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APPLICATIONS OF THE OPEN COLORING AXIOM

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1. INTRODUCTION

Standard forcing axioms are usually stated in the form which asserts the existence of sufficiently generic filters in every partial order $\mathcal{P}$ which belongs to a given class $\mathcal{K}$ of forcing notions. This approach, which is derived by "internalizing" generic extensions, has been very successful in providing strong forcing axioms and proving their consistency; in [FMS] a maximal axiom of this sort is proved consistent for the case when one wishes to consider only generic filters for families of at most $\aleph_1$ dense sets. However, when applying these axioms we need to know when there is a partial order in the class $\mathcal{K}$ which introduces the object we wish to find. Of course, there is no easy general answer to this question and even some of the most basic instances are still open.

Following the realization that many applications of forcing axioms involve finding homogeneous sets in certain kinds of partitions, in [TV] a study of the so-called Ramsey forcing axioms was initiated. The idea is that these statements would provide a combinatorial intermediary between the abstract forcing axioms and their applications. It turned out that in some cases they are equivalent to the axioms from which they are derived.

To be more specific suppose we are given an uncountable set S and a partition of the form:

$$[S]^n = K_0 \cup K_1 \tag{1}$$

or

$$[S]^{<\omega} = K_0 \cup K_1 \tag{2}$$

together with a class $\mathcal{K}$ of partial orders. Let us say that this partition is *$\mathcal{K}$-destructible* if there is a poset $\mathcal{P}$ in $\mathcal{K}$ which forces an uncountable subset H of S which is *0-homogeneous* (i.e. $[H]^n \subseteq K_0$ for (1) and $[H]^{<\omega} \subseteq K_0$ for (2)) and in addition every $s \in S$ is forced by some condition in $\mathcal{P}$ to be in H. Let $\mathrm{RFA}^n(\mathcal{K})$ ($\mathrm{RFA}^{<\omega}(\mathcal{K})$) be the statement that every $\mathcal{K}$-destructible partition of form (1) ((2)) has an uncountable 0-homogeneous set. The following results are proved in [TV].

Theorem 1.1. *If κ is an uncountable cardinal* MA_κ *is equivalent to the statement that for every ccc destructible partition of the form* (2) *with card(S) $\geq \kappa$ there are countably many 0-homogeneous sets whose union covers S.*

Theorem 1.2. $\mathrm{MA}_{\aleph_1}$ *is equivalent to* $\mathrm{RFA}^{<\omega}(\mathrm{CCC})$.

These results raise the following questions.

Question 1.1. Can the assertion in Theorem 1.1 be weakened to say that if κ is regular then for every ccc destructible partition of form *(2)* such that card(S) $\leq \kappa$ there is an 0-homogeneous subset of S of size κ?

Question 1.2. Is there $n < \omega$ such that $RFA^n(CCC)$ is equivalent to $MA_{\aleph_1}$?

It is possible that these statements provide a natural hierarchy of axioms whose limit is $\mathrm{MA}_{\aleph_1}$. These questions were further studied in [To2].

Now, turning to stronger axioms much less is known. When is there a proper poset forcing an uncountable homogeneous set for a partition of the form (1) or (2)? For our purposes we only need to know that iterations of σ-closed and ccc posets are proper. While for a given ccc destructible partition there always exists a ccc poset of size at most $\aleph_1$ which adds an uncountable 0-homogeneous set and, in fact, there is a poset of finite 0-homogeneous sets which does this, there is no known such bound in the case of proper posets. Thus, for example, the following is open.

Question 1.3. If there is a proper poset forcing an uncountable 0-homogeneous set to a partition of form (1) or (2), is there such a poset of size $< \beth_\omega(S)$?

It is known though that $\mathrm{RFA}^{<\omega}$(proper) has roughly the same consistency strength as PFA. Given these limitations of our knowledge we adopt a more modest approach by trying to find sufficient conditions for the existence of proper posets adjoining an uncountable 0-homogeneous set. This approach was taken by Todorcevic in [To1] where it was pursued in connection with the well-known (S) and (L) problems from general topology. The thesis is that this line of work would provide partition-type statements which lie at the core of many diverse problems and are thus more suitable for applications than the abstract forcing axioms. In this paper we offer further evidence for this point of view by focusing on one particular axiom of this kind which has been very successful in resolving questions about sets of reals. We present a survey of applications of this statement, study possible extensions and indicate directions for further research.

Thus, let us consider partitions of form (1) for $n = 2$. The idea is to put a topology on S and require the color classes to be open and closed respectively. It was first formulated explicitly by Abraham, Rubin, and Shelah ([ARS]) who were working on the extension of Baumgartner's consistency result that every two $\aleph_1$-dense sets without endpoints are isomorphic. Their work was further extended and refined by Todorcevic ([To1]) who formulated and proved the relative consistency with ZFC + $MA_{\aleph_1}$ of the following version of the Open Coloring Axiom (OCA):

If S is a set of reals and

$$[S]^2 = K_0 \cup K_1$$

is a partition with K_0 open in the product topology then either there exists an uncountable 0-homogeneous subset of S, or else S can be covered by countably many 1-homogeneous sets.

The statement of the original ARS-axiom was symmetric and required only the existence of an uncountable homogeneous set in one of the colors. As it turns out this amplification yields a much more useful axiom which has a particularly strong influence on $\mathcal{P}(\omega)/\text{fin}$ and related structures. Its additional advantage is that applying it does not require any knowledge of the niceties of forcing and is thus suitable for use by topologists, analysts, and other non-specialists in set theory working on subjects related to $\beta\omega$.

The paper is organized as follows. In section 2 we present a proof of the consistency of OCA, in fact, we derive it from PFA. Sections 3, 4 and 5 consist of applications of OCA. In section 3 we present some combinatorial consequences and show, for example, that OCA has strong influence on the partial order of all functions from ω to ω, ordered under eventual dominance. In particular, it implies that the least size of an unbounded subset of $\omega^\omega, <_*$ is $\aleph_2$. This gives evidence for the conjecture that OCA implies that the continuum is $\aleph_2$. In section 4 we turn to the study of automorphisms of $\mathcal{P}(\omega)/\text{fin}$. We show that OCA can be used to prove that every automorphism of $\mathcal{P}(\omega)/\text{fin}$ is trivial, i.e. is induced by an almost permutation of ω. In section 5 OCA is used to prove that a particular kind of topological space designated by $\gamma\omega$ cannot be completely normal. This implies that under PFA a version of Tychonoff's product theorem holds for countably compact spaces. Finally, in section 6, we consider possible extensions of OCA, and show, for example, that it cannot be generalized to dimensions bigger than 2. Then we raise some open problems and indicate areas for further research.

We believe that our notation is mostly standard, as for example in [Ku], or self explanatory.

2. Consistency of OCA

In this section we present the proof of the consistency of OCA ([To1, Theorems 4.4 and 8.0]) . We start with a ZFC result which is a natural generalization of the classical diagonalization argument of Sierpinski-Zygmund ([SZ]).

Theorem 2.1. *Let S be a set of reals and suppose*

$$[S]^2 = K_0 \cup K_1$$

is a given coloring where K_0 is open in the product topology. Assume that S is not the union of $< 2^{\aleph_0}$ 1-homogeneous sets. Then there is $Y \subseteq S$ of size $2^{\aleph_0}$ such that the poset of finite 0-homogeneous subsets of Y ordered by reverse inclusion has the $2^{\aleph_0}$-chain condition.

Proof. For $p \in S^n$ and open $U \subseteq S^n$ such that $p \in U$ let:

$$U_p = \{q \in U : q_i \neq p_i \text{ and } \{p_i, q_i\} \in K_0, \text{ for all } i < n\}.$$

If f is a function from $A \subseteq S^n$ into S and $p \in S^n$ let:

$$\omega_f(p) = \bigcap \{\mathrm{cl}(f(U_p \cap A)) : U \subseteq S^n \text{ open and } p \in U\}.$$

Let $\{f_\xi : \xi < 2^{\aleph_0}\}$ enumerate all countable functions from a finite power of S into S, and let $\{T_\xi : \xi < 2^{\aleph_0}\}$ enumerate all closed 1-homogeneous subsets of S. Build Y as the set $\{x_\xi : \xi < 2^{\aleph_0}\}$ such that:

(a) $x_\alpha \in S \setminus \{x_\xi : \xi < \alpha\}$,
(b) $x_\alpha \notin T_\xi$, for $\xi < \alpha$,
(c) x_α does not belong to any 1-homogeneous set which has the form $\omega_{f_\xi}(p) \cap S$, where $\xi < \alpha$ and p is a finite sequence from $\{x_\xi : \xi < \alpha\}$.

To prove Y works, assume that $\mathcal{F}$ is a disjoint family of $2^{\aleph_0}$ many finite 0-homogeneous subsets of Y. Without loss of generality we may assume that all elements of $\mathcal{F}$ have the same size $n \geq 1$. We prove, by induction on n, there there are two members of $\mathcal{F}$ whose union is 0-homogeneous. Case $n = 1$ is handled by (b). Suppose $n > 1$. For $s \in \mathcal{F}$ let $s = \{x_{s(0)}, \ldots, x_{s(n-1)}\}_<$ be the enumeration of s in the increasing order of indices, i.e. $s(0) < \ldots s(n-1) < 2^{\aleph_0}$. Identifying each s with an element of S^n we may assume that some fixed basic open set U in S^n separates all elements of $\mathcal{F}$. Thinking of $\mathcal{F}$ as a graph of an $(n-1)$-ary function g, where $g(s \restriction (n-1)) = x_{s(n-1)}$, for all $s \in \mathcal{F}$, let:

$$\mathcal{F}_0 = \{s \in \mathcal{F} : x_{s(n-1)} \in \omega_g(s \restriction (n-1)\}.$$

Claim 1. *$\mathcal{F} \setminus \mathcal{F}_0$ has size* $< 2^{\aleph_0}$.

Proof. Assume otherwise and for each $s \in \mathcal{F} \setminus \mathcal{F}_0$ pick a rational open interval I^s which contains $x_{s(n-1)}$ and is disjoint from $\omega_g(s \restriction (n-1))$. Fix also a basic open subset U^s of S^{n-1} containing $s \restriction (n-1)$ such that if $q \in U^s_{s\restriction(n-1)}$ then $g(q) \notin I^s$. Then there is a subset Z of $\mathcal{F} \setminus \mathcal{F}_0$ of size $2^{\aleph_0}$ such that the I^s for $s \in Z$ are all equal to some I and the U^s for $s \in Z$ are all equal to some U. By the inductive assumption there are $s, t \in Z$ such that $s \cup t$ is 0-homogeneous. But then $t \restriction (n-1) \in U_{s\restriction(n-1)}$ and $g(t \restriction (n-1)) \in I$, a contradiction. □

Let now g_0 be a countable dense subfunction of g. Then $g_0 = f_\xi$ for some ξ. Pick $s \in \mathcal{F}_0$ with all indices above ξ and above all the indices of elements of g_0. Then,

$$x_{s(n-1)} \in \omega_{f_\xi}(s \restriction (n-1))$$

and hence, by (c), $\omega_{f_\xi}(s \restriction (n-1))$ is not 1-homogeneous. We can now pick $u, v \in \omega_{f_\xi}(s \restriction (n-1))$ such that $\{u, v\} \in K_0$ and find open intervals I and J such that $u \in I$, $v \in J$, and $I \times J \subseteq K_0$. By the definition of $\omega_{g_0}(s \restriction (n-1))$, there is $p \in \mathrm{dom}(g_0)$ such that $p \cup s \restriction (n-1)$ is 0-homogeneous and $g_0(p) \in I$. Pick $U \subseteq S^{n-1}$ such that $s \restriction (n-1) \in U$ and for every $q \in U$ $p \cup q$ is 0-homogeneous. Now, pick $q \in U$ such that $g_0(q) \in J$. Then $p \cup \{g_0(p)\}$ and $q \cup \{g_0(q)\}$ are two members of $\mathcal{F}$ whose union is 0-homogeneous. □

Theorem 2.2. PFA *implies* OCA.

Proof. Fix a partition $[S]^2 = K_0 \cup K_1$ as in OCA and assume that S cannot be covered by countably many 1-homogeneous sets. This remains to hold in $V^{\mathcal{P}}$ where $\mathcal{P}$ is the σ-closed collapse of $2^{\aleph_0}$ to $\aleph_1$. In $V^{\mathcal{P}}$ CH holds so there is $Y \subseteq S$ such that the poset $\mathcal{Q}$ of finite 0-homogeneous subsets of Y is ccc. Some conditions in $\mathcal{Q}$ forces the generic homogeneous set to be uncountable and we may assume that the maximal condition does so. Thus, in $V^{\mathcal{P}*\mathcal{Q}}$ there is an uncountable 0-homogeneous set. By forcing internally with $\mathcal{P} * \mathcal{Q}$ we can produce such a set in V. □

Let us point out that although large cardinals are needed to prove the consistency of PFA this is not the case with OCA + $MA_{\aleph_1}$. Namely, we can start with a model of $V = L$ and perform a finite support iteration of ccc posets forcing $MA_{\aleph_1}$. Along the way, we use $\diamondsuit(\{\alpha < \omega_2 : \mathrm{cof}(\alpha) = \omega_1\})$ to guess potential open colorings on a set of reals S and, if possible, force with the poset from Theorem 2.1 to obtain an uncountable 0-homogeneous set. The resulting model then satisfies OCA + $MA_{\aleph_1}$. In [To1] OCA is shown to be equivalent to the following closed set-mapping axiom (CSM):

> *If F is a closed set-mapping on a set of reals, then either there is an uncountable F-free subset of dom(F), or else F is the union of countably many connected subfunction.*

Note that the strength of OCA comes from the fact that, although the partition is assumed to be open, S is allowed to be an arbitrary set of reals. Qi Feng ([Fe]) has studied versions of OCA obtained by restricting the complexity of the set S and has shown that the restriction of OCA to projective sets of reals follows from PD.

3. Combinatorial Applications

We start by presenting some consequences of the weak version of OCA. The following is [To1, Theorem 8.4]; but see also [Ba, Theorems 6.13 and 6.14].

Theorem 3.1. (OCA)

(a) *Every uncountable subset of $\mathcal{P}(\omega)$ contains an uncountable chain or an uncountable antichain.*
(b) *Every function from an uncountable set of reals into the reals in monotonic on an uncountable set.*
(c) *If X and Y are two uncountable sets of reals then there is a strictly increasing mapping from an uncountable subset of X into Y.*
(d) *Every uncountable Boolean algebra contains an uncountable antichain.*
(e) *Every subset of ω^ω of size $\aleph_1$ is bounded under $<_*$.*

Proof. To see (a), (b), and (c) observe that the inclusion is an closed relation on $\mathcal{P}(\omega)$, and that strictly increasing is an open relation in the plain. For (d), first show that if $\mathcal{B}$ is an uncountable Boolean algebra with no uncountable antichains then $\mathcal{B}$ can be embedded into $\mathcal{P}(\omega)$. Then use (a) and (b). For (e) let $\mathcal{F}$ be a subset of ω^ω of size $\aleph_1$. We may assume that each function in $\mathcal{F}$ is strictly increasing and that $\mathcal{F}$ is well-ordered by $<_*$ of order type ω_1. The everywhere dominance $<$ is a closed relation on ω^ω.

Since there are no uncountable linearly ordered sets under $<$, by OCA $\mathcal{F}$ has an uncountable pairwise incomparable subset A. Then by [To1, §1] A and hence $\mathcal{F}$ is bounded under $<_*$. $\square$

The following result is implicit in [To1]. It shows that OCA has a strong influence on the partial ordering ω^ω and gives support for the conjecture that OCA implies that the continuum is $\aleph_2$.

Theorem 3.2. OCA *implies that the least size of an unbounded subset of* ω^ω *under* $<_*$ *is* $\aleph_2$.

Proof (see [To1, Theorem 3.7]). By Theorem 3.1(e) every subset of ω^ω of size $\aleph_1$ is bounded under $<_*$. To produce an unbounded subset of size $\aleph_2$ we shall need the following result which is of independent interest. Recall that a *gap* in ω^ω is a pair $\langle A, B\rangle$ of subsets of ω^ω such that:

(a) the order type of $A, <_*$ is a regular infinite cardinal,
(b) the order type of $B, <_*$ is the converse of a regular infinite cardinal,
(c) $f <_* g$ for all $f \in A$ and $g \in B$,
(d) there is no $h \in \omega^\omega$ such that $f <_* h <_* g$ for all $f \in A$ and $g \in B$.

Lemma 3.1. (OCA) *Let* $\langle A, B\rangle$ *be a gap in* ω^ω. *If* A *and* B *are uncountable then they both have size* $\aleph_1$.

Proof (see [To1, Theorem 8.6]). Suppose, for example, that the size of A is $> \aleph_1$. Given $f, g \in \omega^\omega$ such that $f <_* g$ let:

$$\Gamma(f,g) = \min\{m : f(n) < g(n) \text{ for all } n \geq m\}.$$

By shrinking A if necessary we may assume that there is a fixed n_0 and for all $f \in A$ an unbounded subset B_f of B such that $\Gamma(f,g) = n_0$, for all $g \in B_f$. Let $X = \{\langle f, g\rangle : f \in A \text{ and } g \in B_f\}$ and consider the partition

$$[X]^2 = K_0 \cup K_1$$

defined by

$$\{\langle f, g\rangle, \langle \bar{f}, \bar{g}\rangle\} \in K_0 \text{ iff } \max\{\Gamma(f, \bar{g}), \Gamma(\bar{f}, g)\} > n_0.$$

Then K_0 is open in the product topology. Let us show that X is not the union of countably many 1-homogeneous sets. Suppose towards contradiction that $X = \bigcup_{n<\omega} X_n$, where each X_n is 1-homogeneous. Then for some n the set $\bar{A}$ of all $f \in A$ such that the set

$$\bar{B}_f = \{g \in B_f : \langle f, g\rangle \in X_n\}$$

is unbounded in B is unbounded in A. Let f be a minimal element of $\bar{A}$. Define the function h in ω^ω by:

$$h(k) = \min\{g(k) : g \in \bar{B}_f\}.$$

Then it follows that h splits the gap $\langle A, B\rangle$, a contradiction.

Now, by OCA, there is an uncountable 0-homogeneous subset Y of X. We may assume that Y is of the form $\{\langle f_\alpha, g_\alpha\rangle : \alpha < \omega_1\}$, where the f_α are $<_*$-increasing. Note that the g_α must be distinct and thus we may assume that they are $<_*$-decreasing. Since A has cofinality $> \aleph_1$ there is $f \in A$ above all the f_α. Since $f_\alpha <_* g_\alpha$ for all $\alpha < \omega_1$ we can find an uncountable subset I of ω_1, an $n_1 < \omega$, and $p, q \in \omega^{n_1}$ such that for all $k \geq n_1$ $f_\alpha(k) < f(k) < g_\alpha(k)$, $f_\alpha \restriction n_1 = p$ and $g_\alpha \restriction n_1 = q$, for all $\alpha \in I$. It then follows that for every distinct $\alpha, \beta \in I$ $\{\langle f_\alpha, g_\alpha\rangle, \langle f_\beta, g_\beta\rangle\} \in K_1$, contradicting the fact that Y is 0-homogeneous. □

To finish the proof of Theorem 3.2, following [Ba, Theorem 4.4] fix a subset A of ω^ω such that the order type of $A, <_*$ is $\aleph_2$. Extend A to a $\subseteq$-maximal $<_*$-linearly ordered set L. Then A will determine a gap in L whose coinitiality, by Lemma 3.1, cannot be a regular uncountable cardinal. Also, it cannot be 1 since if g bounds A then so does $g - 1$. Thus the coinitiality of A in L is either 0 or ω. If it is 0 then A is already unbounded. If it is ω then by [Ro1] one can produce an unbounded subset of ω^ω of size $\aleph_2$. This is done as follows. Let $B = \{g_n : n < \omega\}$ be a subset of L such that $\langle A, B\rangle$ forms a gap. We may assume that $n \leq m$ implies $g_m(k) \leq g_n(k)$, for all k. For $f \in A$ let h_f be defined as follows

$$h_f(n) = \min\{k : f(l) < g_n(l) \text{ for all } l \geq k\}.$$

Then the family $\{h_f : f \in A\}$ is unbounded in ω^ω. □

Todorcevic and the author have shown that PFA implies that $2^{\aleph_0} = \aleph_2$ (see [Ve2] for the proof and the history involving this result). Similarly we conjecture that the answer to the following question is positive.

Question 3.1. Does *OCA* imply that $2^{\aleph_0} = \aleph_2$?

4. Automorphisms of $\mathcal{P}(\omega)/\text{fin}$

We now turn to the study of automorphisms of the Boolean algebra $\mathcal{P}(\omega)/\text{fin}$. Under the Continuum Hypothesis $\mathcal{P}(\omega)/\text{fin}$ has $2^{2^{\aleph_0}}$ automorphisms. On the other hand Shelah ([Sh]) proved the consistency that every automorphism φ of $\mathcal{P}(\omega)/\text{fin}$ is *trivial*, i.e. there exist finite sets $a, b \subseteq \omega$

and a bijection $e : \omega \setminus a \to \omega \setminus b$ such that for every $x \subseteq \omega$, $\varphi[x] = [e''(x)]$, where $[y]$ denotes the equivalence class of y modulo the ideal of finite subsets of ω. Clearly, there are only $2^{\aleph_0}$ such automorphisms. Subsequently, Shelah and Steprans ([SS]) have shown that the same conclusion follows from PFA. We now show how OCA was used in [Vel] to derive the same result.

Theorem 4.1. (OCA + $\mathrm{MA}_{\aleph_1}$) *Every automorphism of $\mathcal{P}(\omega)/\mathrm{fin}$ is trivial.*

Proof. We indicate the main parts of the argument. To begin let us fix an automorphism φ and a function $F : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that $\varphi[x] = [F(x)]$, for every subset x of ω. We shall write $\varphi \restriction a$ for $\varphi \restriction \mathcal{P}(a)/\mathrm{fin}$ and say that φ is *trivial on a* provided $\varphi \restriction a$ is induced by some function $e : a \to \omega$. We shall refer ambiguously to $\mathcal{P}(a)$ and 2^a by identifying a set with its characteristic function. We shall need the following ZFC result, for the proof see [Vel].

Theorem 4.2. *Suppose there exist Borel functions $F_n : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$, for $n < \omega$ such that for every $a \subseteq \omega$ there exists $n < \omega$ such that $F(a) =_* F_n(a)$. Then φ is trivial.* □

The first step of the proof is to show that φ is somewhere trivial, i.e. there is an infinite set a such that $\varphi \restriction a$ is trivial. Let us say that a family $\mathcal{A}$ of almost disjoint infinite subsets of ω is *neat* if there is a 1-1 map $e : \omega \to 2^{<\omega}$ such that if $a \in \mathcal{A}$ and $n, m \in a$ then $e(n) \subseteq e(m)$ or $e(m) \subseteq e(n)$. Thus, $\bigcup e''(a)$ is an infinite branch through $2^{<\omega}$, for every $a \in \mathcal{A}$. The following lemma is the key application of OCA in the proof.

Lemma 4.1. *Let $\mathcal{A}$ be a neat almost disjoint family. Then φ is trivial on all but countably many $c \in \mathcal{A}$.*

Proof. Let $e : \omega \to 2^{<\omega}$ be a function witnessing that $\mathcal{A}$ is neat. Let X be the set of all pairs $\langle a, b\rangle$ of subsets of ω such that there exists $c \in \mathcal{A}$ such that $b \subseteq a \subseteq c$, and define the partition:

$$[X]^2 = K_0 \cup K_1$$

by $\{\langle a, b\rangle, \langle \bar{a}, \bar{b}\rangle\} \in K_0$ iff

(a) $\bigcup e''a \neq \bigcup e''\bar{a}$,
(b) $a \cap \bar{b} = \bar{a} \cap b$,
(c) $F(a) \cap F(\bar{b}) \neq F(\bar{a}) \cap F(b)$.

Then K_0 is open in the product of the separable metric topology τ on X obtained by identifying $\langle a, b\rangle$ with $\langle a, b, F(a), F(b)\rangle$. □

Claim: There are no uncountable 0-homogeneous subsets of X.

Proof. Suppose Y is an uncountable 0-homogeneous set. Let d be the union of all b such that for some a the pair $\langle a,b\rangle$ belongs to Y. Let $\langle a,b\rangle$ be such a pair. By (b) in the definition of K_0 it follows that $d\cap a=b$ and hence $F(d)\cap F(a)=_* F(b)$. We can find an uncountable $Z\subseteq Y$ and $n<\omega$ such that for every $\langle a,b\rangle\in Z$, $(F(d)\cap F(a))\Delta F(b)\subseteq n$ and $F(b)\setminus n\subseteq F(a)$. Then there are distinct $\langle a,b\rangle$ and $\langle \bar a,\bar b\rangle$ in Z such that $F(a)\cap n=F(\bar a)\cap n$ and $F(b)\cap n=F(\bar b)\cap n$. It then follows that $F(a)\cap F(\bar b)=F(\bar a)\cap F(b)$ which contradicts the fact that $\{\langle a,b\rangle,\langle \bar a,\bar b\rangle\}\in K_0$. □

Now, by OCA we can find a decomposition $X=\bigcup_{n<\omega}X_n$ where X_n is 1-homogeneous for all n. Fix for each n a countable subset D_n of X_n which is dense in X_n in the sense of τ. For each $\langle a,b\rangle\in X$ pick $\sigma(a)\in\mathcal{A}$ such that $b\subseteq a\subseteq\sigma(a)$. Let

$$\mathcal{B}=\{\sigma(a):\ \langle a,b\rangle\in D_n \text{ and } n<\omega\}.$$

We shall show that φ is trivial on every $c\in\mathcal{A}\setminus\mathcal{B}$. Thus, fix any such c and decompose it into two disjoint sets $c=c_0\cup c_1$ such that for every $i\in\{0,1\}$, $n<\omega$, and $\langle a,b\rangle\in X_n$ if $a\subseteq c_i$ then for every $m<\omega$ there exists $\langle \bar a,\bar b\rangle\in D_n$ such that:

(a) $a\cap\bar b=\bar a\cap b$,
(b) $a\cap m=\bar a\cap m$ and $b\cap m=\bar b\cap m$,
(c) $F(a)\cap m=F(\bar a)\cap m$ and $F(b)\cap m=F(\bar b)\cap m$.

This is done as follows. An increasing sequence $\langle n_i:\ i<\omega\rangle$ is constructed by induction. Let $n_0=0$. Suppose $\langle n_i:\ i\le k\rangle$ has been defined. Then n_{k+1} is chosen sufficiently large such that for every $x,y,u,v\subseteq n_k$ and every $i\le k$ if there exist $\langle a,b\rangle\in X_i$ such that $a\cap n_k=x$, $b\cap n_k=y$, $F(a)\cap n_k=u$ and $F(b)\cap n_k=v$ then there exist $\langle a,b\rangle\in D_i$ with the same property such that in addition $a\cap c\subseteq n_{k+1}$. This is possible since a is almost disjoint from c whenever there is b such that $\langle a,b\rangle\in D_n$. Finally, let

$$c_0=\bigcup\{c\cap[n_k,n_{k+1}):\ k \text{ is even}\}$$

and let $c_1=c\setminus c_0$. Define the function $F_n:\mathcal{P}(c_0)\to\mathcal{P}(\omega)$, for $n<\omega$, by:

$$F_n(b)=\bigcup\{F(c_0)\cap F(\bar b):\ \langle\bar a,\bar b\rangle\in D_n \text{ and } \bar a\cap b=c_0\cap\bar b\}.$$

Clearly, F_n is a Borel function for all n. We claim that if $\langle c_0,b\rangle\in X_n$ then $F_n(b)=_* F(b)$. This follows easily from the properties of the decomposition $c=c_0\cup c_1$. Thus, by Theorem 4.2, φ is trivial on c_0. A similar argument shows that φ is trivial on c_1, and hence it is also trivial on c. □

Now consider the following set:

$$\mathcal{I} = \{\ a \subseteq \omega :\ \varphi \text{ is trivial on } \ a\}.$$

Fix, for the rest of the proof, for each a in $\mathcal{I}$ a function $e_a : a \to \omega$ inducing $\varphi \restriction a$. Recall that an ideal on ω containing all finite sets is called *dense* provided every infinite subset of ω contains an infinite member of the ideal. Then $\mathcal{I}$ is a dense ideal on $\mathcal{P}(\omega)$. An ideal on ω is called a P-*ideal* if it is countably directed under $\subseteq_*$, and, in general, it is called a P_κ-*ideal* if it is $< \kappa^+$-directed. We shall consider two cases according to whether $\mathcal{I}$ is a P-ideal or not.

Case 1: *$\mathcal{I}$ is a dense P-ideal.*

Define the partition

$$[\mathcal{I}]^2 = K_0 \cup K_1$$

by $\{a, b\} \in K_0$ iff there exists $n \in a \cap b$ such that $e_a(n) \neq e_b(n)$. Note that K_0 is open in the topology on $\mathcal{I}$ obtained by identifying a with e_a. Now using $\mathrm{MA}_{\aleph_1}$ one can prove the following (see [Vel, Lemma 4]).

Claim: There are no uncountable 0*-homogeneous subsets.*

By OCA, there is a decomposition $\mathcal{I} = \bigcup_{n<\omega} \mathcal{I}_n$ where for every $n < \omega$ $\mathcal{I}_n$ is 1-homogeneous. Since $\mathcal{I}$ is a P-ideal, there is $n < \omega$ such that $\mathcal{I}_n$ is cofinal in $\mathcal{I}, \subseteq_*$. Let e be the union of the e_a, for $a \in \mathcal{I}_n$. It follows that for every $a \in \mathcal{I}$ $e \restriction a =_* e_a$, and, since $\mathcal{I}$ is dense and φ is an automorphism, that e induces φ. □

Case 2: *$\mathcal{I}$ is not a P-ideal.*

Find a decomposition decomposition $\omega = \bigcup_{n<\omega} a_n$ into disjoint infinite sets from $\mathcal{I}$ such that there does not exist a in $\mathcal{I}$ almost containing a_n for all n. Given $f \in \omega^\omega$ let $b_f = \bigcup\{a_n \cap f(n) :\ n < \omega\}$.

Claim: There exists $f \in \omega^\omega$ such that φ is nontrivial on b_f.

Proof. Assume otherwise and let $\mathcal{J}$ be the collection of all $b \subseteq \omega$ which are almost disjoint from the a_n. Then it follows from either Theorem 3.1(e) or by a simple application of $\mathrm{MA}_{\aleph_1}$ $\mathcal{J}$ is a $\mathrm{P}_{\aleph_1}$-subideal of $\mathcal{I}$. Then as is easily seen the partition considered in Case 1 restricted to $\mathcal{J}$ has no uncountable 0-homogeneous sets. Thus, there exists $e : \omega \to \omega$ such that $e \restriction b =_* e_b$ for every $b \in \mathcal{J}$. We claim that there exists $k < \omega$ such that e induces φ on $\omega \setminus \bigcup_{i<k} a_i$, which contradicts the nontriviality of φ. To see this, it suffices

to show that the set

$$T = \{m < \omega : \ e \restriction a_m \text{ does not induce } \ \varphi \restriction a_m\}$$

is finite. For then e induces $\varphi \restriction a$ for every a in the ideal generated by $\mathcal{J}$ and $\{a_m : m \notin T\}$. Since this ideal is dense in $\mathcal{P}(u)$, where $u = \omega \setminus \{a_m : m \in T\}$, and φ is an automorphism it follows that e induces φ on u.

Now, suppose T were infinite. For each $m \in T$ we pick an infinite subset c_m of a_m such that $e''(c_m) \cap F(c_m) =_* \emptyset$. By shrinking the c_m we can arrange that, furthermore, for every $m, k \in T$ $e''(c_m) \cap F(c_k) =_* \emptyset$. We then find d such that for every $m \in T$ $F(c_m) \subseteq_* d$ and $e''(c_m) \cap d =_* \emptyset$ and let c be such that $F(c) =_* d$. It follows that $c_m \subseteq_* c$, for each $m \in T$ and hence we can pick $i_m \in c_m \cap c$ such that $e(i_m) \notin F(c)$. Let $b = \{i_m : m \in T\}$. Then $b \in \mathcal{J}$ and hence $F(b) =_* e''(b)$. On the other hand $b \subseteq c$ and hence $F(b) \subseteq_* F(c)$. But $e''(b) \cap F(c) = \emptyset$. Contradiction. □

Note that Claim actually shows that for every $f \in \omega^\omega$ there exists $g \in \omega^\omega$ such that $b_g \setminus b_f$ is nontrivial. We can then easily construct an $<_*$-increasing sequence f_α; $\alpha < \omega_1$ in ω^ω such that φ is nontrivial on $b_{f_{\alpha+1}} \setminus b_{f_\alpha}$ for every $\alpha < \omega_1$. Let $a_\alpha = b_{f_{\alpha+1}} \setminus b_{f_\alpha}$. By another application of $\mathrm{MA}_{\aleph_1}$ (see [Vel, Lemma 3]) we can split each a_α into two disjoint sets a_α^0 and a_α^1 such that $\mathcal{A}^i = \{a_\alpha^i : \alpha < \omega_1\}$ is neat, for $i = 0, 1$. By Lemma 4.1 there is $\alpha < \omega_1$ such that φ is trivial on both a_α^0 and a_α^1, and hence on a_α. Contradiction. □

Some of these ideas have been used by Just ([Ju]) in the proof of the following.

Theorem 4.3. (OCA)

(a) *$(\omega^*)^{(n+1)}$ is not a continuous image of $(\omega^*)^n$, for every $n < \omega$.*

(1) *If $\mathcal{I}$ is a dense P-ideal then $\mathcal{P}(\omega)/\mathcal{I}$ is not isomorphic to $\mathcal{P}(\omega)/\mathit{fin}$.*

(b) *If all Σ^1_{n+2} sets are measurable and $\mathcal{I}$ is a Σ^1_n ideal containing all finite sets such that $\mathcal{P}(\omega)/\mathcal{I}$ is embeddable into $\mathcal{P}(\omega)/\mathit{fin}$ then $\mathcal{I}$ is generated over the Fréchet ideal by at most one set.*

(c) *If $\mathcal{I}$ is the ideal of sets density 0 and $\mathcal{J}$ is the ideal of sets of logarithmic density 0, then $\mathcal{P}(\omega)/\mathcal{I}$ and $\mathcal{P}(\omega)/\mathcal{J}$ are not isomorphic.*

5. Complete Normality of $\gamma\omega$

We now present an application of OCA in the study of countably compact topological spaces. Recall that a topological space X is called *completely normal* if for every two subsets A and B of X which are separated (i.e. $\mathrm{cl}A \cap B = \emptyset = A \cap \mathrm{cl}B$) there are disjoint open sets containing A and

B, respectively. Hausdorff spaces satisfying this property are designated T_5. How well-behaved can countably compact T_5 spaces be? Assuming $V = L$ they can be quite pathological, but assuming PFA it was shown in [NV] that every countably compact T_5 space is sequentially compact, in fact every countable subset has compact, Fréchet-Urysohn closure. [A space is called *Fréchet-Urysohn* if whenever a point x is in the closure of a subset A, then there is a sequence from A converging to x.] Hence, in particular, a separable subspace can have cardinality at most $2^{\aleph_0}$. A consequence of this is a version of Tychonoff's theorem for countably compact spaces: under PFA the product of any number of countably compact T_5 spaces is countably compact, although the T_5 property may be lost. The key application of OCA is to show that certain kind of spaces commonly denoted by $\gamma\omega$ cannot be completely normal. Here $\gamma\omega$ is the generic symbol for a locally compact Hausdorff space X with a countable dense set of isolated points, identified with the set ω of positive integers, such that $X \setminus \omega$ is homeomorphic to ω_1. We will also identify $X \setminus \omega$ with ω_1 using a definition of ω that makes it disjoint from ω_1.

Theorem 5.1. *Under* OCA *no version of* $\gamma\omega$ *can be completely normal.*

Proof. For each $\alpha < \omega_1$ let $a_\alpha \subset \omega$ be such that $a_\alpha \cup [0,\alpha]$ is a compact neighborhood of $[0,\alpha]$. It is easily seen that $a_\alpha \subset_* a_\beta$ and $a_\beta \setminus a_\alpha \subset_* U$, for every neighborhood U of $(\alpha,\beta]$ whenever $\alpha < \beta$. Let S be the set of all $\langle a_\xi, a_\eta, a_\mu\rangle$ such that $\xi < \eta < \mu$ and define the partition

$$[S]^2 = K_0 \cup K_1$$

by $\{\langle a,b,c\rangle\langle \bar{a},\bar{b},\bar{c}\rangle\} \in K_0$ iff

$$a \neq \bar{a} \text{ and } [(a \setminus b) \cap (\bar{c} \setminus \bar{b}) \neq \emptyset \text{ or } (c \setminus b) \cap (\bar{b} \setminus \bar{a}) \neq \emptyset].$$

Then K_0 is open in the product topology.

Suppose first that $\{S_n : n < \omega\}$ is a sequence of 1-homogeneous sets whose union covers S. Let T_n be the set of all ξ for which there are uncountably many η such that $\langle a_\xi, a_\eta, a_\mu\rangle \in S_n$, for some μ. Clearly some T_n must be uncountable. Fix such n and some $\xi \in T_n$. Let $\langle a_{\bar{\xi}}, a_{\bar{\eta}}, a_{\bar{\mu}}\rangle \in S_n$ be such that $\xi < \bar{\xi}$ and find $\mu > \eta > \bar{\mu}$ such that $\langle a_\xi, a_\eta, a_\mu\rangle \in S_n$. Since $\xi < \bar{\eta} < \bar{\mu} < \eta$ we have $a_{\bar{\mu}} \setminus a_{\bar{\eta}} \subset_* a_\eta \setminus a_\xi$. Thus, $\{\langle a_\xi, a_\eta, a_\mu\rangle, \langle a_{\bar{\xi}}, a_{\bar{\eta}}, a_{\bar{\mu}}\rangle\} \in K_0$, which contradicts the fact that S_n is 1-homogeneous.

Now, by OCA, there is an uncountable 0-homogeneous subset H of S. By cutting H down if necessary we may assume $\mu < \bar{\xi}$ whenever $\langle a_\xi, a_\eta, a_\mu\rangle$ and $\langle a_{\bar{\xi}}, a_{\bar{\eta}}, a_{\bar{\mu}}\rangle$ are two distinct members of H such that $\xi < \bar{\xi}$. Then

$$A = \bigcup\{(\xi, \eta] : \langle a_\xi, a_\eta, a_\mu \rangle \in H\}$$

and

$$B = \bigcup\{(\eta, \mu] : \langle a_\xi, a_\eta, a_\mu \rangle \in H\}$$

are separated in $\gamma\omega$. If there were an open subset U of $\gamma\omega$ such that $A \subset U$ and $\mathrm{cl}U \cap B = \emptyset$, we could let $c = U \cap \omega$ and have $a_\eta \setminus a_\xi$ almost contained in c and $a_\mu \setminus a_\eta$ almost disjoint from c whenever $\langle a_\xi, a_\eta, a_\mu \rangle \in H$. Now, for every ξ there are at most one η and μ such that $\langle a_\xi, a_\eta, a_\mu \rangle \in H$. If this happens choose $n(\xi) \in \omega$ such that

$$[(a_\eta \setminus a_\xi) \setminus c] \cup [(a_\mu \setminus a_\eta) \cap c] \subseteq [0, n(\xi)].$$

Then there is an uncountable subset I of H, $n \in \omega$, and $a \subseteq [0, n]$ such that whenever $\langle a_\xi, a_\eta, a_\mu \rangle \in I$ then $n(\xi) = n$ and $a_\eta \cap [0, n] = a$. But then any pair of distinct elements of I is in K_1, a contradiction. □

6. Generalizations of OCA

How can the Open Coloring Axiom be strengthened or generalized? It turns out that there are some strong limitations on the possible generalizations. We first present an example from [To3] which shows that one cannot reverse *open* and *closed* in the statement of OCA.

Theorem 6.1. *There is a coloring*

$$[\omega^\omega]^2 = K_0 \cup K_1$$

with K_0 open in the product topology such that there are no uncountable 1-homogeneous sets and ω^ω is not the union of countably many 0-homogeneous sets.

Proof. For every f in ω^ω associate a sequence $\{f_i : i < \omega\}$ converging to f as follows. Let $n_0 < n_1 < \ldots$ be the list of n such that $f(2n+1) \neq 0$. For a given i the real f_i is determined by letting $f_i \restriction n_k = f \restriction n_k$ and

$$f_i(n_k + j) = f(2^{i+1}(2n_k + 2j + 1)),$$

where $k = k(i)$ is minimal such that

$$f(2n_0 + 1) + \ldots f(2n_k + 1) > i$$

if such k exists, otherwise let $f_i = f$. Define the partition $[\omega^\omega]^2 = K_0 \cup K_1$ by:

$$\{f, g\} \in K_0 \text{ iff } f \neq g_i \text{ and } g \neq f_i, \text{ for all } i < \omega.$$

Then K_0 is open in the product topology. □

Claim 1: There are no uncountable 1*-homogeneous sets.*

Proof. Suppose Y is an uncountable subset of ω^ω. Let D be a countable dense subset of Y and let

$$\bar{D} = \{f_i : f \in D \text{ and } i < \omega\}.$$

Pick $g \in Y \setminus \bar{D}$ and find $h \in Y$ such that $h \neq g_i$, for all $i < \omega$. Then there is an open interval I containing h such that $g_i \notin I$, for all $i < \omega$. Since D is dense in Y there is $f \in D \cap I$. Then $\{f, g\} \in K_0$. □

Remark: A similar argument can be used to show that the poset of finite 0-homogeneous sets, ordered under reverse inclusion is ccc.

Claim 2: ω^ω is not the union of countably many 0*-homogeneous sets.*

Proof. Let $\{H_n : n < \omega\}$ be a sequence of 0-homogeneous subsets of ω^ω. Define the function f in ω^ω as follows. First let $f(2i+1) = 1$, for all $i < \omega$. Then define inductively $f_i \in \omega^\omega$ and $f(2i)$, for $i < \omega$. Suppose $f \restriction 2l$ has been defined as well as f_i, for all $i < l$. If $2l = 2^{i+1}(2i + 2j + 1)$ for some $i < l$ and $j < \omega$ let $f(2l) = f_i(i + j)$. Otherwise choose $f(2l)$ to be any number different from $f_i(2l)$, for all $i < l$. If there is $g \in H_l$ such that $g \restriction l = f \restriction l$ let f_l be such a g. Otherwise let f_l be any function such that $f \restriction l \subseteq f_l$. Then thus constructed f does not belong to H_n, for any $n < \omega$. □

Can OCA be generalized to dimensions bigger than two? The following example of Blass shows that it cannot. Given distinct reals x and y in ω^ω let

$$\Delta(x, y) = \min\{n : x(n) \neq y(n)\}$$

and define the partition

$$[\omega^\omega]^2 = K_0 \cup K_1$$

as follows. Given $x, y, z \in 2^\omega$ with $x < y < z$ let

$$\{x, y, z\} \in K_0 \text{ iff } \Delta(x, y) < \Delta(y, z).$$

It is easy to see that both K_0 and K_1 are open in the product topology and that there are no uncountable homogeneous sets in either color. Generalizing this example one can construct an open coloring of n-tuples of reals into $(n-1)!$ colors such that every uncountable set has n-tuples of each of the colors. Is this example in some sense optimal? Is it consistent that for every open coloring of triples of an uncountable set of reals S into finitely many colors there is an uncountable subset of S which hits at most

2 colors? This question was asked in [ARS]. We now present an example which shows that this is not possible.

Theorem 6.2. *There is an uncountable set of reals X and a continuous function $f : [X]^3 \to \omega$ such that if Y is an uncountable subset of X then $f''[Y]^3 = \omega$.*

Proof. Fix a coloring $k : \omega^{<\omega} \times \omega^{<\omega} \to \omega$ such that for every $m > 0$, for every $s \in \omega^m$, every finite $D \subseteq \omega^{<\omega}$, and every function σ which maps D to ω exists n such that for all $t \in D$

$$k(t, s \cup \{\langle \mathrm{lh}(s), n\rangle\}) = \sigma(t).$$

Such a k can be obtained, for example, as follows. Fix an enumeration $\langle \sigma_i : i < \omega \rangle$ of all finite functions from a subset of $\omega^{<\omega}$ to ω. Given $s, t \in \omega^{<\omega}$ such that $\mathrm{lh}(s) = m > 0$ let $n = s(m-1)$ and define $k(t,s)$ to be $\sigma_n(t)$ if $t \in \mathrm{dom}(\sigma_n)$, otherwise let $k(t,s) = 0$. The following lemma is a variation on the main result from [Ro2].

Lemma 6.1. *Suppose a coloring $c : [\omega_1]^2 \to \omega$ is given. Then there exists a sequence of distinct reals $\langle r_\alpha : \alpha < \omega_1 \rangle$ such that for every $\alpha < \beta < \omega_1$ there exists $n < \omega$ such that $k(r_\alpha \restriction m, r_\beta \restriction m) = c(\alpha, \beta)$, for all $m \geq n$.*

Proof. The reals r_α are constructed inductively. Suppose r_ξ has been defined for all $\xi < \alpha$. To construct r_α fix a 1-1 function $e_\alpha : \alpha \to \omega$ and let

$$F_n(\alpha) = \{\xi < \alpha : e_\alpha(\xi) < n\}.$$

Define recursively $r_\alpha(m)$ as follows. Given $r_\alpha \restriction m$ let l be the largest integer $\leq m$ such that if ξ and η are distinct elements of $F_l(\alpha)$ then $r_\xi \restriction (m+1) \neq r_\eta \restriction (m+1)$.

Now, apply the property of k to $r_\alpha \restriction m$ and $\{r_\xi \restriction (m+1) : \xi \in F_l(\alpha)\}$ to find n such that for all $\xi \in F_l(\alpha)$

$$k(r_\xi \restriction (m+1), r_\alpha \restriction m \cup \{\langle m, n\rangle\}) = c(\xi, \alpha).$$

Then let $r_\alpha(m) = n$. Then thus constructed sequence $\langle r_\alpha : \alpha < \omega_1 \rangle$ works. □

Now, fix a coloring $c : [\omega_1]^2 \to \omega$ witnessing $\aleph_1 \not\to [\aleph_1]^2_\omega$, i.e. such that $c''[U]^2 = \omega$, for every uncountable $U \subseteq \omega_1$, (see [To4]). Let $\langle r_\alpha : \alpha < \omega_1 \rangle$ be a sequence of reals as in Lemma 6.1 and let $X = \{r_\alpha : \alpha < \omega_1\}$. Let $f : [X]^3 \to \omega$ be defined as follows. Given $x, y, z \in X$ with $x < y < z$, where $<$ is the lexicographical ordering on ω^ω, let

$$f(\{x,y,z\}) = k(x \restriction \Delta(y,z), y \restriction \Delta(y,z)).$$

Clearly, f is continuous.

Now suppose Y is an uncountable subset of X. We may assume that Y is dense in itself. Given $i < \omega$ we find $x, y, z \in Y$ such that $f(\{x, y, z\}) = i$. Using the fact that c witnesses $\aleph_1 \not\rightarrow [\aleph_1]^2_\omega$ and the property of k, find $x, y \in Y$ and $n \in \omega$ such that $x < y$ and $k(x \restriction m, y \restriction m) = i$, for all $m \geq n$. Since Y is dense in itself there exists $z \in Y$ such that $\Delta(y, z) \geq n$. It follows that $f(\{x, y, z\}) = i$, as desired. □

We finish by posing two open problems concerning generalizations of OCA. The first one, which was stated as a conjecture in [To1 §8], asks to weaken the topological assumptions on the space S to essentially the best possible.

Question 6.1. Is the following version of *OCA* consistent?

> If S is a regular topological space with no uncountable discrete subsets and
>
> $$[S]^2 = K_0 \cup K_1$$
>
> a partition with K_0 open in the product topology then either there is an uncountable 0-homogeneous set or else S can be covered by countably many 1-homogeneous sets.

We have not discussed generalizations of OCA to cardinals bigger than $\aleph_1$ but the following problem would certainly require new techniques.

Question 6.2. Is the following consistent with the negation of the Continuum Hypothesis?

> If S is a set of reals of size $> \aleph_1$ and
>
> $$[S]^2 = K_0 \cup K_1$$
>
> is a partition with both K_0 and K_1 open then there exists a homogeneous subset of S of size $> \aleph_1$.

Clearly, f is continuous.

Now suppose Y is an uncountable subset of X. We may assume that Y is dense in itself. Given $i < \omega$ we find $x, y, z \in Y$ such that $f(\{x, y, z\}) = i$. Using the fact that c witnesses $\aleph_1 \not\to [\aleph_1]^2_\omega$ and the property of k, find $x, y \in Y$ and $n \in \omega$ such that $x < y$ and $k(x \restriction m, y \restriction m) = i$, for all $m \geq n$. Since Y is dense in itself there exists $z \in Y$ such that $\Delta(y, z) \geq n$. It follows that $f(\{x, y, z\}) = i$, as desired. □

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