Randomized Algorithms

Part Three

Announcements

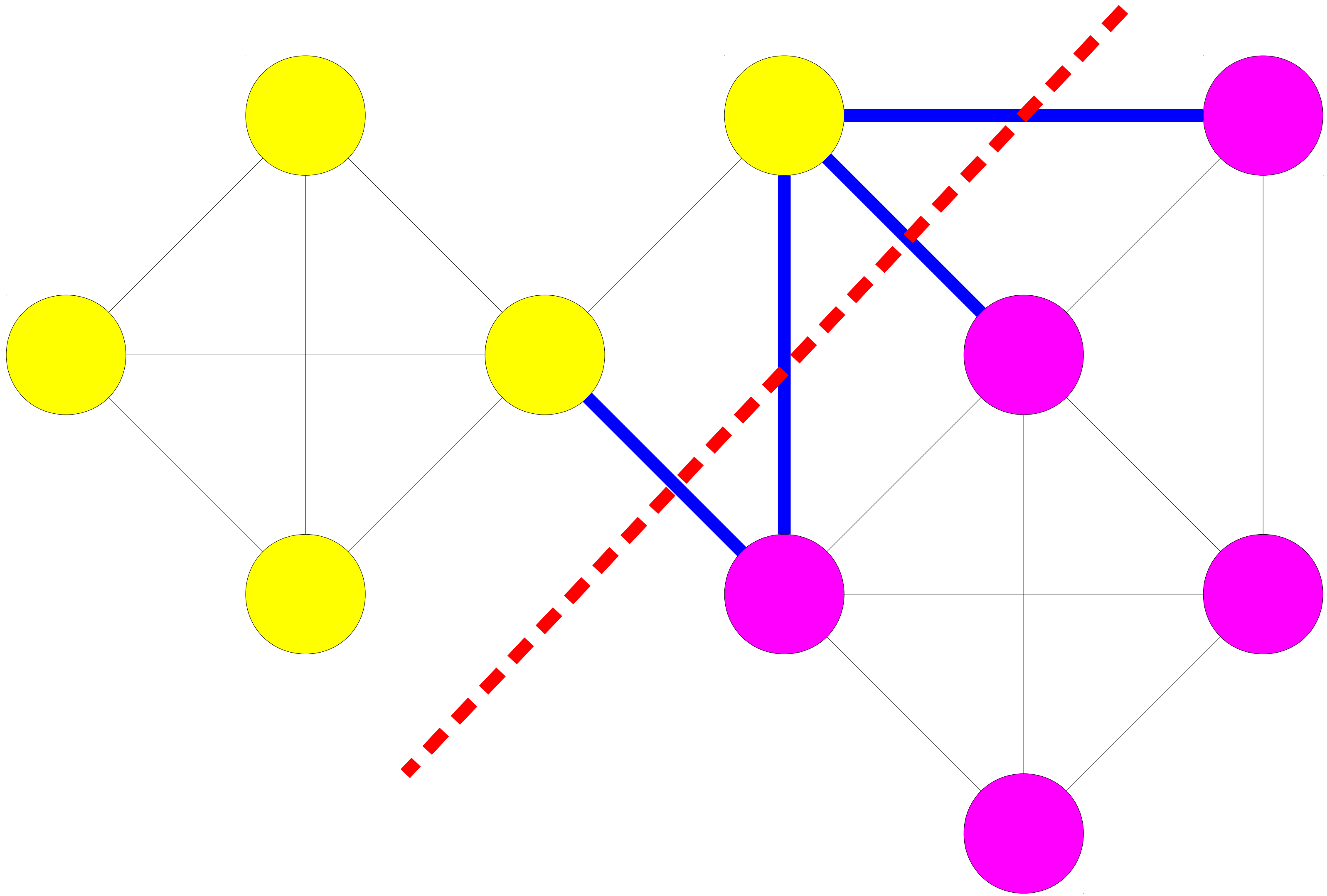
- Problem Set Three due on Monday (or Wednesday using a late period.)
- Problem Set Two graded; will be returned at the end of lecture.

Outline for Today

- **Global Minimum Cut**
  - What is the easiest way to split a graph into pieces?
- **Karger's Algorithm**
  - A simple randomized algorithm for finding global minimum cuts.
- **The Karger-Stein Algorithm**
  - A fast, simple, and elegant randomized divide-and-conquer algorithm.

Recap: Global Cuts

Disconnecting a Graph



Global Min Cuts

- A **cut** in a graph $G = (V, E)$ is a way of partitioning V into two sets S and $V - S$. We denote a cut as the pair $(S, V - S)$.
- The **size** of a cut is the number of edges with one endpoint in S and one endpoint in $V - S$. These edges are said to **cross** the cut.
- A **global minimum cut** (or just **min cut**) is a cut with the least total size.
  - Intuitively: removing the edges crossing a min cut is the easiest way to disconnect the graph.

Source: http://sorreluk.deviantart.com/art/Sunflower-VI-134302826

Image Segmentation

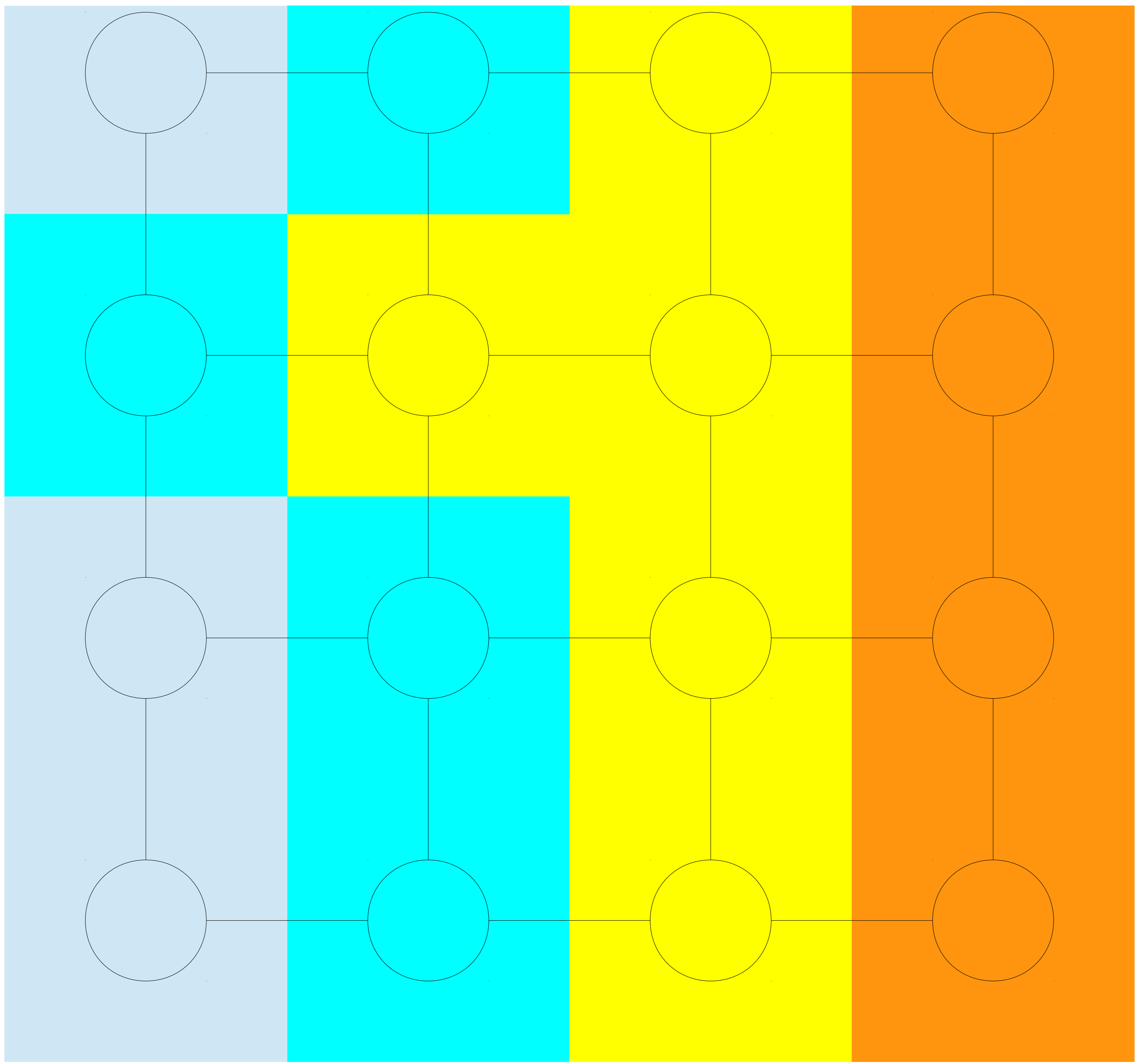
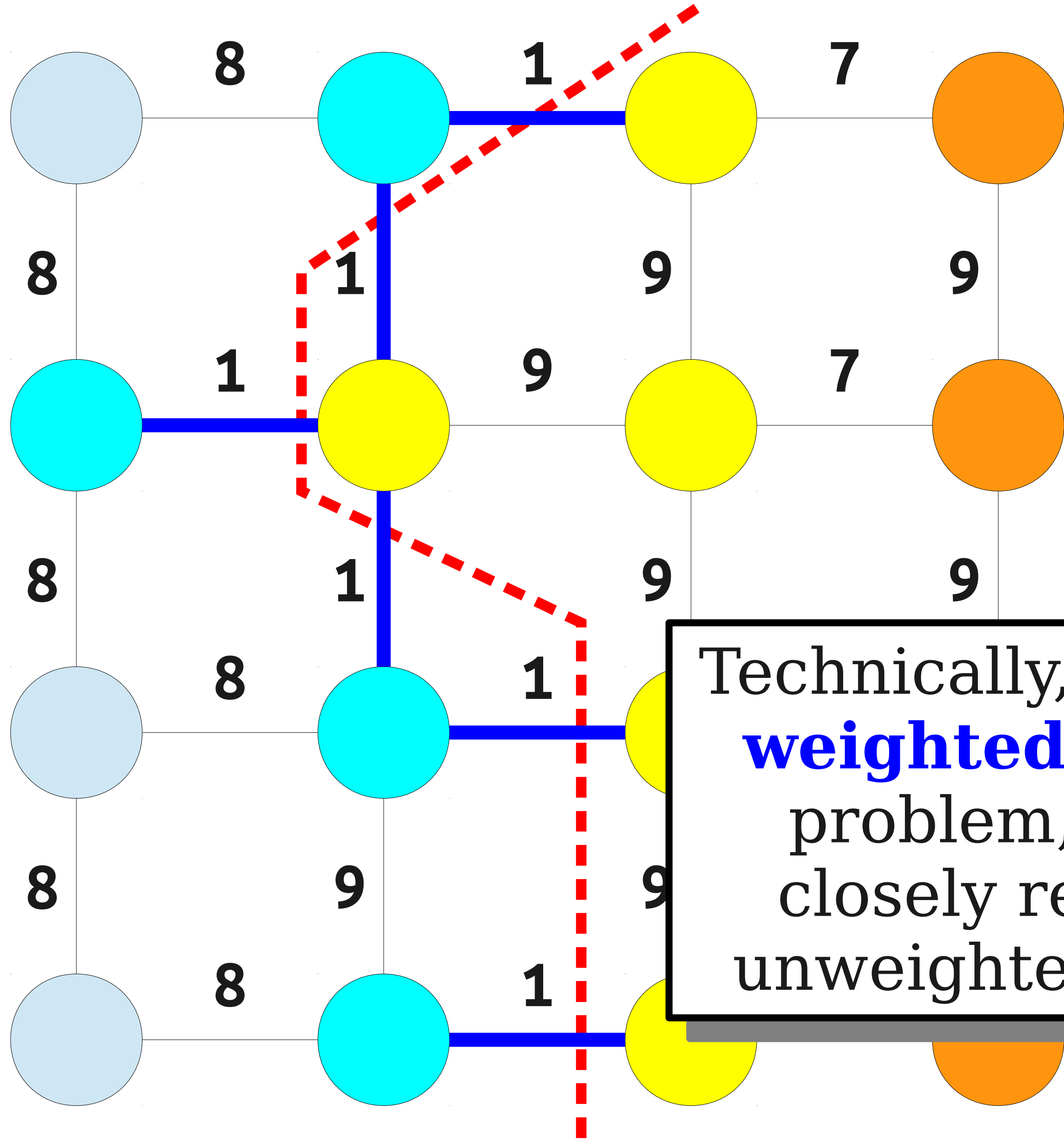
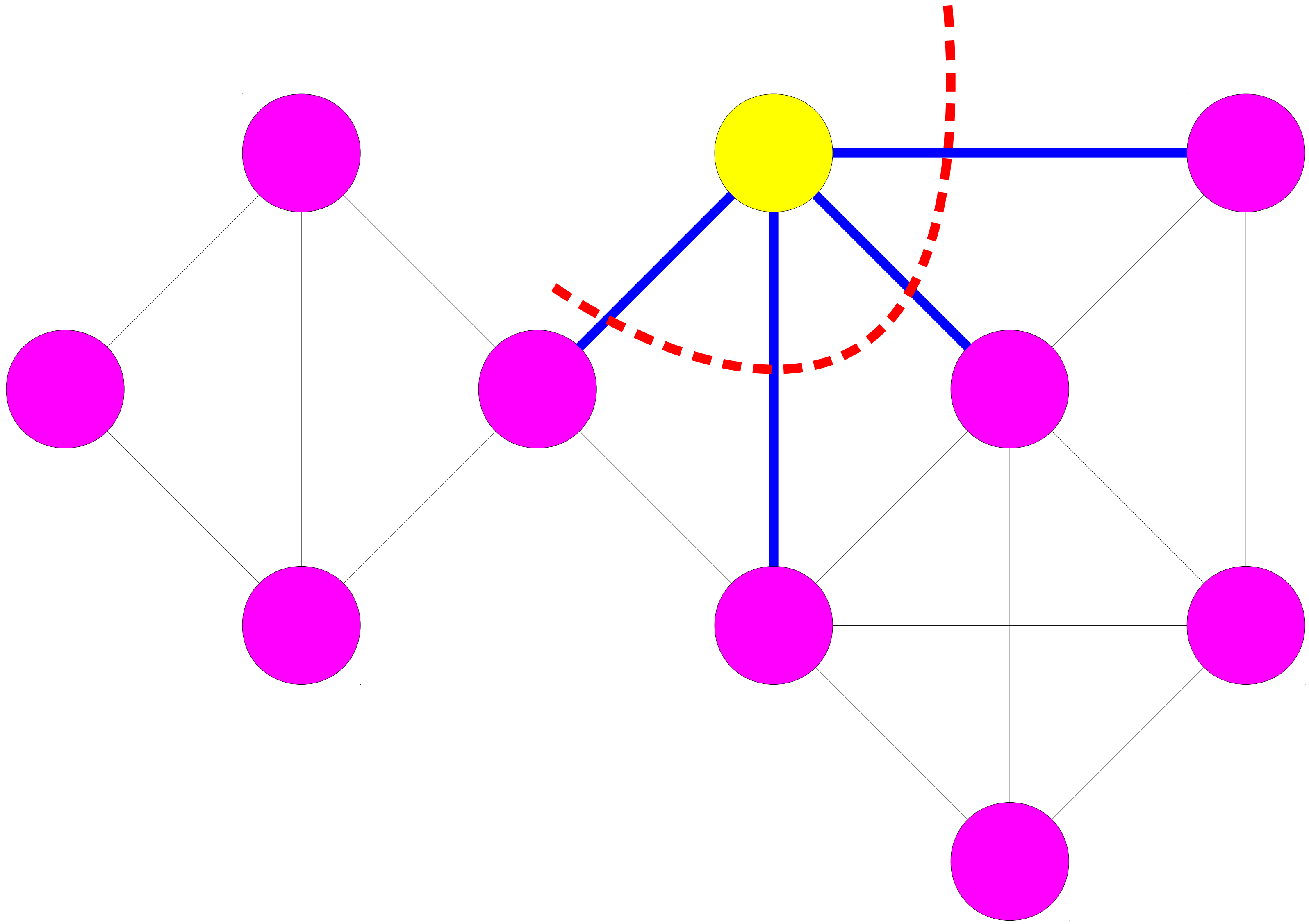


Image Segmentation



Technically, this is the **weighted min cut** problem, but it's closely related to unweighted min cut.

Properties of Min Cuts



Properties of Min Cuts

- **Claim**: The size of a min cut is at most the minimum degree in the graph.
- If v has the minimum degree, then the cut $(\{v\}, V - \{v\})$ has size equal to $\deg(v)$.
- Since the minimum cut is no larger than any cut in the graph, this means that minimum cut has size at most $\deg(v)$ for any $v \in V$.

Properties of Min Cuts

Theorem: In an n-node graph, if there is a min cut with cost k, there must be at least nk / 2 edges.

Proof: If there is a minimum cut with cost k, every node must have degree at least k (since otherwise there would be a cut with cost less than k). Therefore, by the handshaking lemma, we have

$$m = \frac{\sum_{v \in V} deg(v)}{2} \geq \frac{\sum_{v \in V} k}{2} = \frac{nk}{2}$$

And so $m \geq nk$ / 2, as required. ■

Finding a Global Min Cut:
Karger's Algorithm

Karger's Algorithm

- Given an edge (u, v) in a multigraph, we can **contract** u and v as follows:
  - Delete all edges between u and v.
  - Replace u and v with a new “supernode” uv.
  - Replace all edges incident to u or v with edges incident to the supernode uv.
- **Karger's algorithm** is as follows:
  - If there are exactly two nodes left, stop. The edges crossing those nodes form a cut.
  - Otherwise, pick a random edge, contract it, then repeat.

Karger's Algorithm

- Consider any cut $C = (S, V – S)$.
- If we never contract any edges crossing C, then Karger's algorithm will produce the cut C.
  - Initially, all nodes are in their own cluster.
  - Contracting an edge that does not cross the cut can only connect nodes that both belong to the same side of the cut.
  - Stops when two supernodes remain, which must be the sets S and $V – S$.

The Story So Far

- We now have the following:

Karger's algorithm produces cut C iff it never contracts an edge crossing C.

- How does this relate to min cuts?
- Across all cuts, min cuts have the lowest probability of having an edge contracted.
  - Fewer edges than all non-min cuts.
- Intuitively, we should be more likely to get a min cut than a non-min cut.
- What is the probability that we do get a min cut?

Defining Random Variables

- Choose any minimum cut C; let its size be k.
- Define the event $\mathcal{E}$ to be the event that Karger's algorithm produces C.
- This means that on each iteration, Karger's algorithm must not contract any of the edges crossing C.
- Let $\mathcal{E}_k$ be the event that on iteration k of the algorithm, Karger's algorithm does not contract an edge crossing C.
- Then $\mathcal{E} = \bigcap_{i=1}^{n-2} \mathcal{E}_i$

Can anyone explain the summation bounds?

Evaluating the Probability

- We want to know

$$P(\mathcal{E}) = P\left(\bigcap_{i=1}^{n-2} \mathcal{E}_i\right)$$

- These events are *not* independent of one another. *(Why?)*
- By the **chain rule for conditional probability**:

$$P\left(\bigcap_{i=1}^{n-2} \mathcal{E}_i\right) = P(\mathcal{E}_{n-2} | \mathcal{E}_{n-3}, \ldots, \mathcal{E}_1) P(\mathcal{E}_{n-3} | \mathcal{E}_{n-4}, \ldots, \mathcal{E}_1) \ldots P(\mathcal{E}_2 | \mathcal{E}_1) P(\mathcal{E}_1)$$

The First Iteration

- First, let's evaluate $P(\mathcal{E}_1)$, the probability that we don't contract an edge from C.
- For simplicity, we'll evaluate $P(\bar{\mathcal{E}}_1)$, the probability we *do* contract an edge from C on the first round.
- If our min cut has k edges, the probability that we choose one of the edges from C is given by k / m.
- Since the min cut has k edges, $m \geq kn / 2$. Therefore:

$$P(\bar{\mathcal{E}}_1) = \frac{k}{m} \leq \frac{k}{nk/2} = \frac{2}{n}$$

- So

$$P(\mathcal{E}_1) = 1 - P(\bar{\mathcal{E}}_1) \geq 1 - \frac{2}{n} = \frac{n-2}{n}$$

Successive Iterations

- We now need to determine

$$P\left(\mathcal{E}_i | \mathcal{E}_{i-1}\mathcal{E}_{i-2}\ldots\mathcal{E}_1\right)$$

- This is the probability that we don't contract an edge in C in round i, given that we haven't contracted any edge in C at this point.
- As before, we'll look at the complement of this event:

$$P\left(\bar{\mathcal{E}}_i | \mathcal{E}_{i-1}\mathcal{E}_{i-2}\ldots\mathcal{E}_1\right)$$

- This is the probability we *do* contract an edge from C in round i given that we haven't contracted any edges before this.

Successive Iterations

- At iteration i, $n - i + 1$ supernodes remain.
- **Claim:** Any cut in the contracted graph is also a cut in the original graph.
- Since C has size k, all $n - i + 1$ supernodes must have at least k incident edges. *(Why?)*
- Total number of edges at least $k(n - i + 1) / 2$.
- Probability we contract an edge from C is

$$P\left(\bar{\mathcal{E}}_i \mid \mathcal{E}_{i-1}\mathcal{E}_{i-2}\ldots\mathcal{E}_1\right) \leq \frac{k}{k(n-i+1)/2} = \frac{2}{n-i+1}$$

- So

$$P\left(\mathcal{E}_i \mid \mathcal{E}_{i-1}\mathcal{E}_{i-2}\ldots\mathcal{E}_1\right) \geq 1-\frac{2}{n-i+1} = \frac{n-i-1}{n-i+1}$$

$$
\begin{aligned}
P(\mathcal{E}) &= P(\mathcal{E}_{n-2}|\mathcal{E}_{n-3},\ldots,\mathcal{E}_1)\ldots P(\mathcal{E}_2|\mathcal{E}_1)P(\mathcal{E}_1) \\
&\geq \frac{n-(n-2)-1}{n-(n-2)+1}\cdot\frac{n-(n-3)-1}{n-(n-3)+1}\cdot\ldots\frac{n-2}{n} \\
&= \frac{1}{3}\cdot\frac{2}{4}\cdot\ldots\frac{n-2}{n} \\
&= \prod_{i=1}^{n-2}\frac{i}{i+2} \\
&= \prod_{i=1}^{n-2} i \quad / \quad \prod_{i=1}^{n-2} i+2 \\
&= \prod_{i=1}^{n-2} i \quad / \quad \prod_{i=3}^{n} i \\
&= \left(1\cdot 2\cdot\prod_{i=3}^{n-2} i\right) \quad / \quad \left(n\cdot(n-1)\cdot\prod_{i=3}^{n-2} i\right) \\
&= \frac{2}{n(n-1)}
\end{aligned}
$$

The Success Probability

- Right now, the probability that the algorithm finds a minimum cut is at least
$$\frac{2}{n(n-1)}$$
- This number is low, but it's not as low as it might seem.
  - How may total cuts are there?
  - If we picked a cut randomly and there was just one min cut, what's the probability that we would find it?

Amplifying the Probability

- Recall: running an algorithm multiple times and taking the best result can amplify the success probability.
- Run Karger's algorithm for k iterations and take the smallest cut found. What is the probability that we *don't* get a minimum cut?

$$\left(1-\frac{2}{n(n-1)}\right)^k$$

A Useful Inequality

- For any $x \geq 1$, we have

$$\frac{1}{4} \leq \left(1 - \frac{1}{x}\right)^x \leq \frac{1}{e}$$

- If we run Karger's algorithm $n(n - 1) / 2$ times, the probability we don't get a minimum cut is

$$\left(1 - \frac{2}{n(n-1)}\right)^{\frac{n(n-1)}{2}} \leq \frac{1}{e}$$

- If we run Karger's algorithm $(n\ (n - 1) / 2) \ln n$ times, the probability we don't get a minimum cut is

$$\left(1 - \frac{2}{n(n-1)}\right)^{\left(\frac{n(n-1)}{2}\right)\ln n} \leq \left(\frac{1}{e}\right)^{\ln n} = \frac{1}{n}$$

The Overall Result

- Running Karger's algorithm $O(n^2 \log n)$ times produces a minimum cut with high probability.
- ***Claim:*** Using adjacency matrices, it's possible to run Karger's algorithm once in time $O(n^2)$.
- ***Theorem:*** Running Karger's algorithm $O(n^2 \log n)$ times gives a minimum cut with high probability and takes time $O(n^4 \log n)$.

Speeding Things Up:

The Karger-Stein Algorithm

Some Quick History

- David Karger developed the contraction algorithm in 1993. Its runtime was $O(n^4 \log n)$.
- In 1996, David Karger and Clifford Stein (the "S" in CLRS) published an improved version of the algorithm that is *dramatically* faster.
- **The Good News:** The algorithm makes intuitive sense.
- **The Bad News**: Some of the math is really, really hard.

Some Observations

- Karger's algorithm only fails if it contracts an edge in the min cut.
- The probability of contracting the wrong edge increases as the number of supernodes decreases.
  - *(Why?)*
- Since failures are more likely later in the algorithm, repeat just the later stages of the algorithm when the algorithm fails.

Intelligent Restarts

- Interesting fact: If we contract from n nodes down to $n/\sqrt{2}$ nodes, the probability that we don't contract an edge in the min cut is about 50%.
  - Can work out the math yourself if you'd like.
- What happens if we do the following?
  - Contract down to $n/\sqrt{2}$ nodes.
  - Run *two* passes of the contraction algorithm from this point.
  - Take the better of the two cuts.

The Success Probability

- This algorithm finds a min cut iff
  - The partial contraction step doesn't contract an edge in the min cut, and
  - At least one of the two remaining contractions does find a min cut.
- The first step succeeds with probability around 50%.
- Each remaining call succeeds with probability at least 4 / $n(n - 1)$.
  - *(Why?)*

The Success Probability

$$
\begin{aligned}
P(success) &\geq \frac{1}{2}\left(1-\left(1-\frac{4}{n(n-1)}\right)^2\right) \\
&= \frac{1}{2}\left(1-\left(1-\frac{8}{n(n-1)}+\frac{16}{n^2(n-1)^2}\right)\right) \\
&= \frac{1}{2}\left(\frac{8}{n(n-1)}-\frac{16}{n^2(n-1)^2}\right) \\
&= \frac{4}{n(n-1)}-\frac{8}{n^2(n-1)^2}
\end{aligned}
$$

A Success Story

- This new algorithm has roughly twice the success probability as the original algorithm!
- **Key Insight:** Keep repeating this process!
  - Base case: When size is some small constant, just brute-force the answer.
  - Otherwise, contract down to $n / \sqrt{2}$ nodes, then recursively apply this algorithm twice to the remaining graph and take the better of the two results.
- This is the **Karger-Stein** algorithm.

Two Questions

- What is the success probability of this new algorithm?
  - This is extremely difficult to determine.
  - We'll talk about it later.
- What is the runtime of this new algorithm?
  - Let's use the Master Theorem?

The Runtime

- We have the following recurrence relation:

$$\boxed{\begin{array}{ll} T(n) = c & \text{if } n \leq n_0 \\ T(n) = 2T(n/\sqrt{2}) + O(n^2) & \text{otherwise} \end{array}}$$

- What does the Master Theorem say about it?

$$\mathbf{T(\boldsymbol{n}) = O(\boldsymbol{n}^2 \log \boldsymbol{n})}$$

The Accuracy

- By solving a very tricky recurrence relation, we can show that this algorithm returns a min cut with probability Ω(1 / log *n*).
- If we run the algorithm roughly $\ln^2 n$ times, the probability that *all* runs fail is roughly

$$\left(1-\frac{1}{\ln n}\right)^{\ln^2 n} \leq \left(\frac{1}{e}\right)^{\ln n} = \frac{1}{n}$$

- ***Theorem:*** The Karger-Stein algorithm is an $O(n^2 \log^3 n)$-time algorithm for finding a min cut with high probability.

Major Ideas from Today

- You can increase the success rate of a Monte Carlo algorithm by iterating it multiple times and taking the best option found.
  - If the probability of success is $1 / f(n)$, then running it $O(f(n) \log n)$ times gives a high probability of success.
- If you're more intelligent about *how* you iterate the algorithm, you can often do much better than this.

Next Time

- Hash Tables
- Universal Hashing