Asymptotic Refinement of the Berry-Esseen Constant

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December, 2002

Abstract

For sum of independent and identically distributed (i.i.d.) random variables $\{X_i\}_{i=1}^n$, the Berry-Esseen theorem states that

$$\sup_{y\in\Re}\left|\Pr\left\{\frac{1}{s_n}(X_1+X_2+\cdots+X_n)\le y\right\}-\Phi(y)\right|\le C\frac{\rho}{\sigma^3\sqrt{n}},$$

where σ^2 and ρ are respectively the variance and the absolute third moment of the parent distribution, $\Phi(\cdot)$ is the unit normal cumulative distribution function, and C is an absolute constant. In this work, we re-examined the above inequality by following similar procedure as in [3, Sec.XVI.5, Thm. 1]. Instead of targeting an *absolute constant* C, we sought for a *sample-size-dependent coefficient* C_n such that

$$\left|\Pr\left[\frac{1}{\sigma\sqrt{n}}(X_1+\cdots+X_n)\le y\right]-\Phi(y)\right|\le C_n\ \frac{\rho}{\sigma^3\sqrt{n}} \tag{1}$$

hold for every sample size n. Based on the new standpoint, we found that C_n can be made smaller than Shiganov's constant 0.7655 when $n\ge 65$, and can be decreased by further increasing sample size n. As n approaches infinity, the Berry-Esseen constant can be asymptotically improved down to 0.7164.

I. Introduction

THe Berry-Esseen theorem [3, Sec.XVI.5] states that the distribution of the sum of independent and identically distributed zero-mean random variables $\{X_i\}_{i=1}^n$, normalized by its standard deviation, differ from the unit Gaussian distribution by at most $C\,\rho/(\sigma^2\sqrt{n})$, where σ^2 and ρ are respectively the variance and the absolute third moment of the parent distribution, and C is a distribution-independent absolute constant. Specifically, for every $y\in\Re$,

$$\left|\Pr\left[\frac{1}{\sigma\sqrt{n}}(X_1+\cdots+X_n)\le y\right]-\Phi(y)\right|\le C\ \frac{\rho}{\sigma^3\sqrt{n}}, \tag{2}$$

where $\Phi(\cdot)$ represents the unit Gaussian cumulative distribution function (cdf). The remarkable aspect of this theorem is that the upper bound depends only on the variance and the absolute third moment, and therefore, can provide a good probability estimate through the first three moments. A typical estimate of the absolute constant is three [3, Sec.XVI.5, Thm. 1]. Beek sharpened the constant to 0.7975 in 1972 [1]. Later, Shiganov improved the constant down to 0.7655 [5]. Shiganov's result is generally considered to be the best result obtained thus far [4].

In this work, we re-examined inequality (2) by following similar procedure as in [3, Sec.XVI.5, Thm. 1]. We however took different view during our derivation. Instead of targeting an *absolute constant* C, we sought for a *sample-size-dependent coefficient* C_n such that

$$\left|\Pr\left[\frac{1}{\sigma\sqrt{n}}(X_1+\cdots+X_n)\le y\right]-\Phi(y)\right|\le C_n\ \frac{\rho}{\sigma^3\sqrt{n}} \tag{3}$$

hold for every sample size n. This was motivated by observing the numerical behavior of the ratio

$$\frac{\sup_{y\in\Re}\left|\Pr\left[\ (X_1+\cdots+X_n)/\left(\sigma\sqrt{n}\right)\le y\right]-\Phi(y)\right|}{\rho/\left(\sigma^3\sqrt{n}\right)} \tag{4}$$

for several examples, and these examples hint that the ratio in (4) is larger for small sample size, and decreases as sample size increases. It is therefore possible that a tighter bound can be obtained by replacing C by C_n in inequality (2).

Based on the above standpoint, we found that C_n can be made smaller than Shiganov's constant 0.7655 when $n \geq 65$, and can be decreased by further increasing sample size n. As n approaches infinity, the Berry-Esseen constant can be asymptotically improved down to 0.7164.

This paper is organized as follows. Section II covers the basic derivation of the Berry-Esseen theorem based on a filtering function satisfying six pre-specified properties. The selection of a specific filtering function that can asymptotically refine the Berry-Esseen constant is introduced in Section **??**. Also illustrated in Section **??** is the numerical evaluation of the sample-size dependent C_n. Conclusions are drawn in Section **??**.

It is assumed throughout that $\Phi(\cdot)$ denote the unit Gaussian cdf.

II. Berry-Esseen Theorem with sample-size dependent coefficient

In [3], the Berry-Esseen theorem was proved through a properly selected filter

$$v_T(x) = \frac{1-\cos(Tx)}{\pi T x^2} = \frac{2\sin^2(Tx/2)}{\pi T x^2}.$$

This filter satisfies the following properties:

P1. (Symmetry) $v_T(x) = v_T(-x)$ for every $x \in \Re$;

P2. (Integrability) $\int_{-\infty}^{\infty} |v_T(x)|dx < \infty$ so that its Fourier transform

$$\omega_T(\zeta) \triangleq \int_{-\infty}^{\infty} v_T(x) e^{-j\zeta x} dx$$

exists.

P3. (Band-limit to T) $\omega_T(\zeta) = 0$ for $|\zeta| > T$.

P4. (Unity at zero) $\omega_T(0) = 1$.

P5. (Bound by Unity) $|\omega_T(\zeta)| \leq 1$,

P6. Function $h(u) \triangleq \pi u \int_u^{\infty} v_T\left(\frac{t}{T}\right)\frac{dt}{T}$ is independent of T.

P7. (Nonnegativity) $v_T(x) \geq 0$ for every $x \in \Re$.

We noticed that with a slight modification to its original proof, any filter satisfying the above six properties can be used to prove the Berry-Esseen Theorem. We begin with the introduction of the smoothing lemma.

Lemma 1 *Fix a filter $v_T(\cdot)$ satisfying P1–P7. For any cdf $H(\cdot)$ defined on the real line $\Re$ and any real number $\beta > 1$,*

$$\sup_{x\in\Re} |\Delta_T(x)| \geq \frac{(\beta-1)}{\beta}\eta - \frac{2(2\beta-1)}{T\pi\sqrt{2\pi}}\, h\left(\frac{T\sqrt{2\pi}}{\beta}\eta\right),$$

where function $h(\cdot)$ is defined in P6, $\eta \triangleq \sup_{x\in\Re}|H(x)-\Phi(x)|$, and

$$\Delta_T(t) \triangleq \int_{-\infty}^{\infty} [H(t-x) - \Phi(t-x)] \times v_T(x) dx.$$

Proof: The right-continuity of the cdf $H(\cdot)$ and the continuity of the Gaussian unit cdf $\Phi(\cdot)$ together indicate the right-continuity of $|H(x)-\Phi(x)|$, which in turn implies the existence of $x_0 \in \Re$ satisfying

$$\text{either} \quad \eta = |H(x_0) - \Phi(x_0)| \quad \text{or} \quad \eta = \lim_{x\uparrow x_0} |H(x)-\Phi(x)| > |H(x_0)-\Phi(x_0)|.$$

We then distinguish between three cases:

$$\begin{array}{ll} \text{Case A)} & \eta = H(x_0) - \Phi(x_0); \\ \text{Case B)} & \eta = \Phi(x_0) - H(x_0); \\ \text{Case C)} & \eta = \lim_{x \uparrow x_0} |H(x) - \Phi(x)| > |H(x_0) - \Phi(x_0)|. \end{array}$$

Case A) $\eta = H(x_0) - \Phi(x_0)$. In this case, we note that for $s > 0$,

$$\begin{aligned} H(x_0 + s) - \Phi(x_0 + s) &\geq H(x_0) - \left[\Phi(x_0) + \frac{s}{\sqrt{2\pi}}\right] && (5) \\ &= \eta - \frac{s}{\sqrt{2\pi}}, && (6) \end{aligned}$$

where (5) follows from $\sup_{x\in\Re} |\Phi'(x)| = 1/\sqrt{2\pi}$. Observe that (6) implies

$$\begin{aligned} H\left(x_0 + \frac{\sqrt{2\pi}}{\beta}\eta - x\right) - \Phi\left(x_0 + \frac{\sqrt{2\pi}}{\beta}\eta - x\right) &\geq \eta - \frac{1}{\sqrt{2\pi}}\left(\frac{\sqrt{2\pi}}{\beta}\eta - x\right) \\ &= \frac{(\beta - 1)}{\beta}\eta + \frac{x}{\sqrt{2\pi}}, \end{aligned}$$

for $|x| < \eta\sqrt{2\pi}/\beta$. Together with the fact that $H(x) - \Phi(x) \geq -\eta$ for all $x \in \Re$, we obtain

$$\begin{aligned} \sup_{x\in\Re} |\Delta_T(x)| &\geq \Delta_T\left(x_0 + \frac{\sqrt{2\pi}}{\beta}\eta\right) \\ &= \int_{-\infty}^{\infty} \left[H\left(x_0 + \frac{\sqrt{2\pi}}{\beta}\eta - x\right) - \Phi\left(x_0 + \frac{\sqrt{2\pi}}{\beta}\eta - x\right)\right] \times v_T(x)dx \\ &= \int_{[|x|<\eta\sqrt{2\pi}/\beta]} \left[H\left(x_0 + \frac{\sqrt{2\pi}}{\beta}\eta - x\right) - \Phi\left(x_0 + \frac{\sqrt{2\pi}}{\beta}\eta - x\right)\right] \times v_T(x)dx \\ &+ \int_{[|x|\geq\eta\sqrt{2\pi}/\beta]} \left[H\left(x_0 + \frac{\sqrt{2\pi}}{\beta}\eta - x\right) - \Phi\left(x_0 + \frac{\sqrt{2\pi}}{\beta}\eta - x\right)\right] \times v_T(x)dx \\ &\geq \int_{[|x|<\eta\sqrt{2\pi}/\beta]} \left[\frac{(\beta - 1)}{\beta}\eta + \frac{x}{\sqrt{2\pi}}\right] \times v_T(x)dx + \int_{[|x|\geq\eta\sqrt{2\pi}/\beta]} (-\eta) \times v_T(x)dx \\ &= \int_{[|x|<\eta\sqrt{2\pi}/\beta]} \frac{(\beta - 1)}{\beta}\eta \times v_T(x)dx + \int_{[|x|\geq\eta\sqrt{2\pi}/\beta]} (-\eta) \times v_T(x)dx, && (7) \end{aligned}$$

where the last equality holds because of (P1) and (P7) of the filtering function $v_T(\cdot)$.
The quantity of $\int_{[|x|\geq\eta\sqrt{2\pi}/\beta]} v_T(x)dx$ can be derived as follows:

$$\begin{aligned} \int_{[|x|\geq\eta\sqrt{2\pi}/\beta]} v_T(x)dx &= 2\int_{\eta\sqrt{2\pi}/\beta}^{\infty} v_T(x)dx \\ &= 2\int_{\eta T\sqrt{2\pi}/\beta}^{\infty} v_T\left(\frac{u}{T}\right)\frac{du}{T} \\ &= \frac{2\beta}{\eta T\pi\sqrt{2\pi}}\, h\left(\frac{T\sqrt{2\pi}}{\beta}\eta\right). \end{aligned}$$

Continuing from (7),

$$\begin{aligned}\sup_{x\in\Re}|\Delta_T(x)| &\geq \frac{(\beta-1)}{\beta}\eta\left[1-\frac{2\beta}{\eta T\pi\sqrt{2\pi}}\ h\left(\frac{T\sqrt{2\pi}}{\beta}\,\eta\right)\right]\\ &\quad -\eta\cdot\left[\frac{2\beta}{\eta T\pi\sqrt{2\pi}}\ h\left(\frac{T\sqrt{2\pi}}{\beta}\,\eta\right)\right]\\ &= \frac{(\beta-1)}{\beta}\eta-\frac{2(2\beta-1)}{T\pi\sqrt{2\pi}}\ h\left(\frac{T\sqrt{2\pi}}{\beta}\,\eta\right).\end{aligned}$$

Case B) $\eta=\Phi(x_0)-H(x_0)$. Similar to Case A), we first derive for $s>0$,

$$\Phi(x_0-s)-H(x_0-s)\geq\left[\Phi(x_0)-\frac{s}{\sqrt{2\pi}}\right]-H(x_0)=\eta-\frac{s}{\sqrt{2\pi}},$$

and then obtain

$$\begin{aligned}\Phi\left(x_0-\frac{\sqrt{2\pi}}{\beta}\,\eta-x\right)-H\left(x_0-\frac{\sqrt{2\pi}}{\beta}\,\eta-x\right) &\geq \eta-\frac{1}{\sqrt{2\pi}}\left(\frac{\sqrt{2\pi}}{\beta}\,\eta+x\right)\\ &= \frac{(\beta-1)}{\beta}\eta-\frac{x}{\sqrt{2\pi}},\end{aligned}$$

for $|x|<\eta\sqrt{2\pi}/\beta$. Together with the fact that $H(x)-\Phi(x)\geq-\eta$ for all $x\in\Re$, we obtain

$$\begin{aligned}\sup_{x\in\Re}|\Delta_T(x)| &\geq -\Delta_T\left(x_0-\frac{\sqrt{2\pi}}{\beta}\,\eta\right)\\ &\geq \int_{\left[|x|<\eta\sqrt{2\pi}/\beta\right]}\left[\frac{(\beta-1)}{\beta}\eta-\frac{x}{\sqrt{2\pi}}\right]\times v_{_T}(x)dx\\ &\quad+\int_{\left[|x|\geq\eta\sqrt{2\pi}/\beta\right]}(-\eta)\times v_{_T}(x)dx\\ &= \int_{\left[|x|<\eta\sqrt{2\pi}/\beta\right]}\frac{(\beta-1)}{\beta}\eta\times v_{_T}(x)dx+\int_{\left[|x|\geq\eta\sqrt{2\pi}/\beta\right]}(-\eta)\times v_{_T}(x)dx\\ &= \frac{(\beta-1)}{\beta}\eta-\frac{2(2\beta-1)}{T\pi\sqrt{2\pi}}\ h\left(\frac{T\sqrt{2\pi}}{\beta}\,\eta\right).\end{aligned}$$

Case C) $\eta=\lim_{x\uparrow x_0}|H(x)-\Phi(x)|>|H(x_0)-\Phi(x_0)|\geq 0$. In this case, we observe that for any $0<\delta<\eta$, there exists x_0' such that $|H(x_0')-\Phi(x_0')|\geq\eta-\delta\triangleq\eta'$. We can then follow the procedure of the previous two cases to obtain:

$$\sup_{x\in\Re}|\Delta_T(x)| \geq \frac{(\beta-1)}{\beta}\eta'-\frac{2(2\beta-1)}{T\pi\sqrt{2\pi}}\ h\left(\frac{T\sqrt{2\pi}}{\beta}\,\eta'\right).$$

The proof is completed by noting that η' can be made arbitrarily close to η.

■

Lemma 2 *For any cumulative distribution function $H(\cdot)$ with zero-mean and unit variance, its characteristic function $\varphi_{_H}(\zeta)$ satisfies that for any $\beta > 1$,*

$$\eta \leq \frac{\beta}{2\pi(\beta-1)} \int_{-T}^{T} \left| \varphi_{_H}(\zeta) - e^{-(1/2)\zeta^2} \right| \frac{d\zeta}{|\zeta|} + \frac{2\beta(2\beta-1)}{T\pi\sqrt{2\pi}(\beta-1)} h\left(\frac{T\sqrt{2\pi}}{\beta} \eta \right),$$

where η and $h(\cdot)$ are defined in Lemma 1.
Proof: Observe that

$$\Delta_T(t) = \int_{-\infty}^{\infty} [H(t-x) - \Phi(t-x)] \times v_{_T}(x) dx$$

is nothing but a convolution of $v_{_T}(\cdot)$ and $H(\cdot) - \Phi(\cdot)$. By Fourier inversion theorem [3, Sec.XV.3],

$$\begin{aligned} \frac{d\left(\Delta_T(x)\right)}{dx} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-j\zeta x} \left[\varphi_{_H}(\zeta) - e^{-(1/2)\zeta^2} \right] \omega_{_T}(\zeta) d\zeta \\ &= \frac{1}{2\pi} \int_{-T}^{T} e^{-j\zeta x} \left[\varphi_{_H}(\zeta) - e^{-(1/2)\zeta^2} \right] \omega_{_T}(\zeta) d\zeta, \end{aligned}$$

where the second step follows from bandlimit property (P3) of $v_{_T}(\cdot)$. Integrating with respect to x, we obtain

$$\Delta_T(x) = \frac{1}{2\pi} \int_{-T}^{T} e^{-j\zeta x} \frac{\left[\varphi_{_H}(\zeta) - e^{-(1/2)\zeta^2} \right]}{-j\zeta} \omega_{_T}(\zeta) d\zeta, \tag{8}$$

where no integration constant appears since both sides go to zero as $|x| \to \infty$.[1] Accordingly,

$$\begin{aligned}
\sup_{x\in\Re}|\Delta_T(x)| &= \sup_{x\in\Re}\frac{1}{2\pi}\left|\int_{-T}^{T} e^{-j\zeta x}\frac{\left[\varphi_{H}(\zeta)-e^{-(1/2)\zeta^2}\right]}{-j\zeta}\omega_{T}(\zeta)d\zeta\right| \\
&\le \sup_{x\in\Re}\frac{1}{2\pi}\int_{-T}^{T}\left|e^{-j\zeta x}\frac{\left[\varphi_{H}(\zeta)-e^{-(1/2)\zeta^2}\right]}{-j\zeta}\omega_{T}(\zeta)\right|d\zeta \\
&= \sup_{x\in\Re}\frac{1}{2\pi}\int_{-T}^{T}\left|\varphi_{H}(\zeta)-e^{-(1/2)\zeta^2}\right|\cdot|\omega_{T}(\zeta)|\frac{d\zeta}{|\zeta|} \\
&\le \sup_{x\in\Re}\frac{1}{2\pi}\int_{-T}^{T}\left|\varphi_{H}(\zeta)-e^{-(1/2)\zeta^2}\right|\frac{d\zeta}{|\zeta|} \\
&= \frac{1}{2\pi}\int_{-T}^{T}\left|\varphi_{H}(\zeta)-e^{-(1/2)\zeta^2}\right|\frac{d\zeta}{|\zeta|},
\end{aligned}$$

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[1] As

$$|\Delta_{T}(x)| \le \int_{-\infty}^{\infty}|H(x-t)-\Phi(x-t)|\cdot|v_{T}(t)|dt \quad\text{and}\quad |H(x-t)-\Phi(x-t)|\cdot|v_{T}(t)|\le 2|v_{T}(t)|,$$

which is integrable by P2, we obtain from dominated convergence theorem that

$$\lim_{|x|\to\infty}\int_{-\infty}^{\infty}|H(x-t)-\Phi(x-t)|\cdot|v_{T}(t)|dt = \int_{-\infty}^{\infty}\lim_{|x|\to\infty}|H(x-t)-\Phi(x-t)|\cdot|v_{T}(t)|dt = 0.$$

On the other hand, by Taylor's formula with remainder, we obtain:

$$\varphi_H(\zeta) = \varphi_H(0) + \varphi'_H(0)\zeta + \int_0^{\zeta}(\zeta-t)\varphi''_H(t)dt = 1 + \int_0^{\zeta}(\zeta-t)\varphi''_H(t)dt,$$

and

$$e^{-(1/2)\zeta^2} = 1 + \int_0^{\zeta}(\zeta-t)(t^2-1)e^{-(1/2)t^2}dt.$$

Observe that $\lim_{|t|\downarrow 0}[\varphi''_H(t)-(t^2-1)e^{-(1/2)t^2}]=0$ implies the existence of ε, for a given δ, such that

$$\left|\varphi''_H(t)-(t^2-1)e^{-(1/2)t^2}\right| < \varepsilon \text{ for } |t|<\delta.$$

Hence, by bandlimit property (P3) and bounded-by-unity property (P5) of $\omega_{T}(\cdot)$,

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{\left|\varphi_H(\zeta)-e^{-(1/2)\zeta^2}\right|}{|\zeta|}|\omega_{T}(\zeta)|d\zeta &\le \int_{[|\zeta|<\delta]}\frac{\left|\int_0^{\zeta}(\zeta-t)\cdot\left[\varphi''_H(t)-(t^2-1)e^{-(1/2)t^2}\right]dt\right|}{|\zeta|}d\zeta \\
&\quad + \int_{[\delta\le|\zeta|\le T]}\frac{|\varphi_H(\zeta)|+|e^{-(1/2)\zeta^2}|}{|\zeta|}d\zeta \\
&\le \int_{[-\delta<\zeta<0]}\frac{\varepsilon\int_{\zeta}^{0}(t-\zeta)dt}{|\zeta|}d\zeta + \int_{[0\le\zeta<\delta]}\frac{\varepsilon\int_0^{\zeta}(\zeta-t)dt}{|\zeta|}d\zeta \\
&\quad + \int_{[\delta\le|\zeta|\le T]}\frac{2}{|\zeta|}d\zeta \\
&= \frac{\varepsilon}{2}\delta^2 + 4\log\left(\frac{T}{\delta}\right) < \infty.
\end{aligned}$$

The above inequality then guarantees the vanishing of the right-hand-side of (8) by Riemann-Lebesgue lemma, namely, $\int f(x)e^{itx}dx \overset{|t|\to\infty}{\longrightarrow} 0$ for integrable f.

where the last inequality follows from bounded-by-unity property (P5). Together with

$$\sup_{x\in\Re}|\Delta_T(x)| \geq \frac{(\beta-1)}{\beta}\eta - \frac{2(2\beta-1)}{T\pi\sqrt{2\pi}}\, h\left(\frac{T\sqrt{2\pi}}{\beta}\eta\right),$$

we finally have

$$\eta \;\leq\; \frac{\beta}{2\pi(\beta-1)}\int_{-T}^{T}\left|\varphi_{H}(\zeta)-e^{-(1/2)\zeta^2}\right|\frac{d\zeta}{|\zeta|}+\frac{2\beta(2\beta-1)}{T\pi\sqrt{2\pi}(\beta-1)}\,h\left(\frac{T\sqrt{2\pi}}{\beta}\eta\right).$$

■

Theorem 1 (Berry-Esseen theorem) *Let* $Y_n=\sum_{i=1}^n X_i$ *be sum of i.i.d. random variables, where* $n\geq 3$*. Denote the mean and variance of* X_n *by* μ *and* σ^2*, respectively. Define* $\rho \triangleq E\left[|X_n-\mu|^3\right]$*. Also denote the cdf of* $(Y_n-E[Y_n])/(\sigma\sqrt{n})$ *by* $H_n(\cdot)$*. Then for all* $y\in\Re$ *and any* $\beta>1$*,*

$$|H_n(y)-\Phi(y)|\leq \beta\, B_n(\beta)\,\frac{(n-1)}{\sqrt{\pi}\left(2n-3\sqrt{2}\right)}\frac{\rho}{\sigma^3\sqrt{n}},$$

where $B_n(\beta)$ *is the largest positive number* u *satisfying*

$$(\beta-1)\pi u-2(2\beta-1)h(u)\leq\frac{\sqrt{\pi}}{4}\left(\frac{2n-3\sqrt{2}}{n-1}\right)\left(\frac{\sqrt{6\pi}}{\left(3-\sqrt{2}\right)^{3/2}}+\frac{9}{\left(3-\sqrt{2}\right)^{2}}\frac{1}{\sqrt{n}}\right). \tag{9}$$

Proof: From Lemma 2,

$$\frac{2\pi(\beta-1)}{\beta}\eta\leq\int_{-T}^{T}\left|\varphi^n\left(\frac{\zeta}{\sigma\sqrt{n}}\right)-e^{-\zeta^2/2}\right|\frac{d\zeta}{|\zeta|}+\frac{4(2\beta-1)}{T\sqrt{2\pi}}\,h\left(\frac{T\sqrt{2\pi}}{\beta}\eta\right), \tag{10}$$

where $\varphi(\cdot)$ is the characteristic function of $(X_n-\mu)$. Observe that the integrand satisfies

$$\begin{aligned}
&\left|\varphi^n\left(\frac{\zeta}{\sigma\sqrt{n}}\right)-e^{-\zeta^2/2}\right|\\
\leq\;& n\left|\varphi\left(\frac{\zeta}{\sigma\sqrt{n}}\right)-e^{-\zeta^2/(2n)}\right|\gamma^{n-1}, &(11)\\
\leq\;& n\left(\left|\varphi\left(\frac{\zeta}{\sigma\sqrt{n}}\right)-\left(1-\frac{\zeta^2}{2n}\right)\right|+\left|\left(1-\frac{\zeta^2}{2n}\right)-e^{-\zeta^2/(2n)}\right|\right)\gamma^{n-1} &(12)
\end{aligned}$$

where, from [3, Sec.XVI.5, Eg. (5.5)], the quantity γ in (11) requires that

$$\left|\varphi\left(\frac{\zeta}{\sigma\sqrt{n}}\right)\right|\leq\gamma\quad\text{and}\quad\left|e^{-\zeta^2/(2n)}\right|\leq\gamma.$$

By Eq. (26.5) in [2], we upperbound the first and second terms in the parentheses of (12) respectively by

$$\left|\varphi\left(\frac{\zeta}{\sigma\sqrt{n}}\right)-1+\frac{\zeta^2}{2n}\right|\leq\frac{\rho}{6\sigma^3 n^{3/2}}|\zeta|^3\quad\text{and}\quad\left|1-\frac{\zeta^2}{2n}-e^{-\zeta^2/(2n)}\right|\leq\frac{1}{8n^2}\zeta^4.$$

Continuing the derivation of (12),

$$\left|\varphi^n\left(\frac{\zeta}{\sigma\sqrt{n}}\right)-e^{-\zeta^2/2}\right|\;\leq\; n\left(\frac{\rho}{6\sigma^3n^{3/2}}|\zeta|^3+\frac{1}{8n^2}\zeta^4\right)\gamma^{n-1}. \tag{13}$$

It remains to choose γ that bounds both $|\varphi(\zeta/(\sigma\sqrt{n}))|$ and $\exp\{-\zeta^2/(2n)\}$ from above.

From the elementary property of characteristic functions,

$$\left|\varphi\left(\frac{\zeta}{\sigma\sqrt{n}}\right)\right| \leq 1 - \frac{\zeta^2}{2n} + \frac{\rho}{6\sigma^3 n^{3/2}}|\zeta^3|,$$

if

$$\frac{\zeta^2}{2n} \leq 1. \tag{14}$$

For those $\zeta \in [-T, T]$ (which is exactly the range of integration operation in (10)), we can guarantee the validity of the condition in (14) by defining

$$T \triangleq \frac{\sigma^3\sqrt{n}}{\rho}\left(\frac{\sqrt{2}n-3}{n-1}\right),$$

and obtain

$$\frac{\zeta^2}{2n} \leq \frac{T^2}{2n} = \frac{\sigma^6}{2\rho^2}\left(\frac{\sqrt{2}n-3}{n-1}\right)^2 \leq \frac{1}{2}\left(\frac{\sqrt{2}n-3}{n-1}\right)^2 \leq 1,$$

for $n \geq 3$. Hence, for $|\zeta| \leq T$,

$$\begin{aligned}
\left|\varphi\left(\frac{\zeta}{\sigma\sqrt{n}}\right)\right| &\leq 1 + \left(-\frac{\zeta^2}{2n} + \frac{\rho}{6\sigma^3 n^{3/2}}|\zeta^3|\right) \\
&\leq \exp\left\{-\frac{\zeta^2}{2n} + \frac{\rho}{6\sigma^3 n^{3/2}}|\zeta^3|\right\} \\
&\leq \exp\left\{-\frac{1}{2n}\zeta^2 + \frac{\rho}{6\sigma^3 n^{3/2}}T\zeta^2\right\} \\
&= \exp\left\{-\left(\frac{1}{2n} - \frac{\rho T}{6\sigma^3 n^{3/2}}\right)\zeta^2\right\} \\
&= \exp\left\{-\frac{(3-\sqrt{2})}{6(n-1)}\zeta^2\right\}.
\end{aligned}$$

We can then choose

$$\gamma \triangleq \exp\left\{-\frac{(3-\sqrt{2})}{6(n-1)}\zeta^2\right\}.$$

Note that the above selected γ is an upper bound of $\exp\{-\zeta^2/(2n)\}$ for $n \geq 3/\sqrt{2} \approx 2.12$. By taking the chosen γ into (13), the integration part in (10) becomes

$$\begin{aligned}
&\int_{-T}^{T} \left| \varphi^n\left(\frac{\zeta}{\sigma\sqrt{n}}\right) - e^{-\zeta^2/2} \right| \frac{d\zeta}{|\zeta|} \\
\leq\ & \int_{-T}^{T} n\left(\frac{\rho}{6\sigma^3 n^{3/2}}\zeta^2 + \frac{1}{8n^2}|\zeta|^3\right) \cdot \exp\left\{-\frac{(3-\sqrt{2})}{6}\zeta^2\right\} d\zeta \\
\leq\ & \int_{-\infty}^{\infty} \left(\frac{\rho}{6\sigma^3\sqrt{n}}\zeta^2 + \frac{1}{8n}|\zeta|^3\right) \cdot \exp\left\{-\frac{(3-\sqrt{2})}{6}\zeta^2\right\} d\zeta \\
=\ & \frac{\rho}{\sigma^3\sqrt{n}}\left(\frac{\sqrt{6\pi}}{2\left(3-\sqrt{2}\right)^{3/2}} + \frac{9}{2\left(3-\sqrt{2}\right)^2}\frac{\sigma^3}{\rho\sqrt{n}}\right) \\
\leq\ & \frac{\rho}{\sigma^3\sqrt{n}}\left(\frac{\sqrt{6\pi}}{2\left(3-\sqrt{2}\right)^{3/2}} + \frac{9}{2\left(3-\sqrt{2}\right)^2}\frac{1}{\sqrt{n}}\right) \\
=\ & \frac{1}{T}\left(\frac{\sqrt{2}n-3}{n-1}\right)\left(\frac{\sqrt{6\pi}}{2\left(3-\sqrt{2}\right)^{3/2}} + \frac{9}{2\left(3-\sqrt{2}\right)^2}\frac{1}{\sqrt{n}}\right), \qquad (15)
\end{aligned}$$

where the last inequality follows from Lyapounov's inequality, i.e.,

$$\sigma = E^{1/2}\left[|X_n-\mu|^2\right] \leq E^{1/3}\left[|X_n-\mu|^3\right] = \rho^{1/3}.$$

Taking (15) into (10), we finally obtain

$$\begin{aligned}
\frac{2\pi(\beta-1)}{\beta}\eta \ \leq\ & \frac{1}{T}\left(\frac{\sqrt{2}n-3}{n-1}\right)\left(\frac{\sqrt{6\pi}}{2\left(3-\sqrt{2}\right)^{3/2}} + \frac{9}{2\left(3-\sqrt{2}\right)^2}\frac{1}{\sqrt{n}}\right) \\
& + \frac{4(2\beta-1)}{T\sqrt{2\pi}} h\left(\frac{T\sqrt{2\pi}}{\beta}\eta\right),
\end{aligned}$$

or equivalently,

$$(\beta-1)\pi u - 2(2\beta-1)h(u) \leq \frac{\sqrt{\pi}(2n-3\sqrt{2})}{4(n-1)}\left(\frac{\sqrt{6\pi}}{\left(3-\sqrt{2}\right)^{3/2}} + \frac{9}{\left(3-\sqrt{2}\right)^2}\frac{1}{\sqrt{n}}\right) \qquad (16)$$

for $u \triangleq T\sqrt{2\pi}\eta/\beta$ and $\beta > 1$.

Observe that function $(\beta-1)\pi u - 2(2\beta-1)h(u)$, by re-formulated it as:

$$(\beta-1)\pi u - 2(2\beta-1)h(u) = \pi u\left((\beta-1) - 2(2\beta-1)\int_u^\infty v_T\left(\frac{t}{T}\right)\frac{dt}{T}\right),$$

is continuous for $u \geq 0$, and equals 0 at $u = 0$, and goes to ∞ as $u \to \infty$, which guarantees the existence of positive u satisfying (16).

Inequality (16) thus implies

$$u \leq B_n(\beta),$$

where $B_n(\beta)$ is defined in the statement of the theorem. The proof is completed by

$$\begin{aligned} \eta &= u \frac{\beta}{T\sqrt{2\pi}} \\ &\leq B_n(\beta) \frac{\beta}{T\sqrt{2\pi}} \\ &= \beta\, B_n(\beta)\, \frac{(n-1)}{\sqrt{\pi}(2n-3\sqrt{2})} \frac{\rho}{\sigma^3\sqrt{n}}. \end{aligned}$$

■

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