Zeroism and Calculus without Limits

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Introduction: science and religion

Do religious beliefs influence present-day mathematics and science? The connection of mathematics to religion in Western thought can be readily explained to anyone with some rudimentary knowledge of Western philosophy. The very word mathematics derives from *mathesiz* meaning learning. Hence, "mathematics" means, by derivation, the science of learning.[1] Anyone who has read Plato is aware of the thesis that "all learning is recollection"—recollection of eternal ideas of the soul, its innate knowledge and memories of its previous lives, which is forgotten since birth.[2] From Plato to Proclus, mathematics was regarded as particularly suited to learning, since, it was believed, mathematics incorporated eternal truths, and these eternal truths sympathetically moved the eternal soul. (This is the basic principle of sympathetic magic: that "like moves like".) Anyone who knows something of Western philosophy should be familiar with these ideas, considering Whitehead's characterization of Western philosophy as a series of footnotes to Plato.

However, it is somewhat more difficult to understand how this connection of religion and mathematics evolved subsequently, through the *aql-i-kalam* (Islamic rational theology) on the one side, and the attack on it by al Ghazali[3] on the other. The attempt by Aquinas and the schoolmen to appropriate reason, through Christian rational theology, during the Crusades, takes us deeper into this nexus between theology and mathematics.[4] It is at this time that mathematics was reinterpreted in the West. The original meaning of mathematics as the "science of learning" or a means of sympathetically moving the soul was lost. In the West, mathematics now came to mean "a means of compelling argument". Such a "universal" means of compelling argument was then exactly what was needed by the church for its agenda of grabbing Arab wealth, after the strategy of conversion by force, which had worked in Europe, failed with the military failure of the later Crusades. To enable the use of reason as a weapon against Islam (a purpose for which the present pope, Benedict, still uses it[5]), the philosophy of mathematics had to be Christianized, and the heretical doctrine of the soul implicit in the "Neoplatonic" understanding of mathematics had to be eliminated. The post-Crusade understanding of mathematics in Europe is best understood as an adaptation of Islamic rational theology to suit post-Nicene Christian theology.

This Christianization of the philosophy of mathematics was accompanied by the successful attempts to fabricate history on a large scale during the Crusades. These attempts to make the origins of mathematics theologically correct led to concoctions such as Euclid.[6] The name "Euclid" is not mentioned in Greek texts of the *Elements*, which acknowledge other authors, such as Theon, father of Hypatia. The origin of "Euclid" in Latin texts from the 12th c. could well derive from a translation howler at Toledo—"uclides" meant "key to geometry" in Arabic. The key "evidence"[7] for this "Euclid" is an obviously forged passage in a late rendering of Proclus' *Commentary*, which otherwise speaks anonymously of "the author of the Elements", and propagates a contrary Neoplatonist philosophy declared heretical and cursed by the church. The issue has been further complexified by the later-day incorporation of this fabricated history into the studies on the foundations of geometry by Hilbert and even Russell, which studies culminated in the present-day philosophy of formal mathematics or formalism.[8]

Against this background, my attempt has been to "de-theologise" mathematics, to get rid of this theology in what ought to be a secular science.

Time as the interface between science and religion

To understand this attempt to construct a new mathematics and a new science, we need to go a bit deeper into the question of time.

Since formalism disconnects mathematics from the empirical, it rests entirely upon metaphysical beliefs about logic. A deeper aspect of this cross-connection between religion and science is the way logic depends upon the nature of time.[9] Further, *The Eleven Pictures of Time*[10] brought out how time beliefs are fundamental to (a) various religions, (b) to value systems and (c) to science. The attempt to control human behaviour on a large-scale, by transforming value systems, led also to the transformation of time beliefs in religion. Through the religious predilection of people like Newton, these time beliefs have become central to science, as I have explained elsewhere.

My earlier book *Time: Towards a Consistent Theory*[11] had already pointed out what was wrong with the understanding of time in present-day physics, purely from a physics perspective. Correcting that understanding of time leads not only to a new *type* of equations for physics (functional differential equations), but also to a new *sort* of logic: quasi truth-functional logic. In that book, I showed that the resulting quasi truth-functional logic is a quantum logic, thus enabling the derivation, from first principles, of the key postulates of quantum mechanics. So the difficulty in understanding quantum mechanics is primarily due to the excess baggage of theology in science.

The new insight about the time as the interface between religion and science also explained how those erroneous beliefs about time (and the purported universality of logic) had arisen in the first place, and why these errors are so sticky and hard to correct today, even in a subject like physics.

However, unlike the simplistic old tale of the war between science and religion, this new thesis of harmony-discord-remarriage between science and religion demands knowledge of both science and also of various religions. This seems a difficult demand to place even on knowledgeable specialists of either kind, so it is difficult to socially legitimate such truths, and most people still remain trapped in the earlier simplistic account, which places no strain on any faculty.

The new account of the origin of the calculus

Nevertheless, once one sees the connection between science and religion in this new way, it provides an entire new perspective. The science-religion relationship, as seen from this new perspective, naturally cropped up in my next book on the *Cultural Foundations of Mathematics*. One of my objectives here was to explain that Newton's deep and suppressed[12] religious beliefs had influenced not only his physics but also his understanding of calculus.

Contrary to the attempt in the West to glorify itself by attributing the calculus to Newton and Leibniz, the fact is that the calculus developed in India, over a thousand years, in response to the clear economic need of monsoon-driven agriculture. Just as "Arabic numerals" and related Indian algorithms were imported in Europe by Florentine merchants, the calculus was imported in Europe due to the clear economic need of an economy driven by overseas trade, and the difficulties of (and specific to) European navigation.

However, the calculus had developed in Indian under one set of cultural circumstance, and a specific understanding of mathematics, which contrasted with the religious understanding of mathematics then (or now) prevalent in Europe. *Hence*, Europe had so much difficulty in absorbing the calculus imported from India, *just as* it had difficulty in absorbing "Arabic numerals" and related algorithms ("algorismus") from India.

This process of cultural absorption of a foreign technique was misrepresented by Western historians as a processes of invention *de novo*. During the Inquisition, and the related religious intolerance in the rest of Europe, it was understandably a common practice among European scholars to deny any theologically-incorrect origin of their ideas, since that threatened life or livelihood. Thus, Mercator's sources are unknown just because he was arrested by the Inquisition, and his life threatened, while Newton's religious predilections, as is now known, nearly got him thrown out of his lucrative job, like his predecessor, for theological deviance.

Subsequent racist and colonial historians exploited this transformation of history during Crusades and Inquisition. It should be recalled that, in the 15th c., popes had promulgated a "doctrine of Christian Discovery" according to which a piece of land belonged to the first Christian to sight it. *Hence* it was claimed that Vasco da Gama "discovered" India, and, on that basis, ownership of India was assigned to the king of Portugal. Likewise, it was claimed that Columbus "discovered" America, and ownership was assigned to the king of Spain! As observed by the US Supreme Court, while granting legal sanctity to this doctrine of Christian discovery, this doctrine was also accepted by Protestant kings of countries like England (from whom US inherited its laws). The genocide in the Americas, where the atrocities far exceeded anything that Hitler did, was directly instigated by these moral and legal doctrine linking Christian discovery to ownership.

Racist historians, therefore, exploited these legal and moral principles, to set up what might be called the "doctrine of independent Christian rediscovery". The concocted history of science which developed during the Crusades, and which had assigned all scientific knowledge before the Crusades to the theologically correct Greeks, was taken one step further: all knowledge after the Crusades was asserted to have been either developed or "independently (re)discovered" by Christians, such as Copernicus, Mercator, Clavius, Tycho Brahe, Kepler, Newton, etc. Colonialism (and the racism accompanied it) globalised this European account of mathematics and its history.

Because the post-Crusade European understanding of mathematics (and its concocted history) was so deeply influenced by theology, the absorption of the Indian calculus in Europe, and its adaptation to Western theology proved a difficult task. My contention is that mathematics is, today, a difficult subject to understand just because the complexities of theology have got intertwined with the straightforward secular understanding of mathematics that prevailed elsewhere.

This naturally suggests various remedial courses of action. However, practical action along those lines must be preceded by understanding, and understanding this new account requires knowledge of the differences between the different philosophies of mathematics, such as Neoplatonic, Indian and formal mathematics. As stated earlier, this already seems a difficult demand to place on either mathematicians or philosophers in India. And no Indian philosopher, as far as I know, has so far worked also on the interface of Islamic and Christian theology.

Catuskoti: the end of the road for Western philosophy (and formalism)

Therefore, at the 31st Indian Social Science Congress in Mumbai, last year, I tried a different tack to make things clearer. To bring out the intrusion of religious beliefs into science, I asked people to imagine what mathematics would be like if it proceeded on Buddhist principles. I believe this point did get across to some people, though others presumably viewed it with the eurocentric prejudices that our education system inculcates in them.

The key claim is this: formal mathematics crumbles if it is interrogated from a Buddhist perspective.

Technically speaking, there are two key aspects to this proposal. The first is *catuskoti* or the logic of four alternatives. (This is *not* a multi-valued logic, as in Haldane's interpretation of Jain *syadavada*, rather it is a quasi truth-functional logic.[13]) Although commonly associated with Nagarjuna, one finds *catuskoti* being used by the Buddha in the *Brahmajala Sutta*. Now *catuskoti* is clearly incompatible with two-valued logic; present-day formal mathematics, however, is premised on the belief that two-valued logic is universal. Most theorems of present-day formal mathematics would fail if one used *catuskoti* in place of 2-valued logic.

Given this obvious cultural variation in logic, there is no secular way to justify the metaphysical belief system on which current formal mathematics is based. (I am not taking into consideration the sort of "secular" justification conversationally advanced by Nitin Nitsure, that "*they* are willing to *pay* for it, therefore formal mathematics must be worth doing". It is undeniable that Western cultural influences have been promoted by bribing people, or groups of people, since the days of Macaulay, for a handful of Britishers could hardly have ruled India without numerous Indian collaborators.)

The empirically manifest (*pratyaksa*) is undoubtedly secular. But note that one cannot appeal to empirical experience to support the current use of 2-valued logic in formal mathematics. Thus, formal mathematics prides itself on being entirely metaphysical, it prides itself on excluding the empirical on the strange but convenient belief that only metaphysical processes (of a certain culturally biased sort) can grasp certainty—and certainty (or necessary truth) is purportedly the hallmark of formal mathematics and its ritual of theorem-proving. Therefore, the use of 2-valued logic in formal mathematics cannot be defended on empirical grounds which are regarded as weaker[14] than mathematical theorems based on deduction using 2-valued logic. But if one did try to do so, one would have to take into account that quantum logic being quasi truth-functional, on my theory, *catuskoti* could well turn out to be empirically more viable. One does not really have to depend only on quantum logic or Buddhist logic; the proliferation of logics in pre-Buddhist Indian tradition[15] is proof that mundane empirical considerations do *not* lead to a unique logic. On the other hand, regarding the purported uniqueness or universality of logic, we have nothing better to go by than the empty assurances of Western theologians and philosophers.

I should add parenthetically that I call this logic 2-valued and not "Aristotelian" because the text on logic attributed to Aristotle is a very late text. This Arabic text comes from centuries after the Baghdad House of Wisdom where Indian Nyaya texts probably travelled. Given the wide-ranging debates in India, it is understandable why there was a compelling social need to develop various complex syllogisms in India. There was no such social need in Greece. The Aristotelian syllogism is certainly not found in Alexandria, where it was the Stoic syllogism which was used, so the attribution of this text on logic to Aristotle is excessively doubtful, and probably arises from the "bazaar effect" in the Baghdad book bazaar, or the dishonesty with which the market promotes sales. I mention this because Buddhist scholars can easily grasp the difficulty of debating with two different systems of logic as was done by Naiyayikas and Buddhists in India, even down to the time of Udyotkara. A similar thing would happen to the propositions of mathematics, if the underlying logic were changed.

This entire argument has been around for about a decade now:[16] the slogan formulation is that "deduction is less certain than induction" and that the reverse belief is supported only by the tyrannical imposition of Western metaphysics. Noticeably, there has been no answer, although t**his argument destroys the core of Western philosophy (and the philosophy of science), along with the present-day philosophy of mathematics**. The absence of a response suggests that it is time to jettison these beliefs. The roots of much of this philosophy is anchored, like the history of fictions like "Euclid", in the post-Crusade rational theology of Aquinas and the schoolmen, so the world will be a better place if we discard it, like Augustine's views of time, and move on. I have heard some quibbles, but it is a weak understanding of philosophy which thinks that this massive structure of Western philosophy can be saved merely through quibbles. Let us, then, leave them to muddle along with their incoherent and untenable beliefs, and their meaningless ritual of theorem proving (so long as they are not asking for state funding, or trying to manipulate it).

Sunyavada and representability

The second aspect of my proposal on Buddhist mathematics is *sunyavada* proper. My book, *Cultural Foundations of Mathematics*[17] has a chapter on *sunyavada* vs formalism in the context of number representations in the algorismus, calculus, and computers. This emphasizes the differences between a realistic philosophy like *sunyavada* and an idealistic one like Platonism. Of course, this again regrettably makes demands on the reader to know about the history of numbers, the present-day philosophy of calculus, and the relation of formalist philosophy to the present-day theory of computation, but I am now trying to minimize these demands.

On the other hand, during a subsequent discussion on this aspect of the book[18] the possibility emerged that it was possible to avoid all the practical (or theoretical) aspects of this new approach to mathematics, and get bogged down into details of exactly how one ought to interpret what Nagarjuna said, or did not say. Since *sabda pramana* (or authoritative testimony, such as scriptural testimony) is not (or ought not to be) part of Buddhist thought, this is infertile territory. Hence, I have used the terminology of "zeroism". I still believe this was what Nagarjuna had in mind, but I will not refer to the *Mulamadhyamakakarika* for I would like to emphasize that it does not matter whether or not there is textual support; treat zeroism as a separate, new philosophy if you like. What I am claiming about zeroism is, in my opinion, entailed by Buddhist thought, but again it does not matter whether or not that is so, since I am promoting it not for that reason, but for its immediate practical value.

Nevertheless, let me explain the connection with Buddhist thought. The basic point is that any representation of anything real always discards or ignores a certain "non-representable" part. (One might say colloquially that every representation is "approximate", except that one does not know what is "exact".) The understanding, therefore, is that *sunya* does not simply mean emptiness or void, *sunya* refers to the non-representable part which is ignored or zeroed in a representation. The meaningful statement that every representation leaves out some non-representable elements, should not be converted into the meaningless statement that everything is void.

Paticca samuppada and the problem of identity

This understanding of *sunya* or zeroism relates directly to another key notion: *paticca samuppada* (conditioned coorigination). Once more, let me emphasize that, just as I arrived at quasi truth-functional logic first, and noticed the connection with *catuskoti* later, likewise, I arrived at this notion of time and physical time evolution first, purely through an analysis of time in physics, and noticed the similarity with *paticca samuppada* later. Therefore, also I support this notion of time primarily for its practical value. From the practical perspective of present-day physics this notion of time involves history dependence[19] + spontaneity.[20] It would take too long to explain here this notion or its relation to *paticca samuppada* (but I have explained it in the two books cited earlier, and in various other articles). The key point to note is the regrettable frequency with which this notion of time is confounded with a doctrine of flux, or occasionalism, somewhat like Marxism is confounded with Gandhism by its critics.

An immediate consequence of *paticca samuppada* is *anatmavada* (no-soul-ism). The present is not *caused* by the past; it is only *conditioned* by the past. Therefore, the present is *not* implicit in the past (as it would be, for example, on the Samkhya-Yoga view[21]). Knowledge of the past is *not* implicit in the present (as it would be on Newtonian physics), and knowledge of the entire past is inadequate to determine the future. On the other hand, it is manifest (*pratyaksa*) that nothing stays constant for two instants. And the inference is *not* clear that beneath all this "surface" change, a human being has an eternally constant part—the soul. Our immediate concern with the soul relates to representation: if the soul—this supposedly constant part of a human being—exists, it is legitimate to represent events, as in everyday language, as this or that *happening* to a single individual. If the "soul" (in the above sense, as the constant part of a human being) does not exist, this everyday representation is no longer valid. It is a mere figure of speech with no underlying reality. So, this understanding of time directly leads to the problem of representation: how to represent something across time, when nothing about it stays constant?

Note that the issue of *atman* in the Upanishads is exactly this.[22] To be sure, there we are connecting two individuals across cycles of the cosmos which last for an enormous duration. But this duration is really of no consequence as Nietzsche[23] eloquently pointed out, for one has no awareness of it. And the problem of representation is exactly the same whether we consider it across cycles of the cosmos, or across the diastema between two instants. I think this is a very important aspect of the Buddha's notion of time.

Representing a changing entity

Since there is nothing "essential", such as the soul, which stays constant across time, even across two instants, what we commonly speak of as one human being, identified by one name from birth to death, is actually a procession of individuals who differ from each other so "slightly" that we do not care to or are not able to represent the difference.

I do not intend here to plumb the depths of this deep problem (though I have attempted to do so elsewhere). Certainly every Buddhist scholar is familiar with the example of the seed. The seed is not the *cause* of the plant since the seed in the granary is manifestly different from the seed in the ground (which is bloated up). What I wish to talk about is the mundane way in which the problem is resolved in everyday life. *How* do people manage to confound the seed in the granary with the seed in the ground, even though the two are manifestly different? That such confusion is widespread is evident from the empirical existence of patriarchal societies, where the seed (from the father) is regarded as the prime *cause* of the origin of the child, and the mother (or the earth) is regarded as merely a passive carrier.

The point is that this confusion regarding causes arises due to imprecise representation which entails confusion regarding the identity of the seed. I suggest that the two seeds are confounded because people use a representation of the seed, in which the detailed differences between the seed in the granary and the seed in the ground, even though manifest, are neglected as "inessential". To reiterate, such representations are fundamental to the mundane notion of identity across time. Although a person changes from moment to moment, and the child is manifestly different from the adult who is manifestly different from the aged adult, at the mundane level, and in everyday language, all these changes are represented as changes which *happen* to a *single* individual. In this method of representation, the manifest differences in this individual, who supposedly stays constant across time, are ignored or zeroed as somehow inessential to the "true" identity of the individual. Our problem, then, is to understand this mundane method of representation when we no longer believe in any such "true" or "constant" identity of the individual across time.

We can discern two kinds of positions here. It is one thing to say that given this multiplicity of entities and the paucity of names, *as a matter of convenience* we adopt the convention of assigning a single name to entities which differ only "slightly" from each other. This is the realist position. It is another thing to say that there *really* is a single entity. That is the idealist position. The idealist position is critical to religious beliefs, for the notion of one soul which survives after death is based on this position. The realist grants the practical usefulness of incomplete representations or simplifications or abstractions; the idealist asserts that these abstractions really exist out there.

Representing real numbers and supertasks

A similar strategy of representation is used in mathematics. Many things can and have been written about the representation of mathematical or other knowledge in the brain. I prefer to avoid this area, since such claims are rarely directly verifiable or testable, even if we take the brain apart! However, the representation of numbers on a computer is something definite, hence simple and easy to understand; practices here are well known, and theories about this are easy to test. For the purposes of number representation on a computer, it does not matter very much what sort of theory of computation we have: whether we are looking at a traditional computer, or a parallel computer or a quantum computer. Obviously enough, the exact architecture of the computer is even more irrelevant. But before going on to number representations on a computer, let us first consider the mundane representation of numbers.

Consider, therefore, the number labelled π or the number symbolically labelled $\sqrt{2}$. This symbolic representation is an abstraction which does not specify the number. To specify the number one must list out its complete decimal expansion. Since the decimal expansion neither terminates, nor recurs, the process of completely listing out the decimal expansion never terminates. Listing out the complete decimal expansion is a supertask—an infinite series of tasks that cannot be performed in any finite time. Certainly, it would be a reasonable thing to say that an expansion which lists out the first billion digits is likely to be adequate for all practical purposes. This statement is not even required to be timelessly or eternally valid. In future, if some practical purpose were to arise for which we needed the first trillion digits, instead, we could work out those, for we have a process to extract square roots.

However, the idealist is not happy with this situation. He wants to make statements that are *eternally* valid, and *he assumes that it is possible to do so*. Therefore, he regards the process of specifying only a billion or a trillion digits as erroneous. He has no specific practical task in mind for which he regards this as erroneous. But, in his "mind's eye", he sees that this process still leaves out an infinity of unspecified decimal digits. Thus, there are an infinity of numbers which could possibly be confounded with the number we have in mind, and which we have specified only to a billion digits, for what is a billion compared to infinity? The idealist wants to assert that that there really is a *single* number π or a single number $\sqrt{2}$, which can be *uniquely* specified.

The problem is, as we have stated, that such specification requires an infinite process or a supertask. Though this supertask cannot be *avoided*, it can be *hidden*. This is the strategy adopted in present-day formal mathematics: such supertasks are hidden underneath set theory. This set theory is the typical starting point of a math text, and the confusion begins right here: "a set is a collection of objects" is a common piece of nonsense found also in the current NCERT school texts.[24] This is rather like beginning a course on theology by saying, "Grant me a woolly idea of God and everything else will follow." Of course it is easy to make the promise that this woolly idea of God would be clarified in more advanced courses, after a decade. (By that time the person is so deeply drawn into this way of thinking that he finds it difficult to abandon, even if disillusioned!)

It is quite impossible to teach the high abstractions of axiomatic set theory to a school child What the student learns through such mathematics is to accept a confused bunch of rules purely on *fear* of authority, and the *hope* of clarification at some future date. Even most mathematicians do not learn axiomatic set theory. So they remain deficient even from within a purely formalist perspective. Thus, one such mathematician writing a school text proposed to avoid the circularity in the common "definition" of a set by declaring a set to be a primitive undefined notion! Obviously, this gentleman had no idea of what a set is, in formal set theory, nor what is a "primitive undefined notion", and he simply equated these two unknowns!   What a contrast from Proclus' idea of mathematics as the science of learning!

The point of referring to a computer should now be clear: a computer makes manifest the impossibility of performing supertasks. We are not here referring to the axiom of choice or any such fancy transfinite induction principle. We are here referring to a simple process of *specifying* a real number. A formal real number cannot be specified without appeal to set theory (or supertasks of some sort). It is therefore impossible for a formal real number to be represented on a computer. Formalists, therefore, declare the computer representation to be forever erroneous! Nevertheless, the fact is that computer representations of real numbers are adequate for absolutely *all* practical tasks today, without any exception. Therefore, declaring the computer to be forever erroneous reflects only the religious attitude that mathematics is perfect truth, and that this perfect truth can only be grasped metaphysically, for something is wrong with anything real or actually realizable. However, note that that while the Neoplatonic attitude had its merits, that is now made completely barren and devoid of all meaning in formalism. which just manipulates meaningless symbols according to an opaque grammar of supertasks.

Contrast this convoluted formalist attitude with the simplicity of the description of $\sqrt{2}$ in the *sulba sutra*:

$$\sqrt{2} \approx 1 + \frac{1}{3} + \frac{1}{3.4} + \frac{1}{3.4.34}$$

with the added qualification *savisesa*, meaning "this much and something more". The something more does not create any problems from the point of view of any practical applications. We do not have perfection here, but we do have practical value. (Note also that in speaking of $\sqrt{2}$ we are referring to something definite and empirical here, such as the diagonal of a square, which we can *see*, and whose length we can *measure,* both of which are illegal according to present-day mathematics.)

Zeroism

The point of zeroism is to recognize the legitimacy of this process. We do not need formal reals for *any* practical purpose, and can *never* use them in practice: real real numbers are good enough and those are all we have in practice. The sole question is how to represent them. We can certainly use a symbol such as $\sqrt{2}$ with the understanding that it is ambiguous—like the name of a person which can refer to that person as a child, or a youth or an old man. Where this ambiguity creates a practical problem, we can resolve it by moving on to a better description of the person (such as "so-and-so at age 2 years and 3 months"). Likewise, the exact value used for $\sqrt{2}$ might vary with the context, but this causes no confusion, any more than names cause confusion in everyday life.

As a realistic philosophy, zeroism accepts the impossibility of representing things exactly, as being in the nature of the world (and the nature of time). Any representation of any real entity must necessarily neglect some aspect of that entity, which neglected aspect is treated as inconsequential for the purpose at hand.

For many commercial and engineering problems, one usually needs only a fixed precision. In this kind of situation, of fixed-prevision arithmetic, such an incomplete representation corresponds to the usual practice of rounding, as in a commercial transaction, where neither party has adequate change.

Coming now to the actual practice of how numbers are represented on a computer, the IEEE standard 752 is the one used for floating point numbers. The exact details of this standard are not relevant here, and I have explained them in detail elsewhere in my elementary computer courses. The interesting point to note is that these floating-point numbers do *not* obey the stock "laws" of arithmetic: particularly important is the failure of the associative "law" for addition [(a+b) + c = a + (b+c)], as also for multiplication. For the technically informed, this happens because a mantissa-exponent notation is used to represent the numbers, and the addition of the numbers requires equalization of the exponent. Since binary representation is used for numbers, equalization of the exponent requires bit shifting of the mantissa. If the difference of exponents is too large, this causes the mantissa to be bit-shifted to nothing.

Thus, it is possible to find an e different from zero such that $-1 + (1 + e) = 0$ while $(-1+1) + e = e$. For the stock IEEE floating point standard, we can take $e = 0.0000001$ or any smaller number. For double or higher precision arithmetic, this number has to be even smaller.

However, the failure of the associative "law" for floating points numbers does not mean that these numbers are criminals who violate laws! The key point is that it is the grandiose notion of associative "law" which is now to be regarded as erroneous—a useful simplification, but one that is not exactly valid, and admits an infinity of exceptions to the rule. The rule has to be applied with intelligence, as in everyday language, and not mechanically as in formalism. Therefore, no algebraic structure such as a field can be associated with real numbers: the formal mathematics of real numbers is forever erroneous!

Representing integers on a computer

The non-representability of real numbers is clear enough, but a computer also brings out the non-representability of integers as well. If we think about it a bit, it is clear that there is a difficulty in representing very large integers. We don't readily encounter this difficult in practice, because, in practice, we never need to use very large integers: a zillion (howsoever defined) is about as far as most people get!

However, a little thought shows that the problem of representing real or floating point numbers is really equivalent to the problem of representing large integers, for ultimately any (finite) decimal expansion is just a fraction with a large numerator and a large denominator. In fact, this was the explicit way numbers were represented in Indian tradition: all arithmetic was ultimately reduced to integer arithmetic, possibly indefinitely continued integer arithmetic as in an indefinitely continued fraction.

An interesting point here, which I have made earlier, but which has probably not got across, is this. A statement such as 2+2 = 4 seems simple enough, but in formal mathematics, even this simple statement surprisingly involves a supertask. Thus, for the symbol 2 we must specify that it is an integer (and not, for example, the rational number 2/1, or the real number 2.0). On a computer, such a specification is easy enough. However, this formal specification (of integer-ness) itself involves a supertask. The fact that for any particular integer this specification is a finite task is poor consolation: with any finite specification, such as the one given on a computer, this arithmetic would necessarily *fail* to agree with the arithmetic of ideal integers beyond a point. To summarise, from the viewpoint of zeroism, idealisations are erroneous simplifications, and where such "simplification" can only be achieved using abstractions that are complex and difficult to articulate (such as sets), they ought to be abandoned. Simplification by complexification is useless!

The project on calculus without limits is to be understood against this background.

A brief history of limits in calculus

When the calculus first went from India to Europe, its practical value was immediately grasped. It was clear that the calculus could be used to calculate trigonometric values to high precision: and such trigonometric values (tables of secants) were in great demand for the Mercator chart which was indispensable for European navigation. While Mercator's source for his table of secants remains a secret (Mercator was arrested by the Inquisition, and therefore had ample reason to hide his sources), elaborate trigonometric tables were published by Clavius in 1607.

Clavius had ample access to the information about the Indian developments in calculus since Cochin was where the first Roman Catholic mission started in 1500. The school started by the missionaries (which soon turned into a college), and mainly catering to the local Syrian Christians, was taken over by the Jesuits in 1550. The Jesuits had made Cochin into a mini Toledo: for they collected vast amounts of local literature, got it translated and sent it back to Rome. This literature on the Indian calculus was to be found in the local timekeeping (*jyotisa*) texts circulating in the vicinity of Cochin. Clavius' loyal student, Matteo Ricci recorded that he was in Cochin, looking for an "honest Moor or an intelligent Brahmin" to explain to him the Indian methods of timekeeping. This was shortly before the Gregorian calendar reform of 1582, authored by Clavius. (The revised length of the year used in this calendar reform was based on *texts*, not empirical observations.) Of course, Clavius did not really understand any trigonometry, for he didn't know even the elementary trigonometry needed to measure the size of the earth. (This was then a critical parameter for navigation; the globe could not then be used for navigation, since the size of the earth had underestimated by Columbus, and the use of the globe was banned aboard ships by Portugal.)

Eventually, through Tycho Brahe, then Kepler, and Galileo, and his student Cavalieri, this Indian work on calculus started circulating in Europe. Some mathematicians, such as Pascal and Fermat greeted it with enthusiasm. However, others like Descartes complained that this was not mathematics. Descartes wrote in his *Geometry* that

> [T]he ratios between straight and curved lines are not known, and I believe cannot be discovered by human minds, and therefore no conclusion based upon such ratios can be accepted as rigorous and exact.

From the Indian perspective this is a very strange statement, very hard for the human mind to grasp. Thus, Indians used a string or rope since the days of the *sulba sutra*. Measurement using the rope (*rajju*) was part of the mathematics syllabus traditionally taught to Indian children.

A string can obviously be used to measure a curved line. It can be straightened, and thus compared with a straight line. Therefore, it is very easy for any child to grasp the ratio of a curved and a straight line. Why did this major Western thinker find this simple thing so hard to understand?

First, Descartes took it for granted that the straight line was the natural figure, and that curved lines must necessarily be understood in terms of straight lines, and not the other way around. Second, he ruled out empirical procedures as not mathematics. According to his system of religious beliefs only the metaphysical could be perfect, and mathematics being perfect had to be metaphysical. It should be observed that this belief that mathematics ought to be metaphysical was unique to post-Crusade Christianity: Proclus, for example, did not subscribe to it, for he admitted the empirical at the beginning of mathematics, as in the proof of the Side-Angle-Side theorem (*Elements* 1.4, as it is called).

Thus, Descartes, if he at all thought about it, regarded this easy empirical procedure of comparing straight and curved lines as suspect, since physical. Descartes assumed that the only right way to understand the matter was to do so metaphysically. So his question was, how is this procedure to be justified metaphysically?

On a computer screen, a curved line can be easily represented by means of a large number of tiny straight line segments. The eye cannot see the difference, and the digitised representation is adequate for any practical purpose. But Descartes was worried that this was not *exact.* Like a dot on a piece of paper, he thought it involved an approximation, howsoever tiny. Something somehow was being lost, he felt.

As we have seen this difficulty of representation arises even with integers, or with anything else, but Descartes thought this was a problem specific to this new-fangled calculus, which lacked perfection, and *hence* was not quite mathematics. Perfection required, in his opinion, that each such straight-line segment should be infinitesimal, but then there ought to be an infinity of them. So, perfection required a supertask—that of summing the infinity of these infinitesimal lengths—and this, he thought, was beyond the human mind.[25] After vacillating for a few years, Galileo concurred, and hence he left it to his student Cavalieri to take the credit or discredit for this disreputable sort of mathematics which was not perfect.

Given Newton's religious predilections, and his belief in the perfection of mathematics, he mathematized physics to a never-before extent. Calculations with planetary orbits were being done, using the calculus, well before Newton or Kepler. What Newton did was to bring in some "perfection" in this process by introducing the right sort of metaphysics.

This "right sort of metaphysics" had a peculiar consequence for Newtonian physics. Compared to his predecessor, Barrow, who adopted a physical definition of time and time-measurement, and summarily rejected Augustine as a metaphysical "quack", Newton reverted to a metaphysical and *mathematical* notion of time. His definition about "**absolute**, **true**, and **mathematical** time..." is not usually understood correctly, although all three adjectives make it clear that he is referring to a metaphysical and not a physical notion, and he confirms this by saying that it "flows on **without regard to anything external**". Something physical cannot obviously have such a generalized disregard for anything external! This metaphysical notion of time meant that there was no proper way to measure time in Newtonian physics.[26] The absence of a proper definition of time was the reason for the eventual failure of Newtonian mechanics when it clashed with electromagnetic theory, and relativity had to be brought in.

Why did Newton need such a definition of time? We have to understand that he wanted to use calculus, and specifically time derivatives, to explain circular *and* elliptical orbits in terms of straight line motion. (Circular orbits alone could be "explained", without the need of the calculus, by postulating an inverse-square force law, and a "natural" straight line motion; Newton's success lay in extending this procedure to elliptical planetary orbits on the one hand, and to parabolic ballistic trajectories on the other.)

But more than the practical applications, Newton was interested in rigour. It was critical to his understanding of calculus in terms of his theory of fluxions that time should *flow*, or be a "fluent" entity. This, he thought, made time infinitely divisible—at any rate it made "absolute, true and mathematical time" infinitely divisible—and such infinite divisibility was needed to justify that the supertasks needed for making the time-derivative meaningful could be performed. He thought the process would fail if time were discrete (for then the process of subdividing time, needed for taking time derivatives, would stop when subdivisions reached some finite, atomic proportions).

Historically speaking, in his defence against Leibniz's charges of plagiarism, it is rigour for which Newton, writing anonymously about himself, claims credit. (The other thing he claims credit for is the sine series, which was obviously known from earlier, though not in Europe.) And, historians, today, once again credit Newton with the calculus on the grounds that he had rigorously proved the "fundamental theorem of calculus".

Of course, Newton was mistaken in thinking that this "fluency" of time provided a solution to the problem of supertasks. Many discerning people were aware of this, and Berkeley took it upon himself to tear Newton's theory of fluxions to bits, when there seemed a danger that Newton's views on the church would become public. In the event, Newton's *History of the Church* in 8 volumes, a result of 50 years of scholarship, was successfully suppressed, and Berkeley's criticism was subsequently played down.

Regardless of its probable motivation, and regardless of the subsequent attempts to play it down, Berkeley's criticism was valid. His argument was very simple and robust, and interesting. He assumes, along with Newton, Descartes, and Galileo, that mathematics is perfect, and cannot neglect even the smallest quantity. However, he goes along with Newton and allows that a quantity may be neglected if it is infinitesimal (whatever that might mean). But, he asks, if a quantity is to be set to zero at the end of the calculation, why not set it to zero in the beginning itself?

We know the modern answer to Berkeley's objections: that the ratio of two infinitesimals may be finite. A hundred years ago, when neither Non-Standard analysis nor non-Archimedean fields had come in, and infinitesimals were still formally disreputable, the idea was to try and define limits by playing on the space provided by the non-definition of 0/0. Dedekind's formal real numbers, or the continuum, by allowing infinite divisibility, seemed to be just the right framework for such limits. That is how advanced calculus (or elementary analysis) is still taught—by appealing to the completeness of the (formal) real numbers. But, of course, it was evident, even in Dedekind's time, that the construction of formal real numbers required set theory which was suspect for the infinitary processes it involved.

The axiomatisation of set theory has tamed those doubts in an interesting way. From curves, to numbers, the doubts have now been pushed into the domain of set theory which the average mathematician does not care about. So, like the professional theologians who were concerned with establishing the number of angels that could fit on the head of a pin, without bothering about fundamental questions as to the nature of God, the professional mathematician can merrily go on proving theorems without bothering about the supertasks used in set theory.

Secondly, on the philosophy of formalism the only question that mathematicians will accept about the process is that of consistency. The consistency of set theory has not been formally *proved*, of course, but mathematicians *believe* it is consistent. It is interesting to see the double standards involved here.

Thus, if supertasks were really regarded as really admissible, one should be able to apply them also in metamathematics. In that case, it would be a trivial matter to use some transfinite induction principle, such as Zorn's lemma, or Hausdorff maximality principle, to make set theory decidable. In such a case, the theory obviously cannot be consistent, by Godel's theorem, for it easily accommodates a statement which asserts its own negation. Thus, the consistency of set theory is maintained by means of a double standard: allow supertasks within mathematics, but *not* while talking about mathematics. It is interesting to note the theological parallel: it was exactly such a double standard that was needed to claim the consistency of the notion of an omnipotent, and omniscient God with the existence of evil in the world!

Calculus without limits

The proposal on calculus without limits comes against this background. A rigorous formulation of the calculus does *not* need any supertasks. The need for supertasks was just a myth that arose in the West because of the idealistic belief in the "perfection" of mathematics, and the belief that this perfection could only be attained through metaphysics.

Let us start, for example, with a key polemic used by Western historians, which relates to the "Fundamental theorem of Calculus". It is clear from the above considerations, that Newton could hardly have *proved* it, for his theory of fluxions was mistaken and had to be rejected. Also, the idea of the fundamental theorem of calculus supposes that we have an independent definition of the integral and the derivative, and the two are related by the theorem. However, where was Newton's definition of the integral? Such a definition became available only after limits (and this led to various complexities, for the fundamental theorem of calculus obviously does not hold with either the classical derivative and the Lebesgue integral, or with the Schwartz derivative and the Riemann integral). All that Newton had was the naïve idea of the integral as the anti-derivative, which naïve idea is the basis of the calculus today taught to students.

More to the point, the central question is: *just what mathematics is about?* If it is not metaphysics, or a branch of theology, its function is to help carry out calculations with a practical purpose in mind. From this perspective, what is the exact purpose that the fundamental theorem of calculus serves? At best, it enables one to solve the differential equations Newton used to formulate physics. However, the more appropriate thing for the calculus (and for Newtonian physics), then, is to have is a numerical technique for *calculating* the solution of ordinary differential equations. This was what Aryabhata developed, and which later came to be known as Euler's method of solving ordinary differential equations. Thus, instead of a *theorem*, we have a *process*. (Of course, the process can and has been improved since Aryabhata and Euler.)

From the point of view of practical calculation, this process is far superior to the plethora of theorems that are needed to be able to solve the simplest ordinary differential equation. For example, the theory of the simple pendulum cannot be taught in schools, and is missed out even by most physics teachers, since it involves the Jacobian elliptic functions, which are difficult to teach and explain. Consequently, most people confound the simple pendulum with simple harmonic motion, the linear differential equations for which can easily be solved symbolically. However, with the process of numerical solution, the equations for the simple pendulum can be solved just as readily.

With the advent of the computer, there is no question that this is the superior approach, and the one that one will surely adopt for *any* practical task, such as ballistics.

The only question is whether this practical approach can be supported by means of an appropriate philosophy. Zeroism is such an appropriate philosophy which counters both formalism and idealism, and supports this practically superior approach.

Thus, consider the idealist polemic about the "perfection" of mathematics. (This is just a polemic the moment one has divorced mathematics from "Neoplatonic" ideas.) This has generated the belief that symbols are somehow superior to numbers (which are, of course, symbols of another sort). That is, to one trained in formal mathematics, it seems desirable to write π instead of using 3.14, for one might mean 3.1415. Now, so long as a *name*, such as $\sqrt{2}$ refers to a *process* (like the algorithm for square root extraction) by which the quantity in question may be specified (as in the ostensive definition of an individual) this may be regarded as an acceptable practice. However, where there is no such underlying process, or the underlying process involves a supertask, this representation is faulty and deceitful: for merely by assigning a name to a thing one generates a faulty expectation that there is a real and unique entity corresponding to that name. This, as we have seen, is not the case, and cannot be the case, whenever a supertask is involved.

Similarly, there is the belief that a "closed form" solution is superior to a numerical solution. Now, to the extent, and only to the extent, that one is comparing a definite and well-established *process* of calculation (such as the calculation of sine values) with a new, and as-yet unclear numerical method, I am willing to go along with this. Often, such symbolic representation is just a fallback to the old days (namely 30 or 40 years ago) when a desktop-computation of the values of special functions (such as the Jacobian elliptic functions) was a complex task that one did not want to perform.

When there is no reference to any such underlying *process* of calculation or specification, the symbolic representation is just erroneous. To my mind, there is a close analogy between this argument, and the Buddhist (and specially *sunyavada*) argument against Naiyayikas, although this argument is in a new context.

From the formal math angle, let us see how this process works for the case of limits. To fix ideas, consider the case of an infinite series: all other cases can be handled similarly.

Recall also that the first definition of the sum of an infinite geometric series was given in India. Similar notions of convergence were used to sum the sine and cosine series.

From the present-day perspective, limits are idealized constructs, which are required to be unique (even if not specifiable). In practice what we require is something specifiable (even if not unique). On the formal definition, a series $\sum_{n=1}^{\infty} a_n$ is convergent if for all $\epsilon > 0$, there exists a number N such that $|\sum_{n=N}^{N+m} a_n| < \epsilon$ for all m.

Let us compare this formal definition of limit with the earlier definition used in India. Working to a fixed but arbitrary precision is equivalent to having a fixed but arbitrary $\epsilon > 0$. Formally speaking, by the principle of generalisation, anything that is true for such an ϵ will be true for all $\epsilon > 0$. Now the *Yuktidipika* (or *Tantrasangrahavyakhya*) asks us to sum the series up to that value of N beyond which adding additional terms will not make any difference to the sum (up to the precision concerned). This is the same as the requirement that $|\sum_{n=N}^{N+m} a_n| < \epsilon$ although it was not specified that this ought to be done for *all* values of m. Clearly, except where there is a distinct pattern, that would constitute a supertask. Of course, if some value of *m* were to be found at which this condition was violated, no doubt the author of the *Yuktidipika* would have agreed that the series cannot be summed—it was just that he did not formulate things in a "legalistic" or formal way. (Indian law then was not so rigidly codified, and had plenty of room for to apply commonsense and neglect the quibbles that one today associates with legalese.) So, this difference of formulation would not really have been an issue in the case of a slowly divergent series, such as $\sum_{1}^{\infty} \frac{1}{n}$.

So, the *only* difference between the present-day definition "Cauchy criterion of convergence") and the notion of convergence used in India is the absence of supertasks.

Now what *exactly* does this formal ability to do supertasks mean? Consider a series such as 1-1+1-1+.... What is the sum of this series? The partial sums of the series clearly oscillate between the values of 0 and 1 and hence the series is judged to be divergent on conventional analysis. However, in many practical situations it is convenient to *define* the sum of this series as $\frac{1}{2}$. Such a definition may seem puzzling at first sight since the series never attains that value, and the sum to any finite number of terms is always either 0 or 1. This is an example of an "asymptotically convergent" series. The point is that just like 0/0, the sum of an infinite number of terms has no arithmetical meaning of its own. Within the philosophy of formal mathematics, such a meaning has to be assigned *by definition*: and definitions, as everyone knows are bound to have some arbitrariness in them. The above definition is practically convenient and is often used in physics in the more sophisticated form $\theta.\delta=\frac{1}{2}\delta$, where θ is the Heaviside function and δ is the Dirac delta function. So we are back to the situation where there is no need to actually perform any supertasks. When a seeming supertask arises, what one needs, instead, is a process (a finite process) which needs to be carried out. The nature of this process (or, equivalently, the definition of such products of Schwartz distributions) is to be decided by practical value and *empirical* considerations. (The alternative is to rely on mathematical authority.)

There remains the question of the value of teaching symbolic manipulation. Under the current system of education, symbolic manipulation is most of what is taught in a calculus course, and this is the skill that the students are expected to take away. But what is the worth of this? A half-century ago, one could possibly justify this procedure by arguing that it helped people to relate a new process of calculation to an old process. But that is no longer necessary. Moreover, computers can carry out such mechanical symbolic manipulation (by applying rules such as integration by parts) far more easily. Today, no one would dream of doing a complex piece of arithmetic by hand: one would use a computer. Likewise, no one today would dream of doing by hand any complex process of symbolic manipulation: one would use MATHEMATICA, or better, MACSYMA. I am specially reminded of the months of frustrating effort that I have spent in getting a complex piece of symbolic manipulation right, and which could be done in a jiffy today with these programs. Therefore, apart from a knowledge of the basic rules involved, the student does not need to be drilled in such symbolic manipulation, as happens in a normal calculus course: it is enough to teach how to use the open source version: MAXIMA.

1 Proclus, *Commentary*, 47, trans. Glen L. Morrow, Princeton Univ. Press, 1970, p. 38.

2 Plato, *Meno*, 81–83. *The Dialogues of Plato*, trans. B. Jowett, *Great Books of the Western World*, Encyclopaedia Britannica, Chicago, 1996, p. 180.

3 Al Ghazālī, *Tahāfut al-Falāsifā*, trans. S.A. Kamali, Pakistan Philosophical Congress, Lahore, 1958.

4 C. K. Raju, "The Religious Roots of Mathematics", *Theory, Culture & Society* **23**(1–2) Jan-March 2006, Spl. Issue on Problematizing Global Knowledge, ed. Mike Featherstone, Couze Venn, Ryan Bishop, and John Phillips, pp. 95–97.

5 C. K. Raju, "Benedict's Maledicts", *Indian Journal of Secularism*, **10** (3), 2006, pp. 79-90. Also online, http://zmag.org/znet/viewArticle/3109.

6 C. K. Raju, "Teaching racist history", *Indian Journal of Secularism*, **11**(4) (2008) pp. 25-28. Also, "Itihas ke vichalan", Jansatta 23(?) Jan 2008. More details in "Towards Equity in Math Education 1: Good Bye Euclid", paper presented at 31st Indian Social Science Congress, Mumbai, Dec 2007, session on Math Education. Available online at: http://ckraju.net/papers/MathEducation1Euclid.pdf.

7 T. L. Heath, *The Thirteen Books of Euclid's Elements*, Dover, New York, 1956, p. 1.

8 C. K. Raju, "Euclid and Hilbert", chp. 1 in *Cultural Foundations of Mathematics*, Pearson Longman, 2007.

9 C. K. Raju, "Logic", *Springer Encyclopedia of NonWestern Science, Technology, and Medicine*, 2008. Draft available at http://ckraju.net/IndianCalculus/Nonwestern_logic.pdf.

10 C. K. Raju, *The Eleven Pictures of Time: The Physics, Philosophy, and Politics of Time Beliefs*, Sage, 2003.

11 C. K. Raju, *Time: Towards a Consistent Theory*, Kluwer Academic, Dordrecht, 1994; Fundamental Theories of Physics, vol. 65.

12 For the way in which Newton's lifelong work on the history of the church has been suppressed, see "Newton's Secret", chp. 4 in *The Eleven Pictures of Time*, cited above. For details of how the Newton scholar Whiteside continued trying to mislead people about Newton, and could not defend this exposure of Cambridge historians except by abusing me, see,

13 See Encyclopedia article on logic cited above,

14 The actual origin of this notion of "weakness" is al Ghazali's assertion, accepted by Western theologians, that God is bound by logic, not by empirical facts. Thus, logic which binds God is "stronger" than empirical facts which don't. Its present-day articulation is that mathematical theorems are "necessary truths" (true in all possible worlds), while empirical facts are only "contingent truths" (true in some possible worlds). The only difference is the euphemism of "logical worlds" used by logicians, where the theologians purportedly spoke of real worlds. This difference is blurred, the moment one tries to connect logic to empirical facts.

15 B. M. Baruah, *A History of Pre-Buddhistic Indian Philosophy*, Calcutta, 1921; reprint Motilal Banarsidass, Delhi, 1970.

16 C. K. Raju, "Computers, Mathematics Education, and the Alternative Epistemology of the Calculus in the Yuktibhasa", *Philosophy East and West,* **51**(3) 2001, pp. 325-62. Draft available online at http://ckraju.net/IndianCalculus/Hawaii.pdf. Also, "The Mathematical Epistemology of Sunya", in: *The Concept of Sunya*, ed. A. K. Bag and S. R. Sarma, IGNCA, INSA and Aryan Books International, New Delhi, 2002, pp. 168–181. (Summary of interventions during the 1997 conference.)

17 C. K. Raju, *Cultural Foundations of Mathematics: The Nature of Mathematical Proof and the Transmission of the Calculus from India to Europe in the 16th c. CE*, Pearson Longman, 2007 (PHISPC vol X.4).

18 Symposium on Mathematics in Relation to History, Culture, and Technology, India International Centre, New Delhi, Nov, 2007.

19 C. K. Raju, "The Electrodynamic 2-Body Problem and the Origin of Quantum Mechanics", *Foundations of Physics* **34** (2004) pp. 937–62. arxiv.org:0511235.

20 C. K. Raju, "Time Travel and the Reality of Spontaneity", *Foundations of Physics*, **36** (2006) pp. 1099-1113. arxiv.org: 0804.0830.

21 "Time in Indian and Western Traditions and Time in Physics." In: *Mathematics, Astronomy and Biology in Indian Tradition*, PHISPC Monograph Series on History of Philosophy, Science, and Culture in Indian, No. 3 (D. P. Chattopadhyaya and Ravinder Kumar eds) ICPR, New Delhi, 1995, pp. 56–93. Also "Kâla and Dik" chp. 5 in P. K. Sen & P. K. Sen et al. (ed), *Philosophical Concepts Relevant to Science in Early Indian Tradition*, PHISPC vol III.5, pp. 67–92.

22 C. K. Raju, "*Atman*, Quasi-Recurrence and *paticca samuppada*", in *Self, Science and Society, Theoretical and Historical Perspectives*, ed. D. P. Chattopadhyaya, and A. K. Sengupta, PHISPC, New Delhi, 2005, pp. 196-206.

23 F. Nietzsche, *Eternal Recurrence*, 33. O. Levy, ed., *The Complete Works of Friedrich Nietzsche*, vol. 16, Foulis, Edinburgh, 1911, p. 253.

24 J. V. Narlikar et al., *Mathematics Text for Class XI*, NCERT, New Delhi, 2007.

25 René Descartes, *The Geometry*, trans. David Eugene and Marcia L. Latham, Encyclopaedia Britannica, Chicago, 1990, Book 2, p. 544.

26 C. K. Raju, "Time: What is it That it Can be Measured?", *Science and Education*, **15**(6) (2006) pp. 537–551