n)}(z_0)}{n!}$. Again by the

$$c_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{n!} \sum_{j=0}^n \binom{n}{j} g^{(j)}(z_0) h^{(n-j)}(z_0)$$

= $\frac{1}{n!} \sum_{j=0}^n \frac{n!}{j!(n-j)!} j! a_j(n-j)! b_{n-j} = \sum_{j=0}^n a_j b_{n-j}$

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division

Example.

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ConvergenceContinuityDifferentiabilityUniquenessMultiplication and DivisionExample. Expand
$$\frac{e^z}{z^2+1}$$
 into a power series around 0.

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Example	• Expand $\frac{e^z}{z^2+}$	– into a power 1	series around 0.	
$\frac{e^z}{z^2+1} =$	$= e^{z} \frac{1}{1 - (-z^{2})}$)		

ConvergenceContinuityDifferentiabilityUniquenessMultiplication and DivisionExample. Expand $\frac{e^z}{z^2+1}$ into a power series around 0.

$$\frac{e^z}{z^2+1} = e^z \frac{1}{1-(-z^2)} = \sum_{k=0}^{\infty} \frac{1}{k!} z^k \sum_{k=0}^{\infty} (-1)^k z^{2k}$$

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Continuity

Differentiability

Uniqueness

Multiplication and Division

Example. Expand $\frac{e^z}{z^2+1}$ into a power series around 0.

$$\frac{e^{z}}{z^{2}+1} = e^{z} \frac{1}{1-(-z^{2})} = \sum_{k=0}^{\infty} \frac{1}{k!} z^{k} \sum_{k=0}^{\infty} (-1)^{k} z^{2k}$$
$$= \left(1+z+\frac{z^{2}}{2}+\frac{z^{3}}{6}+\frac{z^{4}}{24}+\cdots\right)$$

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Example. Expand
$$\frac{e^{z}}{z^2+1}$$
 into a power series around 0.

$$\frac{e^{z}}{z^{2}+1} = e^{z} \frac{1}{1-(-z^{2})} = \sum_{k=0}^{\infty} \frac{1}{k!} z^{k} \sum_{k=0}^{\infty} (-1)^{k} z^{2k}$$
$$= \left(1+z+\frac{z^{2}}{2}+\frac{z^{3}}{6}+\frac{z^{4}}{24}+\cdots\right) \left(1-z^{2}+z^{4}-z^{6}+\cdots\right)$$

Bernd Schröder

Example. Expand $\frac{e^z}{z^2+1}$ into a power series around 0.

$$\begin{aligned} \frac{e^{z}}{z^{2}+1} &= e^{z} \frac{1}{1-(-z^{2})} = \sum_{k=0}^{\infty} \frac{1}{k!} z^{k} \sum_{k=0}^{\infty} (-1)^{k} z^{2k} \\ &= \left(1+z+\frac{z^{2}}{2}+\frac{z^{3}}{6}+\frac{z^{4}}{24}+\cdots\right) \left(1-z^{2}+z^{4}-z^{6}+\cdots\right) \\ &= 1 \end{aligned}$$

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Example. Expand
$$\frac{e^{z}}{z^2+1}$$
 into a power series around 0.

$$\frac{e^{z}}{z^{2}+1} = e^{z} \frac{1}{1-(-z^{2})} = \sum_{k=0}^{\infty} \frac{1}{k!} z^{k} \sum_{k=0}^{\infty} (-1)^{k} z^{2k}$$
$$= \left(1+z+\frac{z^{2}}{2}+\frac{z^{3}}{6}+\frac{z^{4}}{24}+\cdots\right) \left(1-z^{2}+z^{4}-z^{6}+\cdots\right)$$

= 1 + z

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Example. Expand
$$\frac{e^{z}}{z^2+1}$$
 into a power series around 0.

$$\frac{e^{z}}{z^{2}+1} = e^{z} \frac{1}{1-(-z^{2})} = \sum_{k=0}^{\infty} \frac{1}{k!} z^{k} \sum_{k=0}^{\infty} (-1)^{k} z^{2k}$$
$$= \left(1+z+\frac{z^{2}}{2}+\frac{z^{3}}{6}+\frac{z^{4}}{24}+\cdots\right) \left(1-z^{2}+z^{4}-z^{6}+\cdots\right)$$
$$= 1+z+z^{2} \left(-1+\frac{1}{2}\right)$$

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Example. Expand
$$\frac{e^{z}}{z^2+1}$$
 into a power series around 0.

$$\frac{e^{z}}{z^{2}+1} = e^{z} \frac{1}{1-(-z^{2})} = \sum_{k=0}^{\infty} \frac{1}{k!} z^{k} \sum_{k=0}^{\infty} (-1)^{k} z^{2k}$$
$$= \left(1+z+\frac{z^{2}}{2}+\frac{z^{3}}{6}+\frac{z^{4}}{24}+\cdots\right) \left(1-z^{2}+z^{4}-z^{6}+\cdots\right)$$
$$= 1+z+z^{2} \left(-1+\frac{1}{2}\right) + z^{3} \left(-1+\frac{1}{6}\right)$$

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Example. Expand
$$\frac{e^{z}}{z^2+1}$$
 into a power series around 0.

$$\frac{e^{z}}{z^{2}+1} = e^{z} \frac{1}{1-(-z^{2})} = \sum_{k=0}^{\infty} \frac{1}{k!} z^{k} \sum_{k=0}^{\infty} (-1)^{k} z^{2k}$$
$$= \left(1+z+\frac{z^{2}}{2}+\frac{z^{3}}{6}+\frac{z^{4}}{24}+\cdots\right) \left(1-z^{2}+z^{4}-z^{6}+\cdots\right)$$
$$= 1+z+z^{2} \left(-1+\frac{1}{2}\right) + z^{3} \left(-1+\frac{1}{6}\right) + z^{4} \left(1-\frac{1}{2}+\frac{1}{24}\right)$$

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Example. Expand
$$\frac{e^{z}}{z^2+1}$$
 into a power series around 0.

$$\begin{aligned} \frac{e^z}{z^2+1} &= e^z \frac{1}{1-(-z^2)} = \sum_{k=0}^{\infty} \frac{1}{k!} z^k \sum_{k=0}^{\infty} (-1)^k z^{2k} \\ &= \left(1+z+\frac{z^2}{2}+\frac{z^3}{6}+\frac{z^4}{24}+\cdots\right) \left(1-z^2+z^4-z^6+\cdots\right) \\ &= 1+z+z^2 \left(-1+\frac{1}{2}\right) + z^3 \left(-1+\frac{1}{6}\right) + z^4 \left(1-\frac{1}{2}+\frac{1}{24}\right) + \cdots \end{aligned}$$

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Uniquenes

Multiplication and Division

Example. Expand
$$\frac{e^{z}}{z^2+1}$$
 into a power series around 0.

$$\frac{e^{z}}{z^{2}+1} = e^{z} \frac{1}{1-(-z^{2})} = \sum_{k=0}^{\infty} \frac{1}{k!} z^{k} \sum_{k=0}^{\infty} (-1)^{k} z^{2k}$$
$$= \left(1+z+\frac{z^{2}}{2}+\frac{z^{3}}{6}+\frac{z^{4}}{24}+\cdots\right) \left(1-z^{2}+z^{4}-z^{6}+\cdots\right)$$
$$= 1+z+z^{2} \left(-1+\frac{1}{2}\right) + z^{3} \left(-1+\frac{1}{6}\right) + z^{4} \left(1-\frac{1}{2}+\frac{1}{24}\right) + \cdots$$

= 1

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Uniquenes

Multiplication and Division

Example. Expand
$$\frac{e^{z}}{z^2+1}$$
 into a power series around 0.

$$\frac{e^{z}}{z^{2}+1} = e^{z} \frac{1}{1-(-z^{2})} = \sum_{k=0}^{\infty} \frac{1}{k!} z^{k} \sum_{k=0}^{\infty} (-1)^{k} z^{2k}$$
$$= \left(1+z+\frac{z^{2}}{2}+\frac{z^{3}}{6}+\frac{z^{4}}{24}+\cdots\right) \left(1-z^{2}+z^{4}-z^{6}+\cdots\right)$$
$$= 1+z+z^{2} \left(-1+\frac{1}{2}\right) + z^{3} \left(-1+\frac{1}{6}\right) + z^{4} \left(1-\frac{1}{2}+\frac{1}{24}\right) + \cdots$$

= 1 + z

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Uniquenes

Multiplication and Division

Example. Expand
$$\frac{e^{z}}{z^{2}+1}$$
 into a power series around 0.

$$\begin{aligned} \frac{e^z}{z^2+1} &= e^z \frac{1}{1-(-z^2)} = \sum_{k=0}^{\infty} \frac{1}{k!} z^k \sum_{k=0}^{\infty} (-1)^k z^{2k} \\ &= \left(1+z+\frac{z^2}{2}+\frac{z^3}{6}+\frac{z^4}{24}+\cdots\right) \left(1-z^2+z^4-z^6+\cdots\right) \\ &= 1+z+z^2 \left(-1+\frac{1}{2}\right) + z^3 \left(-1+\frac{1}{6}\right) + z^4 \left(1-\frac{1}{2}+\frac{1}{24}\right) + \cdots \\ &= 1+z-\frac{1}{2} z^2 \end{aligned}$$

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Uniquenes

Multiplication and Division

Example. Expand
$$\frac{e^z}{z^2+1}$$
 into a power series around 0.

$$\frac{e^{z}}{z^{2}+1} = e^{z} \frac{1}{1-(-z^{2})} = \sum_{k=0}^{\infty} \frac{1}{k!} z^{k} \sum_{k=0}^{\infty} (-1)^{k} z^{2k}$$

$$= \left(1+z+\frac{z^{2}}{2}+\frac{z^{3}}{6}+\frac{z^{4}}{24}+\cdots\right) \left(1-z^{2}+z^{4}-z^{6}+\cdots\right)$$

$$= 1+z+z^{2} \left(-1+\frac{1}{2}\right) + z^{3} \left(-1+\frac{1}{6}\right) + z^{4} \left(1-\frac{1}{2}+\frac{1}{24}\right) + \cdots$$

$$= 1+z-\frac{1}{2}z^{2}-\frac{5}{6}z^{3}$$

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Uniquenes

Multiplication and Division

Example. Expand
$$\frac{e^z}{z^2+1}$$
 into a power series around 0.

$$\begin{aligned} \frac{e^z}{z^2+1} &= e^z \frac{1}{1-(-z^2)} = \sum_{k=0}^{\infty} \frac{1}{k!} z^k \sum_{k=0}^{\infty} (-1)^k z^{2k} \\ &= \left(1+z+\frac{z^2}{2}+\frac{z^3}{6}+\frac{z^4}{24}+\cdots\right) \left(1-z^2+z^4-z^6+\cdots\right) \\ &= 1+z+z^2 \left(-1+\frac{1}{2}\right) + z^3 \left(-1+\frac{1}{6}\right) + z^4 \left(1-\frac{1}{2}+\frac{1}{24}\right) + \cdots \\ &= 1+z-\frac{1}{2} z^2 - \frac{5}{6} z^3 + \frac{13}{24} z^4 \end{aligned}$$

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Convergence Continuity Differentiability Uniqueness Multiplication and Division **Example.** Expand $\frac{e^z}{z^2+1}$ into a power series around 0. $\frac{e^{z}}{z^{2}+1} = e^{z} \frac{1}{1-(-z^{2})} = \sum_{k=0}^{\infty} \frac{1}{k!} z^{k} \sum_{k=0}^{\infty} (-1)^{k} z^{2k}$ $= \left(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \cdots\right) \left(1 - z^2 + z^4 - z^6 + \cdots\right)$ $= 1 + z + z^{2} \left(-1 + \frac{1}{2} \right) + z^{3} \left(-1 + \frac{1}{6} \right) + z^{4} \left(1 - \frac{1}{2} + \frac{1}{24} \right) + \cdots$ $= 1 + z - \frac{1}{2}z^2 - \frac{5}{6}z^3 + \frac{13}{24}z^4 + \cdots$

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division

Example.

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Example	• Expand $\frac{1}{\cos(z)}$	– into a power ser	ries around 0.	
Because of	$\cos(z) = 1 - \frac{\dot{x^2}}{2}$	$+\frac{x^4}{24}-\frac{x^6}{720}+\cdots$		



ConvergenceContinuityDifferentiabilityUniquenessMultiplication and DivisionExample. Expand $\frac{1}{\cos(z)}$ into a power series around 0.Because $\cos(z) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots$ we can obtain the first few
terms of the expansion by generalizing the division of polynomials.
So we will not get the full expansion.

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Example. Expand $\frac{1}{\cos(z)}$ into a power series around 0. Because $\cos(z) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots$ we can obtain the first few terms of the expansion by generalizing the division of polynomials. So we will not get the full expansion. The algorithm will show that getting the full expansion by straightforward division would be a bit much to hope for.

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ConvergenceContinuityDifferentiabilityUniquenessMultiplication and DivisionExample. Expand
$$\frac{1}{\cos(z)}$$
 into a power series around 0. $1 - \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720} + \cdots$ 1



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Convergence	Continuity	Differentiability	Uniq	ueness	Multiplication and Division
Example	e. Expand $\frac{1}{\cos(1)}$	$\frac{1}{z}$ into a pov	ver series o	around 0.	
		1 +	$\frac{z^2}{2}$		
$1 - \frac{z^2}{2}$	$+ \frac{z^4}{24} - \frac{z^6}{720}$	+ … 1			
		(-) 1 -	$\frac{z^2}{2} + \frac{z}{2}$	$\frac{z^4}{24}$ - $\frac{z^6}{720}$	+ …
			$\frac{z^2}{2} - \frac{z}{2}$	$\frac{z^4}{24}$ + $\frac{z^6}{720}$	+ …
		(-)	$\frac{z^2}{2}$ –	$\frac{z^4}{4}$ + $\frac{z^6}{48}$	$+ \cdots$

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Convergence	Continuity	Differentiability		Uni	quenes			Multiplication :	and Division
Example	e. Expand $\frac{1}{\cos}$	$\frac{1}{(z)}$ into a po	wer s	eries	aroi	und 0).		
		1 +	$-\frac{z^2}{2}$						
$1 - \frac{z^2}{2}$	$+ \frac{z^4}{24} - \frac{z^6}{720}$	+ … 1							
		(-) 1 -	$-\frac{z^2}{2}$	+	$\frac{z^4}{24}$	-	$\frac{z^6}{720}$	+ …	
			$\frac{z^2}{2}$	_	$\frac{z^4}{24}$	+	$\frac{z^6}{720}$	+ …	
		(-)	$\frac{z^2}{2}$	_	$\frac{z^4}{4}$	+	$\frac{z^6}{48}$	+ …	

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Convergence	Continuity	Differentia	bility		Ur	iquenes			Multiplication	and Division
Exampl	e. Expand $\frac{1}{\cos}$	$\frac{1}{(z)}$ into a	рои	ver s	eries	s aro	und	0.		
		1	+	$\frac{z^2}{2}$						
$1 - \frac{z^2}{2}$	$+ \frac{z^4}{24} - \frac{z^6}{720}$	$\overline{0} + \cdots \boxed{1}$								
		(-) 1	_	$\frac{z^2}{2}$	+	$\frac{z^4}{24}$	_	$\frac{z^{6}}{720}$	$+ \cdots$	
				$\frac{z^2}{2}$	_	$\frac{z^4}{24}$	+	$\frac{z^6}{720}$	+ …	
		(-)		$\frac{z^2}{2}$	_	$\frac{z^4}{4}$	+	$\frac{z^6}{48}$	+ …	
						$\frac{5z^4}{24}$	_	$\frac{7z^6}{360}$	+ …	

Convergence	Continuity	Differentia	bility		Un	iquenes			Multiplication	and Divisior
Example	e. Expand $\frac{1}{\cos^2}$	$\frac{1}{(z)}$ into a	ром	er s	eries	s aro	und	0.		
		1	+	$\frac{z^2}{2}$	+	$\frac{5z^4}{24}$				
$1 - \frac{z^2}{2}$	$+ \frac{z^4}{24} - \frac{z^6}{720}$	$_{5} + \cdots 1$								
		(-) 1	_	$\frac{z^2}{2}$	+	$\frac{z^4}{24}$	_	$\frac{z^{6}}{720}$	$+ \cdots$	
				$\frac{z^2}{2}$	_	$\frac{z^4}{24}$	+	$\frac{z^{6}}{720}$	+ …	
		(-)		$\frac{z^2}{2}$	_	$\frac{z^4}{4}$	+	$\frac{z^6}{48}$	+ …	
						$\frac{5z^4}{24}$	_	$\frac{7z^6}{360}$	+ …	

Convergence	Continuity	Differ	entiab	ility		Un	niquenes			Multiplication	and Divisior
Examp	le. Expand $\frac{1}{\cos^2}$	$\frac{1}{s(z)}$ into	o a j	pow	er s	eries	s aro	und	0.		
			1	+	$\frac{z^2}{2}$	+	$\frac{5z^4}{24}$				
$1 - \frac{z^2}{2}$	$\frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720}$	$\frac{1}{2} + \cdots$	1								
		(-)	1	_	$\frac{z^2}{2}$	+	$\frac{z^4}{24}$	_	$\frac{z^{6}}{720}$	+ …	
					$\frac{z^2}{2}$	_	$\frac{z^4}{24}$	+	$\frac{z^{6}}{720}$	+ …	
		(-)			$\frac{z^2}{2}$	_	$\frac{z^4}{4}$	+	$\frac{z^6}{48}$	+ …	
		-					$\frac{5z^4}{24}$	_	$\frac{7z^6}{360}$	+ …	
		(-)					$\frac{5z^4}{24}$	_	$\frac{5z^{6}}{48}$	+ ···	

Convergence	Continuity	Differentia	bility		Uı	niquenes			Multiplication a	nd Division
Examp	le. Expand $\frac{1}{\cos}$	$\frac{1}{(z)}$ into a	рои	ver s	erie	s aro	und	0.		
		1	+	$\frac{z^2}{2}$	+	$\frac{5z^4}{24}$				
$1 - \frac{z^2}{2}$	$\frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720}$	$_{5} + \cdots 1$								
		(-) 1	_	$\frac{z^2}{2}$	+	$\frac{z^4}{24}$	_	$\frac{z^{6}}{720}$	+ …	
				$\frac{z^2}{2}$	_	$\frac{z^4}{24}$	+	$\frac{z^6}{720}$	+ …	
		(-)		$\frac{z^2}{2}$	_	$\frac{z^4}{4}$	+	$\frac{z^{6}}{48}$	+ …	
						$\frac{5z^4}{24}$	_	$\frac{7z^6}{360}$	+ …	
		(-)				$\frac{5z^4}{24}$	_	$\frac{5z^{6}}{48}$	$+ \cdots$	

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Convergence	Continuity	Differer	ntiability		Uı	niquenes			Multiplication a	nd Divisio
Examp	le. Expand $\frac{1}{\cos^2}$	$\frac{1}{(z)}$ into	a pov	ver s	erie	s aro	und	0.		
			1 +	$\frac{z^2}{2}$	+	$\frac{5z^4}{24}$				
$1 - \frac{z^2}{2}$	$\frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720}$	$_{5} + \cdots$	1							
		(-)	1 –	$\frac{z^2}{2}$	+	$\frac{z^4}{24}$	_	$\frac{z^{6}}{720}$	+ …	
				$\frac{z^2}{2}$	_	$\frac{z^4}{24}$	+	$\frac{z^{6}}{720}$	+ …	
		(-)		$\frac{z^2}{2}$	_	$\frac{z^4}{4}$	+	$\frac{z^6}{48}$	+ …	
						$\frac{5z^4}{24}$	_	$\frac{7z^{6}}{360}$	+ …	
		(-)				$\frac{5z^4}{24}$	_	$\frac{5z^{6}}{48}$	+ …	
								$\frac{61z^6}{720}$	+ …	

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Convergence	Continuity	Differ	entiab	ility		Uı	iquenes			Multipli	cation a	nd Divisio
Exampl	e. Expand $\frac{1}{\cos^2}$	$\frac{1}{(z)}$ into	o a	pow	er s	erie	s aro	und	0.			
			1	+	$\frac{z^2}{2}$	+	$\frac{5z^4}{24}$	+	$\tfrac{61z^6}{720}$	+		
$1 - \frac{z^2}{2}$	$\frac{z^4}{24} + \frac{z^4}{24} - \frac{z^6}{720}$	$\frac{1}{5} + \cdots$	1									
		(-)	1	_	$\frac{z^2}{2}$	+	$\frac{z^4}{24}$	_	$\frac{z^{6}}{720}$	+		
					$\frac{z^2}{2}$	_	$\frac{z^4}{24}$	+	$\frac{z^6}{720}$	+		
		(-)			$\frac{z^2}{2}$	_	$\frac{z^4}{4}$	+	$\frac{z^6}{48}$	+		
							$\frac{5z^4}{24}$	_	$\frac{7z^6}{360}$	+		
		(-)					$\frac{5z^4}{24}$	_	$\frac{5z^{6}}{48}$	+		
									$\frac{61z^6}{720}$	+		

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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division		
Graphica Blue)	l Double Cl	heck (Polynor	nial: Red, 1	$/\cos(x)$:		

Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division
Graphical Blue)	Double C	Check (Polyno)	mial: Red, 1	$1/\cos(x)$:



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Convergence	Continuity	Differentiability	Uniqueness	Multiplication and Division

Graphical Double Check (Polynomial: Red, $1/\cos(x)$: Blue)



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