§2. Complex 2-Forms: Cauchy-Pompeiu's Formula

In §3.1 we have seen $df = f_z dz + f_{\bar{z}} d\bar{z}$, (where $f_z = \partial f/\partial z$ and $f_{\bar{z}} = \partial f/\partial \bar{z}$,) for a complex (smooth) function f. Recall: $z = x + iy$, $\bar{z} = x - iy$, $dz = dx + idy$ and $d\bar{z} = dx - idy$. In general, a complex 1-form on a set in the complex plane is an expression of the form $\omega = gdz + hd\bar{z}$, where g and h are complex functions. By using exterior products, we can build complex 2-forms from complex 1-forms.

Example 2.1. Verify the following identities:

$$dz \wedge dz = 0, \quad d\bar{z} \wedge d\bar{z} = 0, \quad \text{and} \quad d\bar{z} \wedge dz = -dz \wedge d\bar{z} = 2i\ dx \wedge dy. \tag{2.1}$$

(Suggestion: keep these identities in mind.)

Solution. Here we go:

$$\begin{aligned} dz \wedge dz &= (dx + idy) \wedge (dx + idy) \\ &= dx \wedge dx + i\ dy \wedge dx + i\ dx \wedge dy - dy \wedge dy \\ &= 0 + i(-dx \wedge dy) + i\ dx \wedge dy - 0 = 0, \\ d\bar{z} \wedge dz &= (dx - idy) \wedge (dx + idy) \\ &= dx \wedge dx - i\ dy \wedge dx + i\ dx \wedge dy + dy \wedge dy \\ &= 0 - i(-dx \wedge dy) + i\ dx \wedge dy + 0 = 2i\ dx \wedge dy. \end{aligned}$$

In the same way we can show that $d\bar{z} \wedge d\bar{z} = 0$ and $-dz \wedge d\bar{z} = 2i\ dx \wedge dy$.

Given 1-forms $\omega_1 = f_1 dz + g_1 d\bar{z}$ and $\omega_2 = f_2 dz + g_2 d\bar{z}$, using (2.1), we have

$$\begin{aligned} \omega_1 \wedge \omega_2 &= (f_1 dz + g_1 d\bar{z}) \wedge (f_2 dz + g_2 d\bar{z}) = f_1 g_2\ dz \wedge d\bar{z} + g_1 f_2\ d\bar{z} \wedge dz \\ &= (f_1 g_2 - f_1 h_2)\, dz \wedge d\bar{z} \equiv \begin{vmatrix} f_1 & g_1 \\ f_2 & g_2 \end{vmatrix} dz \wedge d\bar{z}. \end{aligned} \tag{2.2}$$

The **differential** $d\omega$ of a complex 1-form $\omega = gdz + hd\bar{z}$ is the complex 2-form given by $d\omega = dg \wedge dz + dh \wedge d\bar{z}$. More explicitly,

$$\begin{aligned} d\omega &= dg \wedge dz + dh \wedge d\bar{z} = (g_z dz + g_{\bar{z}} d\bar{z}) \wedge dz + (h_z dz + h_{\bar{z}} d\bar{z}) \wedge d\bar{z} \\ &= g_{\bar{z}}\ d\bar{z} \wedge dz + h_z dz \wedge d\bar{z} = (g_{\bar{z}} - h_z) d\bar{z} \wedge dz. \end{aligned} \tag{2.3}$$

(There is no need to memorize (2.2) and (2.3).)

Example 2.2. Verify that 1-form $\omega = zd\bar{z} + \bar{z}dz$ is closed, that is, $d\omega = 0$.

Solution: $d\omega = dz \wedge d\bar{z} + d\bar{z} \wedge dz = dz \wedge d\bar{z} - dz \wedge d\bar{z} = 0$. Alternatively, we can proceed as follows: $d(zd\bar{z} + \bar{z}dz) = d(d(z\bar{z})) = 0$, in view of $d^2 = 0$.

It is convenient to introduce the following "partial differentials":

$$\partial f = \frac{\partial f}{\partial z}\, dz, \quad \overline{\partial} f = \frac{\partial f}{\partial \bar{z}}\, d\bar{z}.$$

(Note: in some books $\overline{\partial} f$ stands for $\partial f / \partial \bar{z}$.) Then identity $df = f_z dz + f_{\bar{z}} d\bar{z}$ becomes $df = \partial f + \overline{\partial} f$. So we simply write $d = \partial + \overline{\partial}$. For a 1-form $\omega = f dz + g d\bar{z}$, we have

$$\overline{\partial}\omega = \overline{\partial} f \wedge dz + \overline{\partial} g \wedge d\bar{z} = f_{\bar{z}} d\bar{z} \wedge dz + g_{\bar{z}} d\bar{z} \wedge d\bar{z} = f_{\bar{z}}\, d\bar{z} \wedge dz.$$

Similarly we have $\partial\omega = g_z\, dz \wedge d\bar{z}$. Notice that $\partial^2 f = \partial(\partial f) = \partial(f_z\, dz) = 0$. Thus we write $\partial^2 = 0$. Similarly we have $\overline{\partial}^2 = 0$. We already know that $d^2 = 0$ (Rule E6). So

$$0 = d^2 = (\partial + \overline{\partial})(\partial + \overline{\partial}) = \partial^2 + \partial\overline{\partial} + \overline{\partial}\partial + \overline{\partial}^2 = \partial\overline{\partial} + \overline{\partial}\partial.$$

So we have $\partial\overline{\partial} + \overline{\partial}\partial = 0$. This makes us curious about $\overline{\partial}\partial$. Well,

$$\overline{\partial}\partial f = \overline{\partial}(f_z dz) = \overline{\partial} f_z \wedge dz = f_{\bar{z}z}\, d\bar{z} \wedge dz = \frac{1}{4}\Delta f.\ 2i dx \wedge dy = \frac{i}{2}\Delta f\ \, dx \wedge dy,$$

where $\Delta f = f_{xx} + g_{yy}$, the Laplacian of f; (see (1.6) in §3.1 for $f_{z\bar{z}} = \frac{1}{4}\Delta f$).

In §3.1 we have introduced the complex 1-form $z^{-1}dz$ to study the complex logarithm $\log z$. Here we entertain a slightly more general expression $\chi = (z - z_0)^{-1} dz$ for some fixed $z_0 = x_0 + iy_0$, which is defined for all z except z_0. Notice that

$$d\chi = d(z - z_0)^{-1} \wedge dz = (-1)(z - z_0)^{-2} dz \wedge dz = 0.$$

Thus χ is a closed 1-form. However, χ is not an exact 1-form, that is, there is no function f defined on the complex plane with the point z_0 deleted such that $df = \chi$. We have seen the kind of argument at the end of §1.1 leading to this conclusion. But let us repeat here using different symbols. Take a loop around z_0, say $\gamma(t) = z_0 + re^{it}$ $(0 \le t \le 2\pi)$, where r is any positive number and e^{it} stands for $\cos t + i \sin t$. Then

$$\int_\gamma \chi = \int_\gamma \frac{dz}{z - z_0} = \int_0^{2\pi} \frac{d(z_0 + re^{it})}{(z_0 + re^{it}) - z_0} = \int_0^{2\pi} \frac{r \times ie^{it} dt}{re^{it}} = \int_0^{2\pi} i dt = 2\pi i.$$

Thus we have:

$$\frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0} dz = 1 \tag{2.4}$$

(Suggestion: keep this identity in mind.) If χ were exact, say $\chi = df$, we would have $\int_\gamma \chi = \int_\gamma df = f(\gamma(2\pi)) - f(\gamma(0)) = f(z_0 + r) - f(z_0 + r) = 0$. So χ is not exact.

Take any complex function f and differentiate the 1-form $f\chi$:

$$\begin{aligned} d(f\chi) &= df \wedge \chi + f d\chi = df \wedge \chi \quad \text{[because of the closedness of } \chi] \\ &= (f_z dz + f_{\bar z} d\bar z) \wedge (z - z_0)^{-1} dz \\ &= f_{\bar z}(z - z_0)^{-1} d\bar z \wedge dz \quad [\text{because } dz \wedge dz = 0] \\ &\equiv \frac{f_{\bar z}}{z - z_0} d\bar z \wedge dz = 2i \frac{f_{\bar z}}{z - z_0} dx \wedge dy. \end{aligned} \tag{2.5}$$

Now we take a domain D in the complex plane and apply Stokes' Theorem to $d(f\chi)$ over D. We have to be careful in doing this in view of the singularity of χ presented at z_0. More precisely, D is a bounded open set in the complex plane with (positively oriented) boundary ∂D consisting of finitely many smooth curves. Let f be a smooth function defined on its closure $\bar D$ $(= D \cup \partial D)$. Take a point z_0 in D and consider the 1-form $\omega = f\chi \equiv f(z)(z - z_0)^{-1} dz$, which is defined on $\bar D$ except at z_0. Due to the singularity of ω at z_0, we remove a tiny disk $D(z_0; r)$ centered at z_0 from D; here $D(z_0; r) = \{z|\ |z - z_0| \leq r\}$, the disk centered at z_0 with radius r. Assume that the radius r is so small that $D(z_0; r)$ is contained in D. Let $D_r = D \setminus D(z_0; r)$, which is the remaining part of D when this disk is removed. Notice that ∂D_r, the boundary of D_r, consists of the boundary ∂D of D, together with the boundary $\partial D(z_0; r)$ of $D(z_0; r)$ oriented in the opposite way. Thus we write $\partial D_r = \partial D \cup (-\partial D(z_0;\ r))$. Now ω has no singularity in D_r, we may apply Stokes' Theorem to ω over D_r:

$$\iint_{D_r} d\omega = \int_{\partial D_r} \omega = \int_{\partial D} \omega - \int_{\partial D(z_0;\ r)} \omega. \tag{2.6}$$

Here $\partial D(z_0; r)$ is a circle γ around the point z_0 we have seen before: it is described by the parametric equation $\gamma(t) = z_0 + re^{it}$ $(1 \leq t \leq 2\pi)$. Since $f(z_0 + re^{it})$ converges to $f(z_0)$ as the radius r shrinks to 0, the last line integral in (2.6)

$$\int_{\partial D(z_0;\ r)} \omega = \int_0^{2\pi} \frac{f(z_0 + re^{it})}{(z_0 + e^{it}) - z_0} de^{it} = i \int_0^{2\pi} f(z_0 + re^{it}) dt$$

converges to $i \int_0^{2\pi} f(z_0) dt = 2\pi i f(z_0)$ as r decreases to 0. Letting $r \to 0$ in (2.6), the disk $D(z_0; r)$ diminishes and D_r fills up D. Consequently we have

$$\iint_D d\omega = \int_{\partial D} \omega \ - \ 2\pi i f(z_0) \tag{2.7}$$

Now $\omega = (z - z_0)^{-1} f(z) dz$ and $d\omega = (z - z_0)^{-1} f_{\bar{z}} d\bar{z} \wedge dz$; see (2.5). Substituting them in (2.7) and rearranging terms, we arrive at the celebrated **Cauchy-Pompeiu's formula**:

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - z_0} dz - \frac{1}{2\pi i} \iint_D \frac{f_{\bar{z}}}{z - z_0} d\bar{z} \wedge dz. \tag{2.8}$$

We discuss two special cases. First, when f is analytic, $f_{\bar{z}} = 0$ and hence we have

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - z_0} dz \tag{2.9}$$

which is the well-known **Cauchy formula** in complex analysis. (Many elementary textbooks in complex analysis set are written in such a way that as though the authors designate half of the time for establishing this formula, and treat the rest as applications of it.) This formula tells us that the value of f at any point z_0 in D is completely determined by the values of f on the boundary of D. Note that Cauchy's formula tells us that (2.4) holds for any closed curve γ surrounding z_0, not just for a circle with z_0 as its center. Secondly, when f vanishes on the boundary of D, that is, $f\big|_{\partial D} = 0$, (2.8) becomes

$$f(\zeta) = \frac{1}{2\pi i} \iint_D \frac{f_{\bar{z}}(z)}{\zeta - z} = \frac{1}{\pi} \iint_D \frac{f_{\bar{z}}(z)}{\zeta - z}\, dx \wedge dy, \tag{2.10}$$

where z_0 is switched to ζ. This identity suggests how to find a solution to the following so-called $\bar{\partial}$**-equation** (pronounced as "dee bar equation")

$$\frac{\partial u}{\partial \bar{z}} = g \tag{2.11}$$

(which is a first order elliptic partial differential equation). Here g is a continuous function on the complex plane with a compact support (that is $g(z) = 0$ if $|z|$ is large enough.) Take a large circular disk D in the complex plane so that g vanishes outside D. Let us restrict our attention to solutions on D, that is, those u defined on D so that the identity $\partial u/\partial \bar{z} = g$ holds on D. Notice that if u is a solution and v is an analytic function on D, then $u + v$ is another solution; indeed $(\partial/\partial \bar{z})(u + v) = (\partial/\partial \bar{z})u + (\partial/\partial \bar{z})v = g + 0 = g$. We can reverse the above argument to show that the general solution to (4.11) is of the form $u + v$, where v is an analytic function. Thus, it is enough to find a special solution to (2.11). Suppose that u is indeed a solution. Then, by Cauchy-Pompeiu's formula, for all $\zeta \in D$,

$$u(\zeta) = \frac{1}{2\pi i} \int_{\partial D} \frac{u(z)}{z - \zeta}\, dz - \frac{1}{\pi} \iint_D \frac{u_{\bar{z}}}{z - \zeta}\, dx \wedge dy = v(\zeta) + \frac{1}{\pi} \iint_D \frac{g(z)}{\zeta - z}\, dx \wedge dy,$$

where $v(\zeta) = \frac{1}{2\pi i} \int_{\partial D} \frac{u(z)}{z-\zeta}\, dz$ can be shown to be analytic as a function of $\zeta \in D$: indeed, by the "punching and kicking" principle, we have

$$\frac{\partial}{\partial \bar{\zeta}}\, v(\zeta) = \frac{1}{2\pi i} \int_{\partial D} f(z) \frac{\partial}{\partial \bar{\zeta}} \left(\frac{1}{z-\zeta} \right) dz = 0.$$

Thus, assuming (2.11) has a solution, the function u_0 given by

$$u_0(\zeta) = \frac{1}{\pi} \iint_D \frac{g(z)}{\zeta - z}\, dx \wedge dy \tag{2.12}$$

is a solution to this $\bar{\partial}$-equation. Note that we have not established the existence of a solution to (2.11) yet. What we have done is that, if solutions do exist, then one of them can be explicitly written down as (2.12). In order to show that (2.11) has a solution, the most natural thing to do is to check directly that u_0 given by (2.12) is indeed a solution. This needs some delicate work and we do not pursue this endeavor. The function u_0 in by (2.12) is called the **Cauchy transform** of g and

$$C(\zeta, z) \equiv \frac{1}{\zeta - z}$$

is called the **Cauchy kernel** of this integral transform.

Exercises

1. The conjugate of a 1-form $\omega = f dx + g dy$ is defined to be $\overline{\omega} = \bar{f} dx + \bar{g} dy$. Verify the following identities: $\overline{dz} = d\bar{z}$, $\overline{d\bar{z}} = dz$, $\overline{\partial f} = \bar{\partial}\, \bar{f}$, $\overline{\bar{\partial} f} = \partial \bar{f}$.
2. (a) Verify that $(\partial - \bar{\partial})^2 = 0$. (b) Verify the product rule: $\bar{\partial}(f\omega) = \bar{\partial} f \wedge \omega + f \bar{\partial}\omega$.
3. Is the identity $\bar{\partial}(g^* f) = g^* \bar{\partial} f$ true ?

   (Hint: Take any analytic function f and any g so that $f \circ g$ is not analytic.)
4. Let f be an analytic function and write $w = u + iv = f(z) = f(x+iy)$ as usual. Then $dw = f'(z)\, dz$ and $d\bar{w} = \overline{f'(z)}\, d\bar{z}$. (a) Verify $d\bar{w} \wedge dw = |f'(z)|^2\, d\bar{z} \wedge dz$ and compare it to $dudv = \{\partial(u,v)/\partial(x,y)\}\, dxdy$ to deduce

   $$|f'(z)|^2 = \frac{\partial(u,v)}{\partial(x,y)}. \tag{2.13}$$

   (b) Use the Cauchy-Riemann equation to check (2.13) directly.

5. We extend Hodge's $*$-operator (see (2.1) in §1.2) to complex differential forms $\omega = Pdx + Qdy$ (where P and Q are complex-valued functions of x and y) by putting

$$*\omega = *(Pdx + Qdy) = \overline{P} * dx + \overline{Q} * dy \equiv \overline{P}\, dy - \overline{Q}\, dx. \tag{2.14}$$

   (a) Check that $**\omega = -\omega$.

   (b) Check that $*dz = id\bar{z}$ and $*d\bar{z} = -idz$.

   (c) Verify that the Cauchy Riemann equations $u_x = v_y$, $u_y = -v_x$ (see (1.8) in §3.1) can be rewritten as $*du = dv$.

   (d) Check that a real valued function $w = w(x, y)$ is harmonic if and only if $d{*}dw = 0$.

6. We say that a real-valued function $v = v(x, y)$ is a **harmonic conjugate** of a real-valued function $u = u(x, y)$ if $dv = *du$. From part (c) of the previous exercise we know that the last condition is equivalent to $u + iv$ being a holomorphic function of $z = x + iy$. In each of the following parts, show that the given function $u = u(x, y)$ is harmonic and find a harmonic conjugate of it.

   (a) $u = x^2 - y^2$, (b) $u = x^3 - 3xy^2$ (c) $u = e^x \cos y$ (d) $u = \log\sqrt{x^2 + y^2}$.

7. Let $u = u(x, y)$ be a harmonic function on $\mathbf{D}^o = \{z = x + iy \in \mathbf{C} : |z| < 1\}$ such that $u(0, 0) = 0$. Check that $\partial u/\partial z$ is an analytic function and the complex 1-form $(\partial u/\partial z)\, dz$ is closed. For each z in $\mathbf{D}^o$, take any path linking 0 to z and define

$$f(z) = 2\int_\gamma \frac{\partial u}{\partial z}\, dz.$$

   Show that f is a well defined analytic function, with $\partial f/\partial z = 2\partial u/\partial z$. Finally, check that the real part of f is u.

8. The punctured plane $\mathbf{C}_0$ is defined to be the complex plane with the origin removed, that is, $\mathbf{C}_0 = \mathbf{C}\backslash\{0\}$. The **reflection** of the punctured plane (with respect to the unit circle) is defined to be the transformation R of $\mathbf{C}_0$ given by $R(z) = 1/\bar{z}$.

   (a) Verify that R is idempotent in the sense that $R(R(z)) = z$.

   (b) Verify that three points 0, z and $R(z)$ are collinear, and $|z||R(z)| = 1$. Make a simple sketch (including the unit circle, the origin, the ray from the origin containing z and $R(z)$) to understand the effect of R. Also check that R leaving all points on the unit circle fixed, that is, $R(z) = z$ if $|z| = 1$.

(c) Check that if we identify the complex variable $z = x + iy$ with (x, y), then

$$R(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right)$$

(The above identity suggests how to generalize R to higher dimensional spaces.)

(d) Check that $R^*(z^{-1}dz) = -\bar{z}^{-1}d\bar{z}$ and $R^*(\bar{z}^{-1}d\bar{z}) = -z^{-1}dz$; (recall that R^* stands for pull back by R). Deduce that the 2-form $\Omega \equiv |z|^{-2}dz \wedge d\bar{z}$ is invariant under R in the sense that $R^*\Omega = \Omega$ and consequently the 2-form $(x^2+y^2)^{-1}dx\wedge dy$ is also invariant under R.

(e) Verify the following remarkable identity the reflection R:

$$\frac{|e^{i\theta} - z|}{|e^{i\theta} - R(z)|} = |z|,$$

that is, the LHS is independent of which point $e^{i\theta}$ on the unit circle is chosen. (Remark: the image $R(z)$ of z under the reflexion R is often denoted by z^*. We avoid using this notation because it crashes with pullbacks.)

9. Let γ be the elliptic path given by $z(t) = a\cos t + ib\sin t$ $(0 \le t \le 2\pi)$, where a and b are positive constants. Use the identity $\int_\gamma dz/z = 2\pi i$ to verify

$$\int_0^{2\pi} \frac{\cos t \sin t}{a^2\cos^2 t + b^2 \sin^2 t}\, dt = 0 \quad \text{and} \quad \int_0^{2\pi} \frac{dt}{a^2\cos^2 t + b^2\sin^2 t} = \frac{2\pi}{ab}.$$

(The first identity can also be obtained by changing the variable t to $s = 2\pi - t$.)

10. Let $\mathbf{D}$ be the unit disk and let m, n be integers ≥ 0. Consider

$$I = \iint_{\mathbf{D}} \bar{z}^m z^n \, d\bar{z} \wedge dz.$$

Compute this double integral by two different methods: first, use Green's theorem (2.2); second, use polar coordinates: $d\bar{z} \wedge dz = 2i\, dx \wedge dy = 2i\, r dr d\theta$. (You need to use the identity $\int_0^{2\pi} e^{i(m-n)t}\, dt = 2\pi\delta_{mn}$, which can be easily checked.)

11. Let $\mathbf{D}$ be the unit disk and let χ be its characteristic function: $\chi(z) = 1$ if z belongs to $\mathbf{D}$ and $= 0$ otherwise. Consider $\varphi(\zeta) = (2\pi i)^{-1} \iint (\zeta - z)^{-1}\chi\, d\bar{z} \wedge dz$, the Cauchy transform of χ.

(a) Use polar coordinates to compute $\varphi(0)$ directly.

(b) Use Cauchy-Pompeiu's formula to find $\varphi(\zeta)$ for ζ in the interior of $\mathbf{D}$: $|\zeta| < 1$.

(c) Can you find $\varphi(\zeta)$ for general ζ?