The Geometry and Pigmentation of Seashells

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Figure 1: Some typical shells and their common names. From top to bottom and left to right: Soldier (California), Pelican's Foot (Mediterranean), Striped Whelk (Adriatic), Honey Whelk (Greece), Screw (Italy), Moon (Philippines), Rock Snail (Mombasa), Ring-Top Cowry (Africa), Dove (Taiwan).

Seashells are beautiful objects that are admired for both their intricate shapes and the patterns on their surfaces. Despite their complexity these shapes are easily described using only elementary tools from geometry. Indeed a wide variety of natural shell shapes can be composed as surfaces in a 3-space and rendered using computer graphic imagery. Moreover, the pigmentation motifs that decorate many shells in the form of wavy stripes and checks as well as chaotic and tent designs can be generated by cellular automaton models, and in particular by the famous "Rule 30".

The photograph in Fig. 1 shows some typical seashell shapes, all of which can be described in terms of a spirally coiled cone. From a mathematical perspective a natural description may be given in terms of a generating spiral and the shape of the opening or generating curve. There are now several algorithms for generating realistic seashell shapes, such as those described in the wonderfully illustrated book [1]. To give the main idea behind these algorithms it is enough to consider the surface generated by rotating an expanding semi-circle in an upward direction as in Fig. 1 (right). Other mathematical shell surfaces can be generated by rotating more realistic shell openings around helico-spirals. To learn about such mathematical shapes we first need to know more about circles, spirals and parametric descriptions of surfaces.

Planar spirals. Points in the plane may be specified with a pair of numbers, such as those of the Cartesian coordinate system. Alternatively one may use the planar polar coordinate system as shown in Fig. 2. A one-armed spiral is then described by the equation $\theta = f(r)$. A classic example is the Archimedean spiral with $f(r) = r$.

The parametric equation of a circle. A circle of radius R may be described in terms of a single parameter $\theta \in [0, 2\pi)$ as

$$x = R\cos\theta, \qquad y = R\sin\theta.$$

If we let θ range over $[0, 2\pi)$ then we generate a

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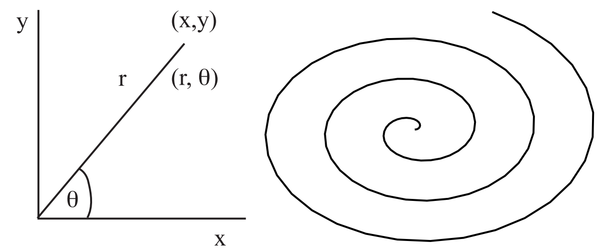

Figure 2: Left: Cartesian and polar coordinate systems. $(x, y) = (r \cos\theta, r \sin\theta)$. Right: An Archimedean spiral with $f(r) = r$.

circle. If θ ranges over $[0, \pi]$ then we generate a semi-circle.

3D spirals. A point in 3D may be described using the 3D Cartesian coordinate system. In Cartesian coordinates a point is specified with the triple (x, y, z). A helico-spiral may be expressed in terms of a single parameter θ by writing $r = F(\theta)$ and $z = G(\theta)$. For a 3D Archimedean spiral these functions are simply $F(\theta) = a\theta$ and $G(\theta) = b\theta$ for given constants a and b.

Armed with the above geometric notions we are now in a position to generate a simple seashell shape formed by the rotation of a generating curve along a helico-spiral. As an example let us take the generating curve to be a semi-circle and the helico-spiral to be a 3D Archimedean spiral as in Fig. 3. To label a point on the surface we need to specify how far along the spiral we are (using θ) and how far round the semi-circle we are (using ϕ). This is easily calculated by letting

$$r \to r + R\sin\phi, \qquad z \to z + R(1 - \cos\phi).$$

In terms of the 3D Cartesian system the coordinates of the shell are given by $(x, y, z) = (r\cos\theta, r\sin\theta, z)$. Hence, the surface of the shell is completely specified in terms of two parame-

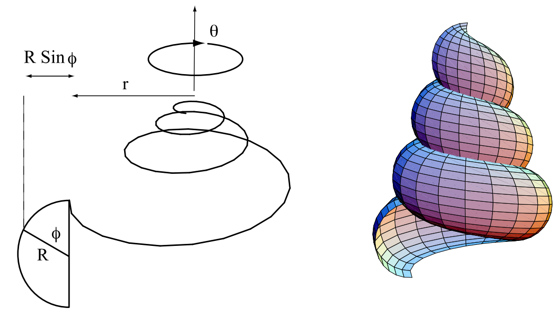

Figure 3: Left: The generator of a simple seashell shape. Right: A mathematical seashell obtained by rotating a semi-circular generating curve around an Archimedean generating spiral.

ters, θ and $\phi \in [0, \pi]$:

$$x(\theta, \phi) = (a\theta + R\sin\phi)\cos\theta,$$
$$y(\theta, \phi) = (a\theta + R\sin\phi)\sin\theta,$$
$$z(\theta, \phi) = b\theta + R(1 - \cos\phi).$$

To make more interesting shapes we can use different helico-spirals, make the radius of the semi-circle depend upon θ and ϕ ($R = R(\theta, \phi)$) or choose more complicated shapes for the generating curve. In Fig. 4 we show some examples of shells generated with a logarithmic helico-spiral and various choices of the generating curve [2].

As well as having interesting shapes many seashells also exhibit exotic patterns on their surfaces, such as that of the widespread species Conus textile, shown in Fig. 5. These patterns arise from the secretion of pigment from cells which lie in a narrow band along the shell's lip. Each cell secretes pigments according to the activity of its neighbouring pigment cells and leaves behind a coloured pattern as the shell grows. In fact these secretions are controlled in part by the mollusc nervous system and can be modelled with mathematical tools for describing dynam-

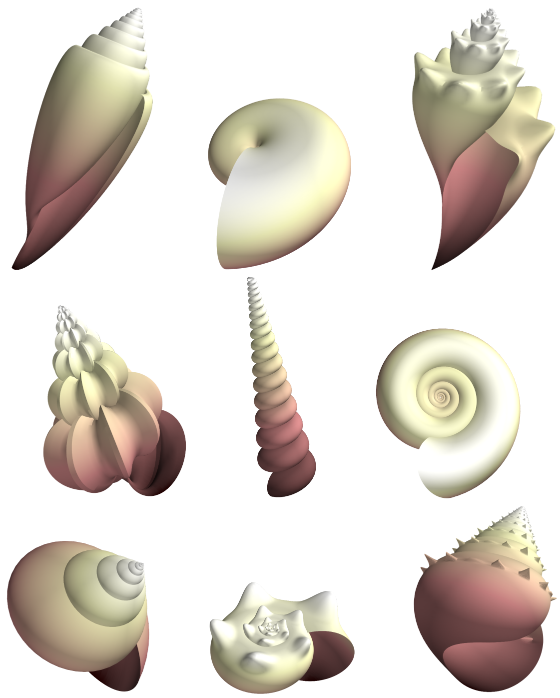

Figure 4: Mathematical seashell shapes. From top to bottom and left to right: Conus, Nautilus, Lyria, Epitonium, Turritella, Planorbis, Oxystele, Turbo, Struthiolaria.

ical systems [4]. However, many shell patterns can be described by simpler so-called cellular automaton (CA) models that do not track the details of the neurosecretory process. A cellular automaton is a discrete model often studied in mathematics in the context of computability theory. It consists of a regular grid of cells, each in one of a finite number of states. Time is also discrete, and the state of a cell at time $t+1$ is a function of the states of the cells in its neighbourhood at time t. The state update rules that define the creation of a new generation can be specified in terms of a simple table. For example "Rule 30" can be listed in the form:

$$\begin{array}{cccccccc} 111 & 110 & 101 & 100 & 011 & 010 & 001 & 000 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array},$$

Figure 5: Conus textile (found in the waters of the Indo-Pacific) exhibits a cellular automaton pattern on its shell.

which tells us, for example, that if three adjacent cells in the CA currently have the pattern 100 (on-off-off), then the middle cell will become 1 (on) in the next time step. The output 00011110 is interpreted as an 8-bit binary number, equal to 30, and hence the name "Rule 30" [3]. Indeed there are $2^8 = 256$, possible CA rules of this type, though this rule is of particular interest because it produces complex, seemingly-random patterns like those in Fig. 6. Here a black cell represents the

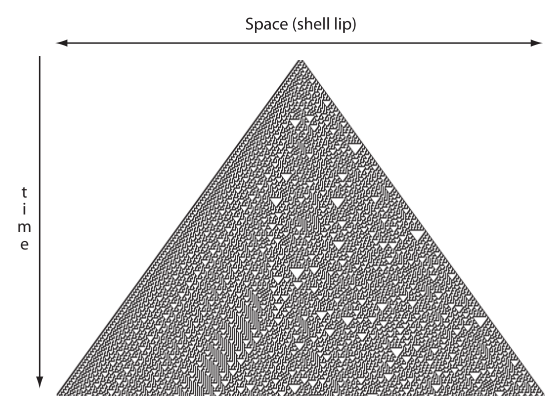

Figure 6: "Rule 30" cellular automaton – time decreasing down the page.

state 1 (on) and a white cell the state 0 (off). If we imagine colouring the seashell lip at time t of its growth with the pattern state obtained from the

CA at time t then we would recover something like the texture of the cone snail Conus textile, with its "cloth of gold" pattern.

References

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Biography

Stephen Coombes has a degree in Theoretical Physics from the University of Exeter and a PhD in Neurocomputing from King's College London. He is currently a Professor of Applied Mathematics at the University of Nottingham, where he is applying the tools of dynamical systems theory to problems in neuroscience, including trying to figure out how his own brain works.