The Analysis of Power Series

Bernd Schröder

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3. It turns out that functions defined by power series are very well behaved.

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2. Whereas so far we had a function and found from it the power series, now we start with a power series and want to know the properties of the function it defines.
3. It turns out that functions defined by power series are very well behaved.
4. But to prove the requisite results, we first must more closely investigate the convergence of power series.

Theorem.

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Recall that absolute convergence meant that $\sum_{n=0}^{\infty} \left| a_n(z-z_0)^n \right|$ converges.

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$$\sum_{n=N+1}^{\infty} \left|a_n(z - z_0)^n\right| = \sum_{n=N+1}^{\infty} \left|a_n(z_1 - z_0)^n\right| \left|\frac{z - z_0}{z_1 - z_0}\right|^n$$

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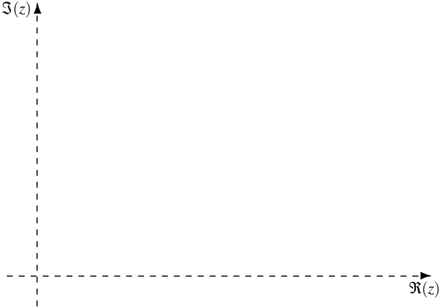

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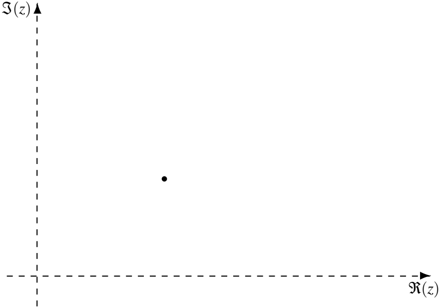

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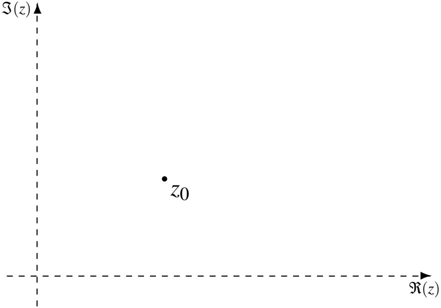

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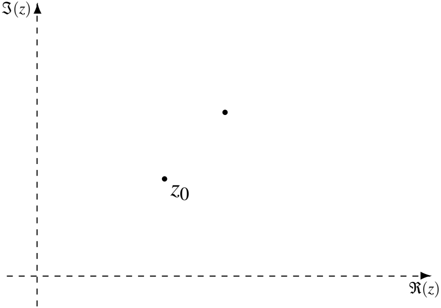

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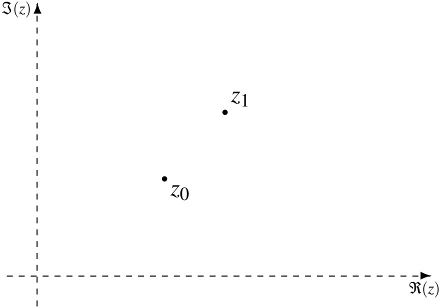

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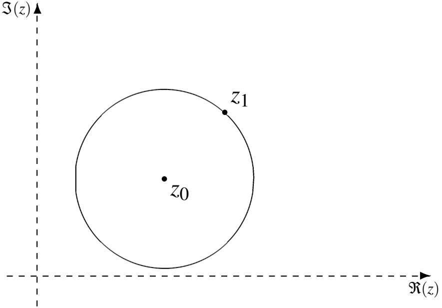

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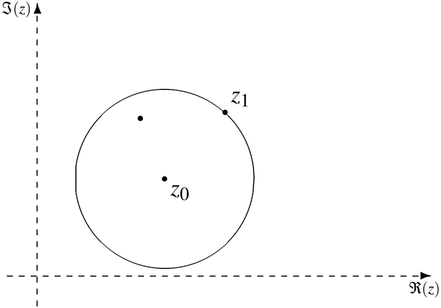

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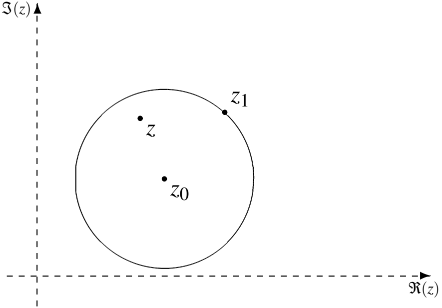

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3. There are no general theorems about what happens when $|z-z_0|=c$.
4. And problems in the complex plane can influence the behavior of functions on the real line.

$f(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} \left(-x^2\right)^n$ Has Radius of Convergence 1

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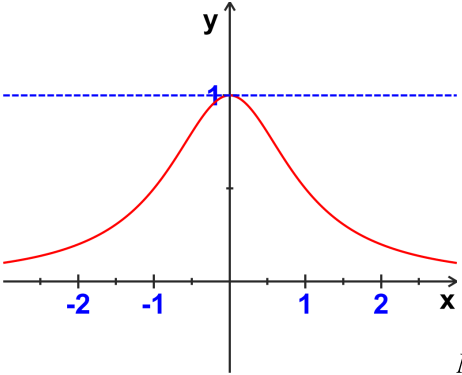

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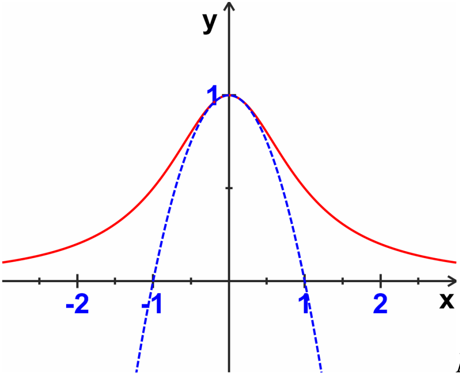

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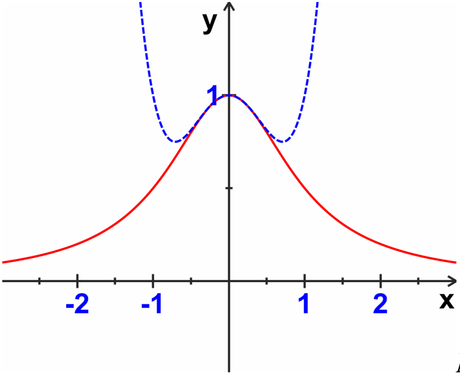

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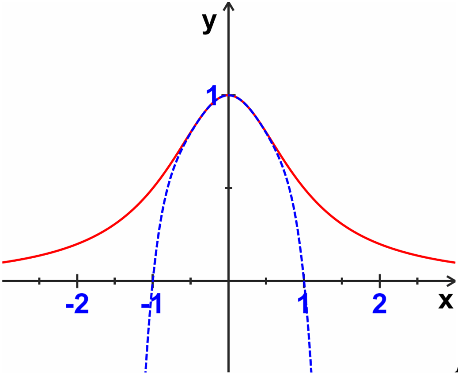

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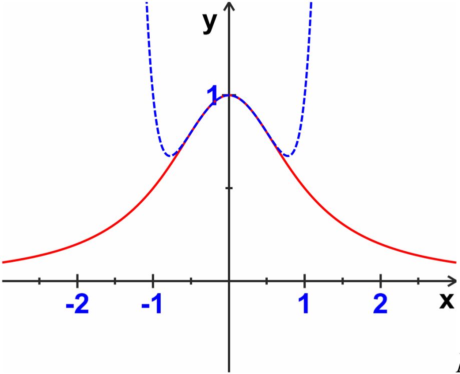

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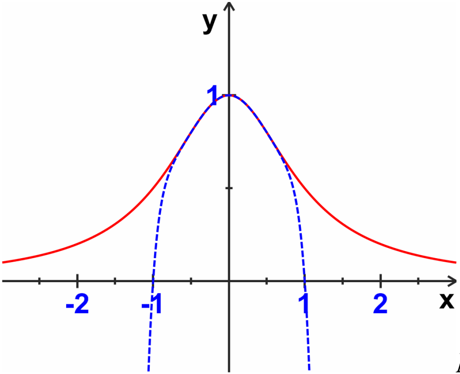

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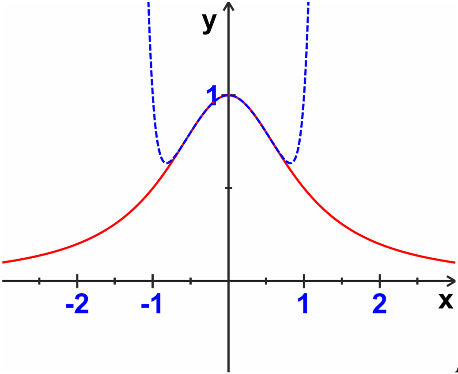

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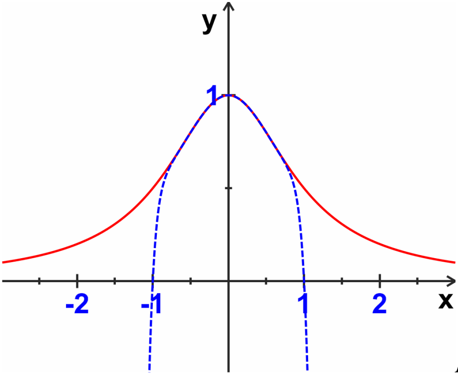

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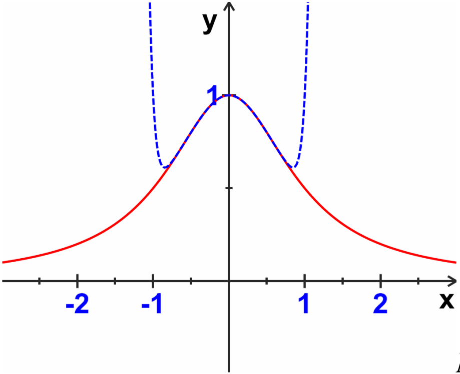

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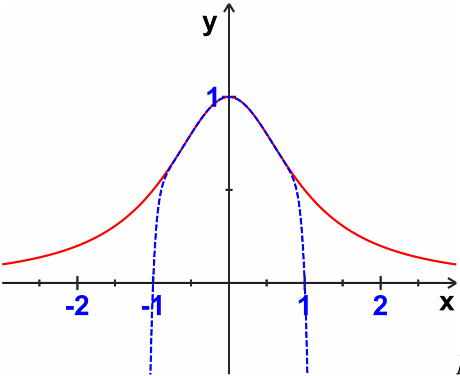

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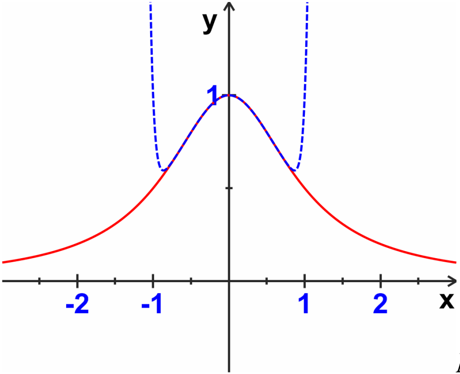

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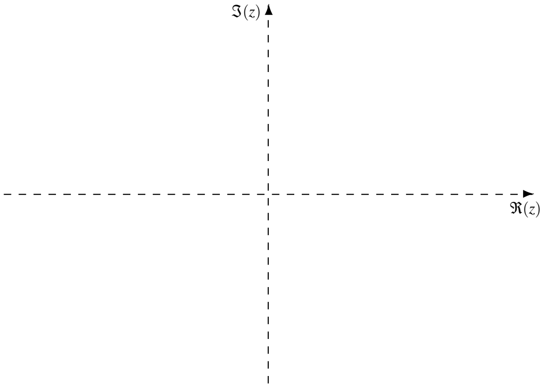

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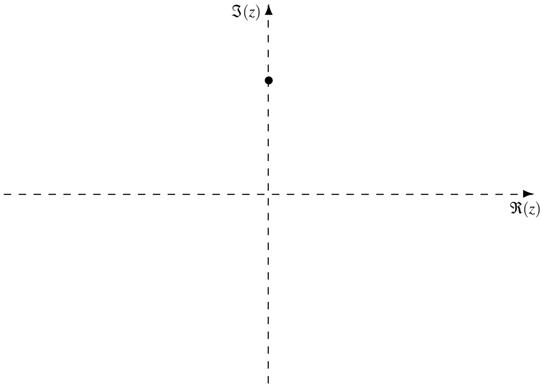

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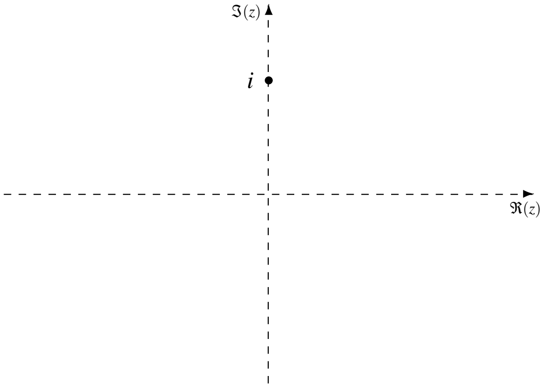

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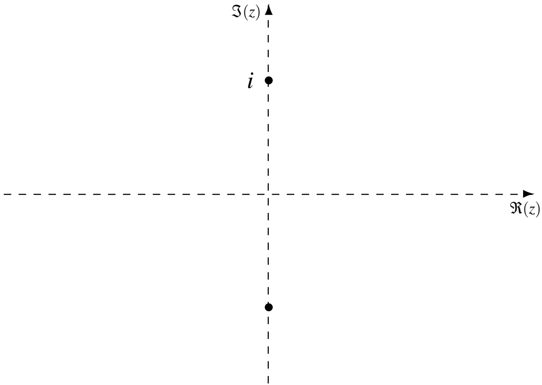

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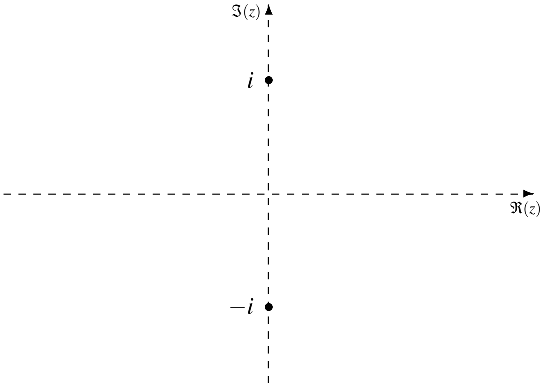

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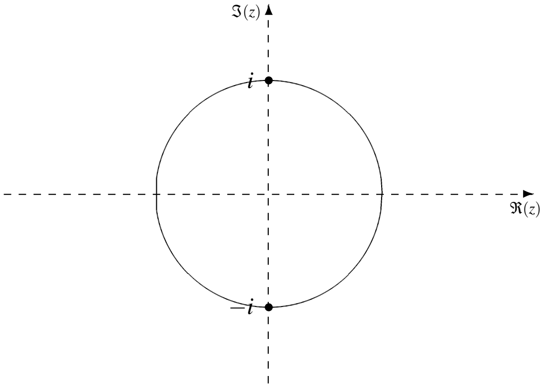

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2. In particular, if $0 < r < R$, then the remainder can be bounded uniformly for all z so that $|z-z_0| \leq r$.
3. That, in turn, means that on such sub-disks $|z-z_0| \leq r$ there is a uniform minimum speed of convergence.

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Proof (Visualization).

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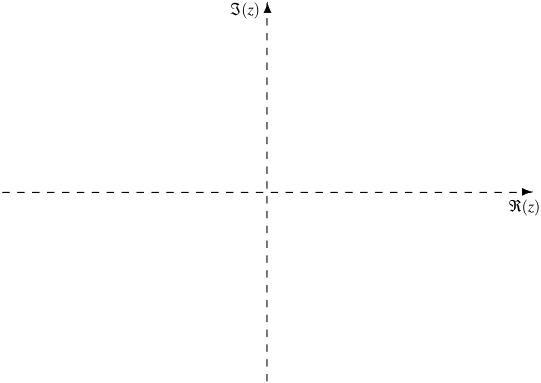

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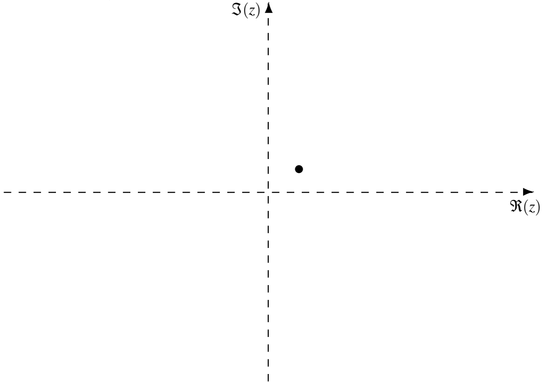

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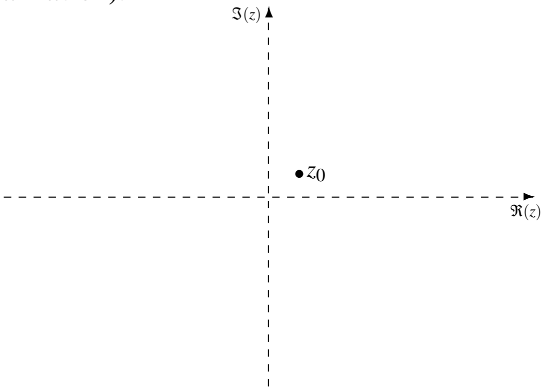

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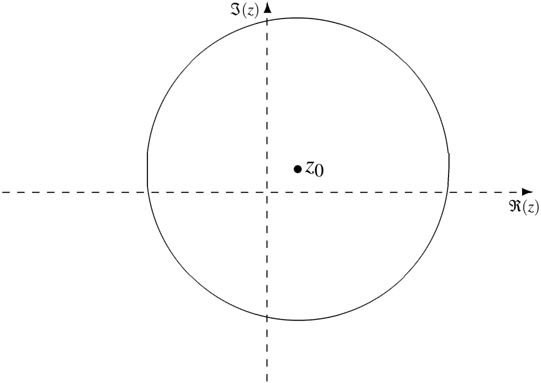

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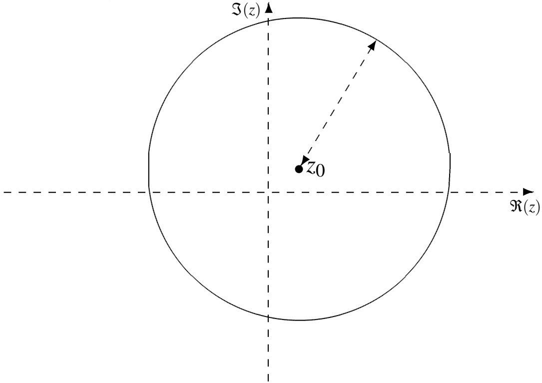

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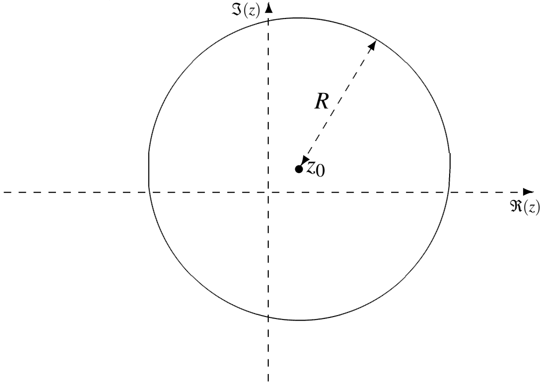

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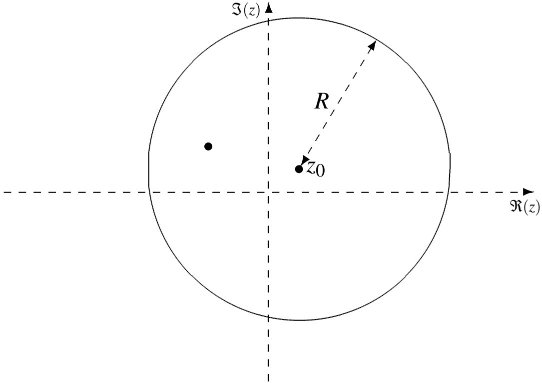

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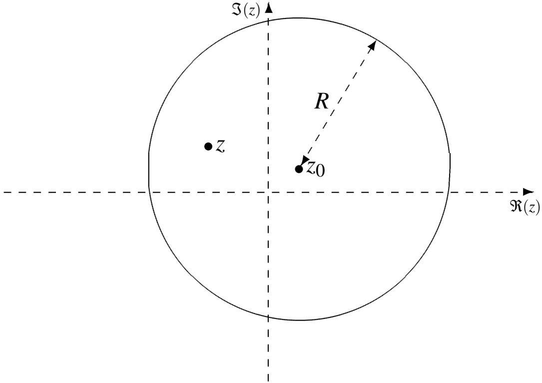

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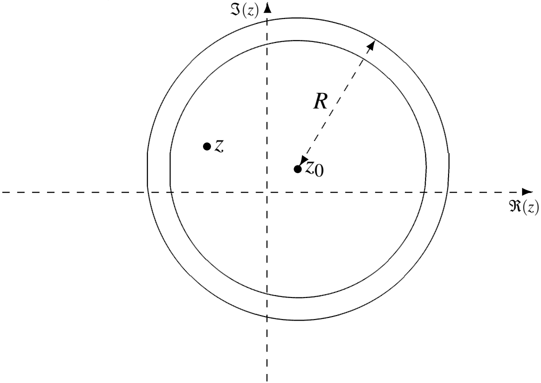

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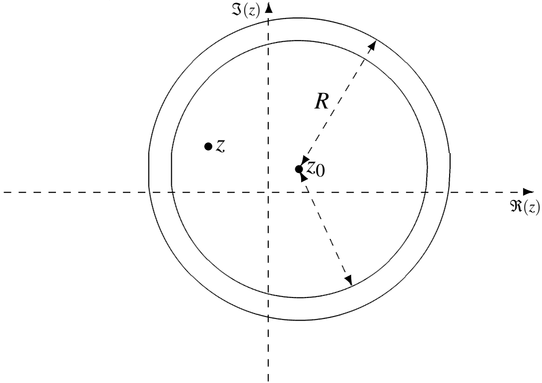

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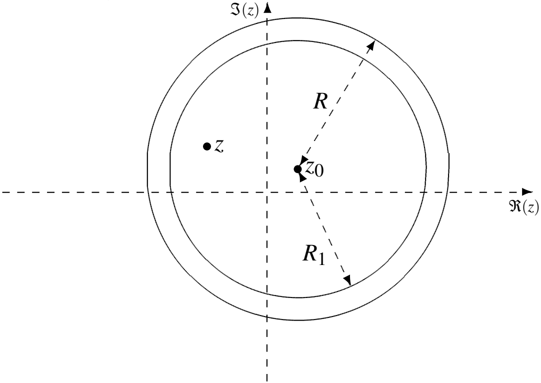

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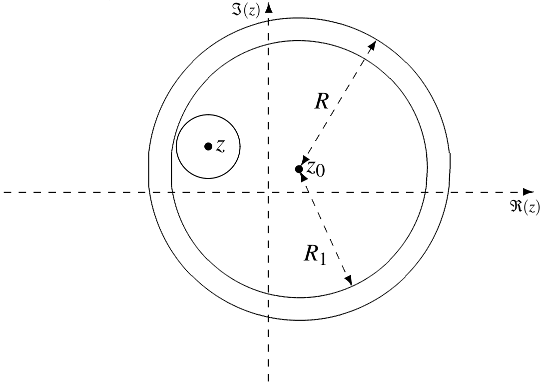

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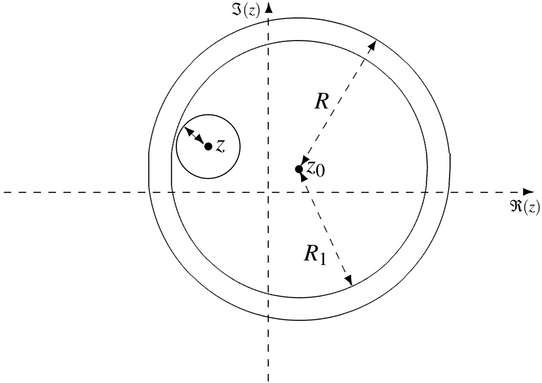

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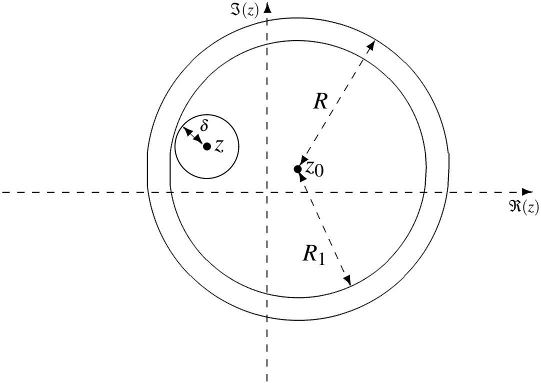

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$$\int_C g(z)f(z)\,dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z-z_0)^n\,dz.$$

Visualization.

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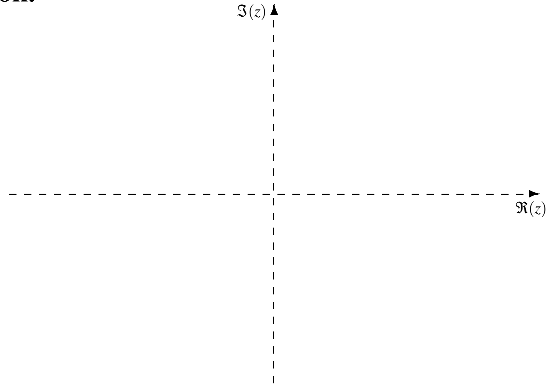
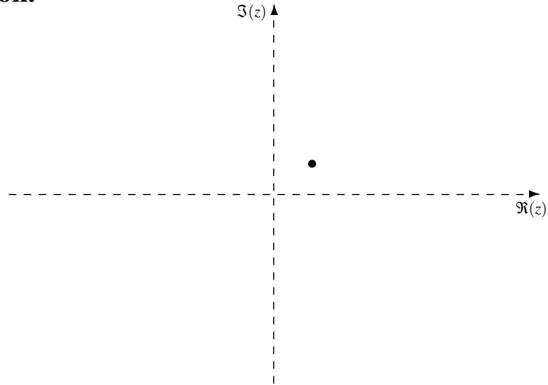
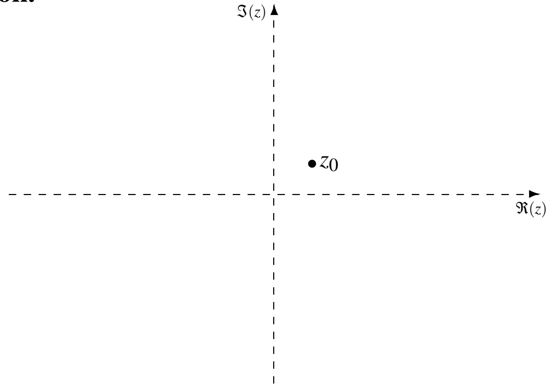


Visualization.



Visualization.



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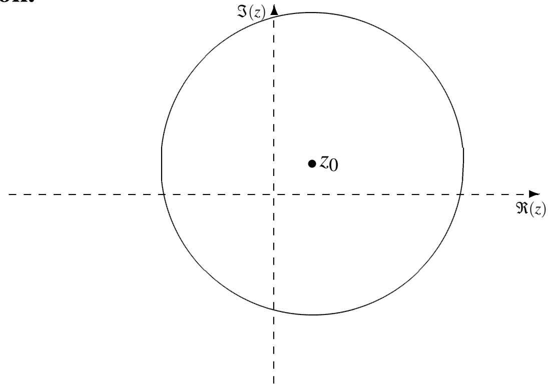

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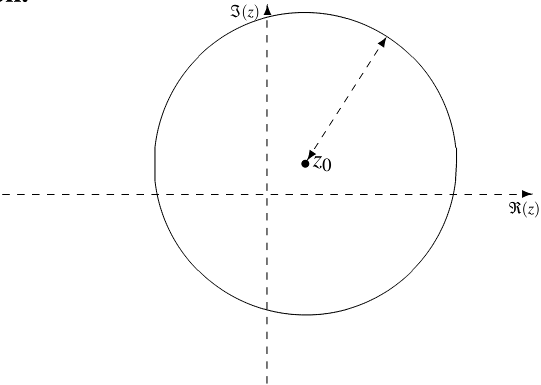

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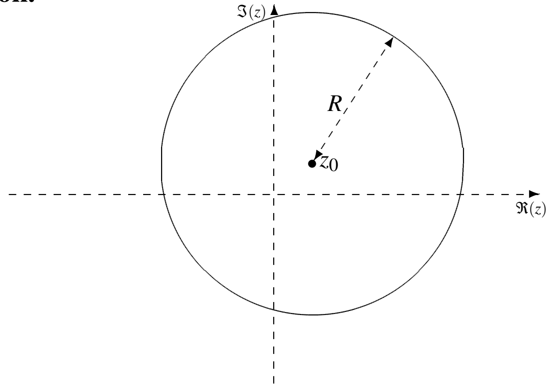

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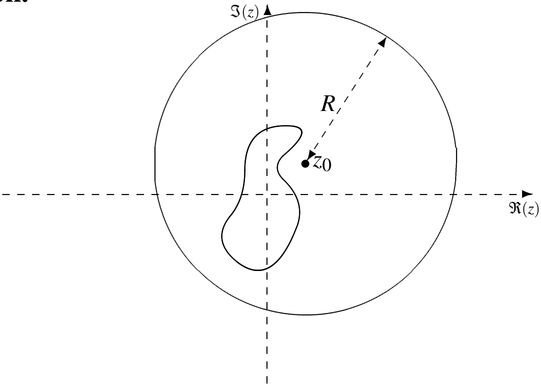

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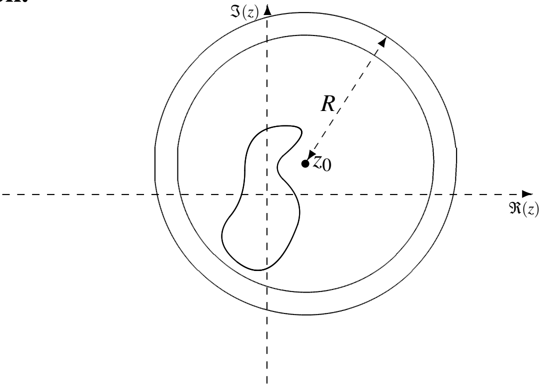

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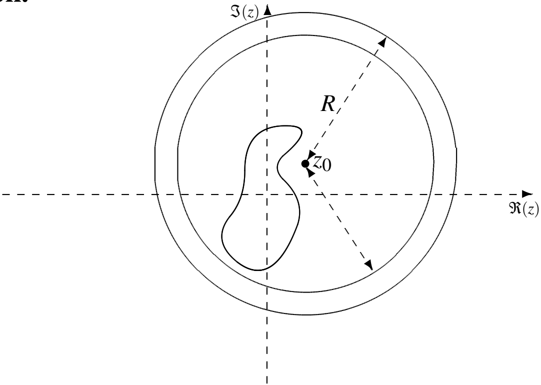

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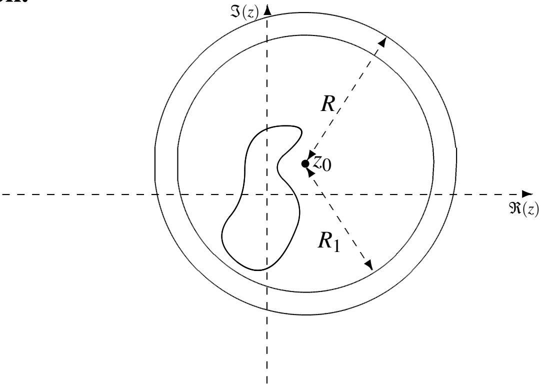

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$$\begin{aligned}\int_C g(z)f(z)\,dz &= \int_C g(z)\sum_{n=0}^{\infty} a_n(z-z_0)^n\,dz \\ &= \int_C g(z)\left(\sum_{n=0}^{N} a_n(z-z_0)^n + \sum_{n=N+1}^{\infty} a_n(z-z_0)^n\right)\,dz\end{aligned}$$

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&= \sum_{n=0}^{N} a_n \int_C g(z)(z-z_0)^n\,dz + \int_C g(z) \sum_{n=N+1}^{\infty} a_n(z-z_0)^n\,dz
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$$f'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}$$

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$$
\begin{aligned}
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&= \sum_{k=0}^{n}\binom{n}{k}\left(g^{(k+1)}(z)h^{(n-k)}(z)+g^{(k)}(z)h^{(n-k+1)}(z)\right)
\end{aligned}
$$

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&= \sum_{k=0}^{n}\binom{n}{k} g^{(k+1)}(z)h^{(n-k)}(z)+\sum_{k=0}^{n}\binom{n}{k} g^{(k)}(z)h^{(n-k+1)}(z)
\end{aligned}
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Induction step, $n \to n+1$.

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&= \binom{n}{n} g^{(n+1)}(z)h^{(0)}(z)+\sum_{j=1}^{n}\binom{n}{j-1} g^{(j)}(z)h^{(n+1-j)}(z) \\
&\quad +\sum_{k=1}^{n}\binom{n}{k} g^{(k)}(z)h^{(n+1-k)}(z)+\binom{n}{0} g^{(0)}(z)h^{(n+1)}(z)
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Example.

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&= 1+z+z^2\left(-1+\frac{1}{2}\right)+z^3\left(-1+\frac{1}{6}\right)+z^4\left(1-\frac{1}{2}+\frac{1}{24}\right)+\cdots \\
&= 1+z-\frac{1}{2}z^2-\frac{5}{6}z^3
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Example. *Expand* $\frac{e^z}{z^2+1}$ *into a power series around* 0.

$$\begin{aligned}
\frac{e^z}{z^2+1} &= e^z \frac{1}{1-(-z^2)} = \sum_{k=0}^{\infty} \frac{1}{k!} z^k \sum_{k=0}^{\infty} (-1)^k z^{2k} \\
&= \left(1+z+\frac{z^2}{2}+\frac{z^3}{6}+\frac{z^4}{24}+\cdots\right)\left(1-z^2+z^4-z^6+\cdots\right) \\
&= 1+z+z^2\left(-1+\frac{1}{2}\right)+z^3\left(-1+\frac{1}{6}\right)+z^4\left(1-\frac{1}{2}+\frac{1}{24}\right)+\cdots \\
&= 1+z-\frac{1}{2}z^2-\frac{5}{6}z^3+\frac{13}{24}z^4
\end{aligned}$$

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&= 1+z+z^2\left(-1+\frac{1}{2}\right)+z^3\left(-1+\frac{1}{6}\right)+z^4\left(1-\frac{1}{2}+\frac{1}{24}\right)+\cdots \\
&= 1+z-\frac{1}{2}z^2-\frac{5}{6}z^3+\frac{13}{24}z^4+\cdots
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Example.

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Because $\cos(z) = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} - \dfrac{x^6}{720} + \cdots$

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Because $\cos(z) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots$ we can obtain the first few terms of the expansion by generalizing the division of polynomials. So we will not get the full expansion.

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Because $\cos(z) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots$ we can obtain the first few terms of the expansion by generalizing the division of polynomials. So we will not get the full expansion. The algorithm will show that getting the full expansion by straightforward division would be a bit much to hope for.

Example. *Expand* $\dfrac{1}{\cos(z)}$ *into a power series around* 0.

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$$1 - \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720} + \cdots \Big| \; 1$$

Example. *Expand* $\dfrac{1}{\cos(z)}$ *into a power series around* 0.

$$\begin{array}{l|l} & 1 \\ \hline 1 - \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720} + \cdots & 1 \end{array}$$

Example. *Expand* $\frac{1}{\cos(z)}$ *into a power series around* 0.

$$\begin{array}{l|l} & 1 \\ \hline 1 - \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720} + \cdots & 1 \\ (-) & \end{array}$$

Example. *Expand* $\frac{1}{\cos(z)}$ *into a power series around* 0.

$$
\begin{array}{r|l}
 & 1 \\
\hline
1 - \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720} + \cdots & 1 \\
(-) & 1 - \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720} + \cdots
\end{array}
$$

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$$\begin{array}{r|l} & 1 \\ \hline 1 - \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720} + \cdots & 1 \\ (-) & 1 - \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720} + \cdots \\ \hline & \frac{z^2}{2} - \frac{z^4}{24} + \frac{z^6}{720} + \cdots \end{array}$$

Example. *Expand* $\frac{1}{\cos(z)}$ *into a power series around* 0.

$$
\begin{array}{r|l}
 & 1 + \frac{z^2}{2} \\
\hline
1 - \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720} + \cdots & 1 \\
(-) & 1 - \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720} + \cdots \\
\hline
 & \frac{z^2}{2} - \frac{z^4}{24} + \frac{z^6}{720} + \cdots
\end{array}
$$

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$$
\begin{array}{r|l}
 & 1 + \frac{z^2}{2} \\
\hline
1 - \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720} + \cdots & 1 \\
(-) & 1 - \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720} + \cdots \\
\hline
 & \frac{z^2}{2} - \frac{z^4}{24} + \frac{z^6}{720} + \cdots \\
(-) & \frac{z^2}{2} - \frac{z^4}{4} + \frac{z^6}{48} + \cdots
\end{array}
$$

Example. *Expand* $\frac{1}{\cos(z)}$ *into a power series around* 0.

$$
\begin{array}{r|l}
 & 1 + \frac{z^2}{2} \\
\hline
1 - \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720} + \cdots & 1 \\
(-) & 1 - \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720} + \cdots \\
\hline
 & \frac{z^2}{2} - \frac{z^4}{24} + \frac{z^6}{720} + \cdots \\
(-) & \frac{z^2}{2} - \frac{z^4}{4} + \frac{z^6}{48} + \cdots \\
\hline
\end{array}
$$

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$$
\begin{array}{r|l}
 & 1 + \frac{z^2}{2} \\
\hline
1 - \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720} + \cdots & 1 \\
(-) & 1 - \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720} + \cdots \\
\cline{2-2}
 & \frac{z^2}{2} - \frac{z^4}{24} + \frac{z^6}{720} + \cdots \\
(-) & \frac{z^2}{2} - \frac{z^4}{4} + \frac{z^6}{48} + \cdots \\
\cline{2-2}
 & \frac{5z^4}{24} - \frac{7z^6}{360} + \cdots
\end{array}
$$

Example. *Expand* $\frac{1}{\cos(z)}$ *into a power series around* 0.

$$
\begin{array}{r|l}
 & 1 + \frac{z^2}{2} + \frac{5z^4}{24} \\
\hline
1 - \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720} + \cdots & 1 \\
(-) & 1 - \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720} + \cdots \\
\cline{2-2}
 & \quad \frac{z^2}{2} - \frac{z^4}{24} + \frac{z^6}{720} + \cdots \\
(-) & \quad \frac{z^2}{2} - \frac{z^4}{4} + \frac{z^6}{48} + \cdots \\
\cline{2-2}
 & \qquad \frac{5z^4}{24} - \frac{7z^6}{360} + \cdots
\end{array}
$$

Example. *Expand* $\dfrac{1}{\cos(z)}$ *into a power series around* 0.

$$
\begin{array}{rrrrrrrrrrrr}
 & & & & & & 1 & + & \frac{z^2}{2} & + & \frac{5z^4}{24} & \\
\cline{7-12}
1 - \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720} + \cdots & \multicolumn{5}{r|}{} & 1 & & & & & \\
 & & & & & (-) & 1 & - & \frac{z^2}{2} & + & \frac{z^4}{24} & - \frac{z^6}{720} + \cdots \\
\cline{7-12}
 & & & & & & & & \frac{z^2}{2} & - & \frac{z^4}{24} & + \frac{z^6}{720} + \cdots \\
 & & & & & (-) & & & \frac{z^2}{2} & - & \frac{z^4}{4} & + \frac{z^6}{48} + \cdots \\
\cline{7-12}
 & & & & & & & & & & \frac{5z^4}{24} & - \frac{7z^6}{360} + \cdots \\
 & & & & & (-) & & & & & \frac{5z^4}{24} & - \frac{5z^6}{48} + \cdots
\end{array}
$$

Example. *Expand* $\dfrac{1}{\cos(z)}$ *into a power series around* 0.

$$1+\frac{z^2}{2}+\frac{5z^4}{24}$$

$$1-\frac{z^2}{2}+\frac{z^4}{24}-\frac{z^6}{720}+\cdots \;\Big|\; 1$$

$$(-)\quad 1-\frac{z^2}{2}+\frac{z^4}{24}-\frac{z^6}{720}+\cdots$$

$$\frac{z^2}{2}-\frac{z^4}{24}+\frac{z^6}{720}+\cdots$$

$$(-)\quad \frac{z^2}{2}-\frac{z^4}{4}+\frac{z^6}{48}+\cdots$$

$$\frac{5z^4}{24}-\frac{7z^6}{360}+\cdots$$

$$(-)\quad \frac{5z^4}{24}-\frac{5z^6}{48}+\cdots$$

Example. *Expand* $\frac{1}{\cos(z)}$ *into a power series around* 0.

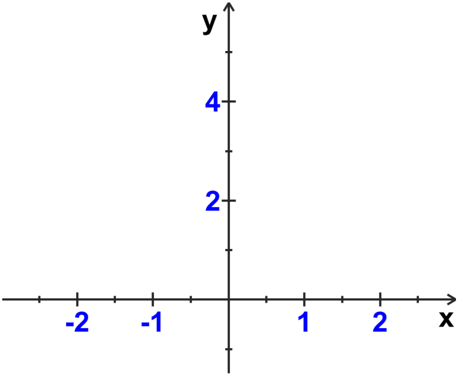
$$
\begin{array}{rrrrrrr}
 & 1 & + \frac{z^2}{2} & + \frac{5z^4}{24} & & & \\
\cline{2-7}
1 - \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720} + \cdots & \multicolumn{1}{|r}{1} & & & & & \\
(-) & 1 & - \frac{z^2}{2} & + \frac{z^4}{24} & - \frac{z^6}{720} & + \cdots & \\
\cline{2-7}
 & & \frac{z^2}{2} & - \frac{z^4}{24} & + \frac{z^6}{720} & + \cdots & \\
(-) & & \frac{z^2}{2} & - \frac{z^4}{4} & + \frac{z^6}{48} & + \cdots & \\
\cline{2-7}
 & & & \frac{5z^4}{24} & - \frac{7z^6}{360} & + \cdots & \\
(-) & & & \frac{5z^4}{24} & - \frac{5z^6}{48} & + \cdots & \\
\cline{2-7}
 & & & & \frac{61z^6}{720} & + \cdots &
\end{array}
$$

Example. *Expand* $\dfrac{1}{\cos(z)}$ *into a power series around* 0.

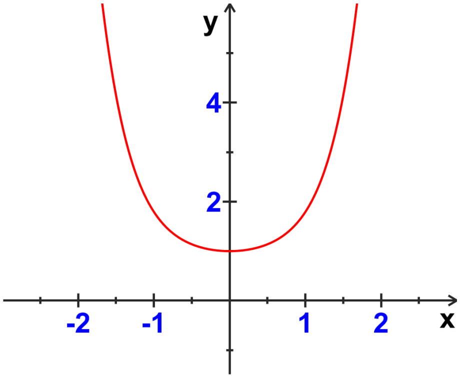
$$
\begin{array}{r|lllllllll}
\multicolumn{1}{r}{} & 1 & + & \frac{z^2}{2} & + & \frac{5z^4}{24} & + & \frac{61z^6}{720} & + & \cdots \\
\cline{2-10}
1-\frac{z^2}{2}+\frac{z^4}{24}-\frac{z^6}{720}+\cdots & 1 & & & & & & & & \\
\multicolumn{1}{r}{(-)} & 1 & - & \frac{z^2}{2} & + & \frac{z^4}{24} & - & \frac{z^6}{720} & + & \cdots \\
\cline{2-10}
\multicolumn{1}{r}{} & & & \frac{z^2}{2} & - & \frac{z^4}{24} & + & \frac{z^6}{720} & + & \cdots \\
\multicolumn{1}{r}{(-)} & & & \frac{z^2}{2} & - & \frac{z^4}{4} & + & \frac{z^6}{48} & + & \cdots \\
\cline{2-10}
\multicolumn{1}{r}{} & & & & & \frac{5z^4}{24} & - & \frac{7z^6}{360} & + & \cdots \\
\multicolumn{1}{r}{(-)} & & & & & \frac{5z^4}{24} & - & \frac{5z^6}{48} & + & \cdots \\
\cline{2-10}
\multicolumn{1}{r}{} & & & & & & & \frac{61z^6}{720} & + & \cdots
\end{array}
$$

Graphical Double Check (Polynomial: Red, $1/\cos(x)$: Blue)

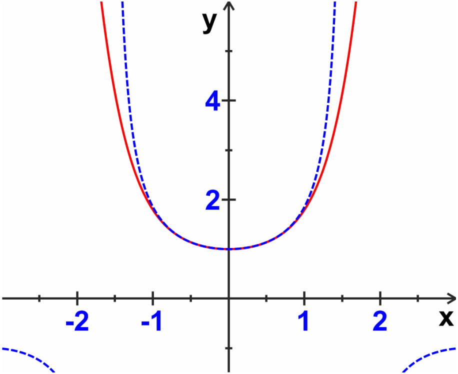
Graphical Double Check (Polynomial: Red, $1/\cos(x)$: Blue)



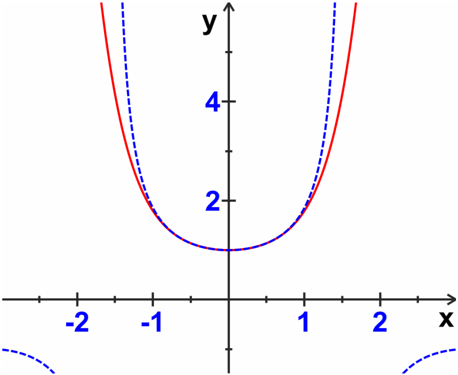
Graphical Double Check (Polynomial: Red, $1/\cos(x)$: Blue)



Graphical Double Check (Polynomial: Red, $1/\cos(x)$: Blue)



Graphical Double Check (Polynomial: Red, $1/\cos(x)$: Blue)



YEAH!