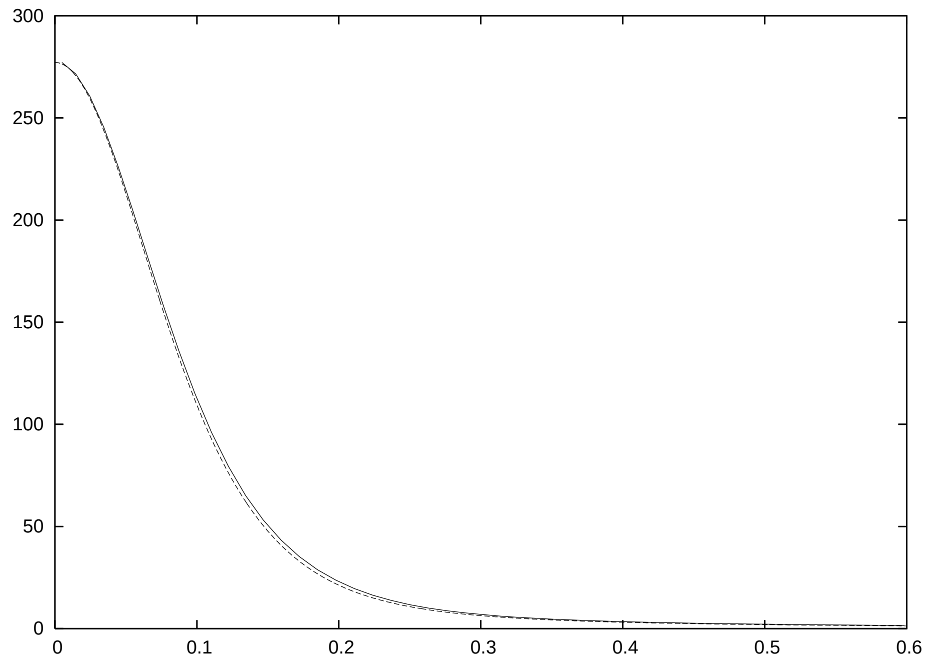

FIGURE 10. $R_\perp$ for critical Ricci flow compared to Bryant steady soliton

Although no mathematician would argue that numerical simulations of the sort described here can be used to replace mathematical proof, it is becoming increasingly clear that such studies are helpful both for testing the plausibility of a conjecture, and for discovering new phenomena to explore. A good example of the former is provided by the studies of the BKL conjecture in general relativity. This conjecture, which was proposed during the 1960's and 1970's and concerns the behavior of gravitational fields in the neighborhood of a singularity was relatively inaccessible for many years. Then, during the 1990's, a number of researchers led by Berger and Moncrief [**?, ?**] began to test the conjecture numerically, and found strong support for the conjecture in various families of space-time solutions of Einstein's equations. This work in turn has led to serious efforts to prove the conjecture in these families, and during the past few years there have indeed been successful proofs of the BKL conjecture in certain cases [**?**].

3. Continuous dependence on the initial data

Let $g_{1,0}$ and $g_{2,0}$ be Riemannian metrics on a closed manifold $\mathcal{M}^n$. Let $\hat{g}_1(t)$, $t \in [0, T_1)$, and $\hat{g}_2(t)$, $t \in [0, T_2)$, be the maximal solutions to the Ricci flow with $\hat{g}_1(0) = g_{1,0}$ and $\hat{g}_2(0) = g_{2,0}$, where $T_1 < \infty$. We wish to show that for any $t_1 \in (0, T_1)$ and $\varepsilon > 0$, if $g_{2,0}$ is sufficiently close to $g_{1,0}$, then $\hat{g}_2(t)$ is ε-close in the $C^{\lceil \varepsilon^{-1} \rceil}$-topology to $\hat{g}_1(t)$ for all $t \in [0, t_1]$.

First, we consider the corresponding solutions to the Ricci–DeTurck flow, taking advantage of its strict parabolicity as a system of partial differential equations. Let $g_1(t)$, $t \in [0, T_1)$, and $g_2(t)$, $t \in [0, T_2)$, be solutions to the Ricci–DeTurck flow, both with respect to the choice of background metric $\tilde{g} \doteqdot g_{1,0}$.[11] Furthermore, assume that for some constant $A < \infty$ we have:

(1)
$$|\mathrm{Rm}_{\tilde{g}}| \leq A, \tag{36.63}$$

(2)
$$A^{-1}\tilde{g} \leq g_a(t) \leq A\tilde{g}, \tag{36.64}$$

(3)
$$\left|\tilde{\nabla} g_a(t)\right| + \sqrt{t}\left|\tilde{\nabla}\tilde{\nabla} g_a(t)\right| \leq A \tag{36.65}$$

for $a = 1, 2$, where $\tilde{\nabla}$ denotes the covariant derivative with respect to $\tilde{g}$ and where the norms are measured with respect to $\tilde{g}$.

Then the calculation on pp. 283–284 of [**196**] implies that

$$\left(\frac{\partial}{\partial t} - g_1(t)^{j\ell}\tilde{\nabla}_j\tilde{\nabla}_\ell\right)|g_1(t) - g_2(t)|^2_{\tilde{g}} \leq \left(\frac{C_1 A}{\sqrt{t}} + \left(\frac{C_1 A}{2}\right)^2\right)|g_1(t) - g_2(t)|^2_{\tilde{g}}, \tag{36.66}$$

where $C_1 < \infty$ is a universal constant.[12] Assume that $g_1(t) \not\equiv g_2(t)$; then by forward and backward uniqueness of the Ricci–DeTurck flow, we have $g_1(t) \neq g_2(t)$ for all $t \in [0, T)$, where $T \doteqdot \min\{T_1, T_2\}$ (see Kotschwar [**435**] for backward uniqueness of both the Ricci–DeTurck flow and the Ricci flow).

Applying the weak maximum principle to (36.66) yields

$$\begin{aligned}\log \frac{|g_1(\bar{t}) - g_2(\bar{t})|^2_{\tilde{g}}}{|g_1(0) - g_2(0)|^2_{\tilde{g}}} &\leq \int_0^{\bar{t}} \left(\frac{C_1 A}{\sqrt{t}} + \left(\frac{C_1 A}{2}\right)^2\right) dt \\ &= 2C_1 A\sqrt{\bar{t}} + \left(\frac{C_1 A}{2}\right)^2 \bar{t}\end{aligned}$$

for all $\bar{t} \in [0, T)$. Hence

$$|g_1(t) - g_2(t)|^2_{\tilde{g}} \leq |g_{1,0} - g_{2,0}|^2_{\tilde{g}} \exp\left(2C_1 A\sqrt{T} + \left(\frac{C_1 A}{2}\right)^2 T\right)$$

for all $t \in [0, T)$. In other words,

$$|g_1(t) - g_2(t)|_{\tilde{g}} \leq C|g_{1,0} - g_{2,0}|_{\tilde{g}} \tag{36.67}$$

for some $C < \infty$. This yields the continuous dependence on its initial data for the Ricci–DeTurck flow in the C^0 sense.

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[11] For each $a = 1, 2$, the time intervals of existence for $\hat{g}_a(t)$ and $g_a(t)$ are the same.

[12] Note that the assumption on p. 283 of [**196**], that the initial metrics are the same, is not necessary for this calculation.

To see continuous dependence in the C^1 sense, we differentiate equation (7.44) of [**196**]; that is, defining the time-dependent symmetric 2-tensor

$$J \doteqdot g_1 - g_2, \tag{36.68}$$

we take the covariant derivative of the equation:

$$\begin{aligned}\frac{\partial J}{\partial t} &= g_1^{j\ell}\tilde{\nabla}_j\tilde{\nabla}_\ell J + g_1^{-1} * g_2^{-1} * \tilde{\nabla}\tilde{\nabla} g_2 * J \\ &\quad + g_1^{-1} * g_2^{-1} * g_1 * \tilde{g}^{-1} * \widetilde{\mathrm{Rm}} * J \\ &\quad + \left(g_1^{-1} + g_2^{-1}\right) * g_1^{-1} * g_2^{-1} * \tilde{\nabla} g_1 * \tilde{\nabla} g_1 * J \\ &\quad + g_2^{-1} * g_2^{-1} * \left(\tilde{\nabla} g_1 + \tilde{\nabla} g_2\right) * \tilde{\nabla} J.\end{aligned} \tag{36.69}$$

In particular, by applying $\tilde{\nabla}$ to (36.69) and using the formula

$$\begin{aligned}-g_1^{j\ell}\left[\tilde{\nabla}_j\tilde{\nabla}_\ell, \tilde{\nabla}\right] J &= g_1^{-1} * \tilde{\nabla}\left[\tilde{\nabla}, \tilde{\nabla}\right] J + g_1^{-1} * \left[\tilde{\nabla}, \tilde{\nabla}\right]\tilde{\nabla} J \\ &= g_1^{-1} * \tilde{g}^{-1} * \tilde{\nabla}\widetilde{\mathrm{Rm}} * J + g_1^{-1} * \tilde{g}^{-1} * \widetilde{\mathrm{Rm}} * \tilde{\nabla} J,\end{aligned}$$

we compute that

$$\frac{\partial}{\partial t}\left(\tilde{\nabla} J\right) = g_1^{j\ell}\tilde{\nabla}_j\tilde{\nabla}_\ell\left(\tilde{\nabla} J\right) + \mathrm{A} * J + \mathrm{B} * \tilde{\nabla} J + \mathrm{C} * \tilde{\nabla}\tilde{\nabla} J, \tag{36.70}$$

where A, B, and C are defined by

$$\begin{aligned}\mathrm{A} &\doteqdot g_1^{-1} * \tilde{g}^{-1} * \tilde{\nabla}\widetilde{\mathrm{Rm}} + \tilde{\nabla}\left(g_1^{-1} * g_2^{-1} * \tilde{\nabla}\tilde{\nabla} g_2\right) \\ &\quad + \tilde{\nabla}\left(g_1^{-1} * g_2^{-1} * g_1 * \tilde{g}^{-1} * \widetilde{\mathrm{Rm}}\right) \\ &\quad + \tilde{\nabla}\left(\left(g_1^{-1} + g_2^{-1}\right) * g_1^{-1} * g_2^{-1} * \tilde{\nabla} g_1 * \tilde{\nabla} g_1\right),\end{aligned} \tag{36.71}$$

$$\begin{aligned}\mathrm{B} &\doteqdot g_1^{-1} * \tilde{g}^{-1} * \widetilde{\mathrm{Rm}} + g_1^{-1} * g_2^{-1} * \tilde{\nabla}\tilde{\nabla} g_2 \\ &\quad + g_1^{-1} * g_2^{-1} * g_1 * \tilde{g}^{-1} * \widetilde{\mathrm{Rm}} \\ &\quad + \left(g_1^{-1} + g_2^{-1}\right) * g_1^{-1} * g_2^{-1} * \tilde{\nabla} g_1 * \tilde{\nabla} g_1 \\ &\quad + \tilde{\nabla}\left(g_2^{-1} * g_2^{-1} * \left(\tilde{\nabla} g_1 + \tilde{\nabla} g_2\right)\right),\end{aligned} \tag{36.72}$$

and

$$\mathrm{C} \doteqdot g_2^{-1} * g_2^{-1} * \left(\tilde{\nabla} g_1 + \tilde{\nabla} g_2\right). \tag{36.73}$$

From (36.70) we calculate

$$\begin{aligned}\frac{\partial}{\partial t}\left|\tilde{\nabla} J\right|_{\tilde{g}}^2 &= g_1^{j\ell}\tilde{\nabla}_j\tilde{\nabla}_\ell\left|\tilde{\nabla} J\right|_{\tilde{g}}^2 - 2\left|\tilde{\nabla}\tilde{\nabla} J\right|_{g_1,\tilde{g}}^2 \\ &\quad + \mathrm{A} * J * \tilde{\nabla} J + \mathrm{B} * \tilde{\nabla} J * \tilde{\nabla} J + \mathrm{C} * \tilde{\nabla}\tilde{\nabla} J * \tilde{\nabla} J,\end{aligned} \tag{36.74}$$

where we absorbed some factors of 2 into the $*$ notation. Now, we qualitatively strengthen the assumption (36.63) to the assumption that

$$|\mathrm{Rm}_{\tilde{g}}| + \left|\tilde{\nabla}\,\mathrm{Rm}_{\tilde{g}}\right| \leq B \tag{36.75}$$

and we qualitatively strengthen the assumption (36.65) to the assumption that

$$\left|\tilde{\nabla} g_a(t)\right| + \left|\tilde{\nabla}\tilde{\nabla} g_a(t)\right| + \left|\tilde{\nabla}\tilde{\nabla}\tilde{\nabla} g_a(t)\right| \leq B \tag{36.76}$$

for $a = 1, 2$.[13] Since, by (36.64), (36.75), and (36.76), we have

$$|\mathrm{A}| \leq C, \quad |\mathrm{B}| \leq C, \quad |\mathrm{C}| \leq C \tag{36.77}$$

for some constant $C < \infty$, we conclude from (36.74) that

$$\frac{\partial}{\partial t}\left|\tilde{\nabla} J\right|_{\tilde{g}}^2 \leq g_1^{j\ell}\tilde{\nabla}_j\tilde{\nabla}_\ell\left|\tilde{\nabla} J\right|_{\tilde{g}}^2 + C\left(|J|_{\tilde{g}}^2 + \left|\tilde{\nabla} J\right|_{\tilde{g}}^2\right) \tag{36.78}$$

for some other constant $C < \infty$.

From this and the weak maximum principle, we may easily deduce that for any $t_1 \in (0, T_1)$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if

$$|g_{1,0} - g_{2,0}|_{\tilde{g}} + |\tilde{\nabla}(g_{1,0} - g_{2,0})|_{\tilde{g}} \leq \delta \tag{36.79}$$

on $\mathcal{M}$, then

$$|g_1(t) - g_2(t)|_{\tilde{g}} + |\tilde{\nabla}(g_1(t) - g_2(t))|_{\tilde{g}} \leq \varepsilon \tag{36.80}$$

on $\mathcal{M} \times [0, t_1]$. This yields the continuous dependence for the Ricci–DeTurck flow in the C^1 sense.

It seems rather clear that one should be able to continue in this way to estimate the higher covariant derivatives of $g_1(t) - g_2(t)$ with respect to $\tilde{g}$. This will prove continuous dependence of the Ricci–DeTurck flow in the C^k sense for all $k \in \mathbb{N}$. Since the Ricci flow is obtained from the Ricci–DeTurck flow by solving a system of ODE, we expect that it should not be too difficult to solve the following.

Problematic Exercise 36.21. (*Formulate and*) *prove continuous dependence of the Ricci flow in the* C^k *sense for all* $k \in \mathbb{N}$.

Hint: Given $k \in \mathbb{N}$, make the assumption that there exists a constant $A_k < \infty$ such that

$$\sum_{i=0}^{k}\left|\tilde{\nabla}^i\,\mathrm{Rm}_{\tilde{g}}\right| \leq A_k \tag{36.81}$$

and, for $a = 1, 2$,

$$\sum_{i=1}^{k+2}\left|\tilde{\nabla}^i g_a(t)\right| \leq A_k. \tag{36.82}$$

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[13]Of course, (36.75) implies (36.63) with $A = B$. Note also that (36.76) implies (36.76) with $A = B\left(1 + \sqrt{T}\right)$.

Show that there exists $C < \infty$ such that

$$\frac{\partial}{\partial t}\left|\tilde{\nabla}^k J\right|_{\tilde{g}}^2 \leq g_1^{j\ell}\tilde{\nabla}_j\tilde{\nabla}_\ell\left|\tilde{\nabla}^k J\right|_{\tilde{g}}^2 + C\sum_{i=0}^{k}\left|\tilde{\nabla}^i J\right|_{\tilde{g}}^2. \tag{36.83}$$

Furthermore, as in (3.35) and (3.29) of Volume One, for each $a = 1, 2$, consider the 1-parameter family of diffeomorphisms

$$\begin{aligned} \frac{\partial}{\partial t}\varphi_{a,t}(p) &= -W_a(\varphi_{a,t}(p), t), \\ \varphi_{a,0} &= \mathrm{id}_{\mathcal{M}^n}, \end{aligned} \tag{36.84}$$

where the time dependent vector field $W_a(t)$ is defined by

$$W_a(t)^k = g_a(t)^{pq}\left(\Gamma(g_a(t))_{pq}^k - \tilde{\Gamma}_{pq}^k\right). \tag{36.85}$$

Then by Step 4 in the proof of Theorem 3.13 in Volume One, we have that

$$\hat{g}_a(t) \doteqdot \varphi_{a,t}^* g_a(t) \tag{36.86}$$

for $a = 1, 2$. It is not hard to see that it just remains to show, for any $t_1 \in (0, T_1)$ and $\varepsilon > 0$, that $\varphi_{1,t}$ and $\varphi_{2,t}$ are ε-close on $\mathcal{M} \times [0, t_1]$ provided that $g_{1,0}$ and $g_{2,0}$ are sufficiently close. In view of the continuous dependence on initial data for systems of ODE, this, in turn, follows from showing that $W_1(t)$ and $W_2(t)$ are sufficiently close.

4. Notes and commentary

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