SOME QUESTIONS OF UNIFORM TOPOLOGY

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My report contains three parts: the proximity, the uniformity and the general metrics. I should like to tell you about some themes which are interesting for Moscow's topologists. I should like to suggest a few open questions also.

I. Proximity

The first axioms of the proximity were given by F. Riesz at the mathematical congress in Bologna in 1908. He gave the definition of general proximity-space. The complete axiomatic was given by W. Ephremovitsch [1] in 1935. He added a separation axiom. With this axiom the apparatus of real-valued functions and pseudometrics can be applied. I proved that the notion of proximity-space is closely connected and is indeed equivalent to the notion of compactification. Thus the proximity-theory can be considered as a part of the theory of topological spaces. In fact, every compact **C** has one and only one proximity:

$$A\delta B(^1) \quad \text{if and only if} \quad \bar{A}^c \cap \bar{B}^c \neq \phi,(^2) \qquad (1)$$

$$\text{where} \quad A \subseteq C, B \subseteq C.$$

Therefore every compactification **C** of the given space **X** defines this natural proximity (1) on the X. The received correspondence between all compactifications of a given space X and all its proximities is one-to-one and onto [2]. Thus every proximity-space P can be defined as a completely regular space considered together with some compactification u**P**. From this point of view the bounded real-valued function g on the **P** is proximity-continuous if and only if it has a continuous extension on the u**P**. Moreover the map f of the proximity-space X into the proximity-space Y is proximity coninuous if and only if it has *a* continuous extension f mapping the compactification u**X** into u**Y** [2]. These facts have been reproved by many authors for example [3], [10].

Therefore we have connection between the proximity-spaces and subrings of the ring $C(X)$ of all bounded real-valued continuous functions. The classical Weierstrass-Stone's theorem for proximity-spaces is a consequence of this situation [4].

The metric spaces have the following natural proximity:

$$A\delta B \quad \text{if and only if} \quad \varrho(A,B)=0.(^3) \qquad (2)$$

I note that the notion of proximity-continuity and the notion of uniform continuity are equivalent for the mappings of the metric spaces. For

(1) $A\ \delta\ B = A$ is near to B.
(2) $\bar{A}^c$ is the closure of A.
(3) $\varrho(A,B) = \inf \varrho(x,y)$, where $x \in A$, $y \in B$ and $\varrho(x,y)$ is the given metric.

proximity-spaces I have given notions of completeness, of the completion and of full-boundness [5]. These notions are founded on the consideration of some kind of coverings or of pseudometrics [6]. All these notions are equivalent to the corresponding metrical notions for metric spaces considered with the natural proximity (2). We can get the completion $c\mathbf{P}$ of the given metric space P from the corresponding compactification $u\mathbf{P}$ by removing all the points with first countable-axiom [2]. Therefore, the metric space P is complete if and only if no one point of the difference $uP \setminus P$ satisfies the first countable-axiom. Thus some uniform properties of the metric space P can be characterised by topological properties of the difference $uP \setminus P$. The second example: The metric space $\mathbf{P}$ is full-bounded in usual sense if and only if the difference $uP \setminus P$ is hereditary-normal [17]. I note that not all uniform properties of the metric space $\mathbf{P}$ can be characterized in this manner (for example, the property being zero-uniform-dimensional).

In topological groups there are also a natural proximity:

$A\delta B$ if and only if $A \cap O \cdot B \neq \phi$ for every neigbourhood O of the unity.(¹) (3)

The proximity of groups was studied very little.

Question 1 (A. N. Kolmogoroff). Is it possible to define the completeness of a topological group P using only the properties of this natural proximity? In other words: are there proximity-homeomorphic groups G and H where G is complete and H is not?

Question 2. Are there uniform properties of topological groups which can not be characterized by proximity-properties?

I note that for metric spaces each uniform property is a proximity-property. It follows that all the theorems of the uniform topology of metric spaces can be formulated and proved in terms of proximity.

II. Uniformity

The theory of uniform spaces has been constructed by A. Weil in 1935 earlier than the theory of proximity [7]. The uniform space can be defined by many ways: the Bourbaki's way [8] by a certain system of neighbourhoods of diagonal of square $X \times X$, the Tuckey's way [9] by a certain system of coverings of X, the very general Császàr's way [10] by a certain ordering of set of all subsets of X. The natural notions of completeness and full-boundness (precompactness) are equivalent to the corresponding metric properties for metric space.

For uniform spaces there are given generalizations of some theorems of analysis (for instance, theorem of Ascoli). The apparatus of uniform topology is very convenient for linear topologic spaces [11].

Every uniform space U has a natural proximity: Let the uniformity of topological space X be defined by the structure $\Sigma = \{\omega_\lambda\}$ of coverings ω_λ. Then

$A\delta B$ if and only if each covering ω_λ has an element O_λ that $A \cap O_\lambda \neq \phi$ and $B \cap O_\lambda \neq \phi$. (4)

(¹) We can get the other proximity putting $A \cap B.O \neq \phi$ for non-commutative groups.

Let P be some proximity-space and P_α is some uniform space on the set P for which the proximity (4) coincide with the proximity of P. Let Σ_α be the uniform structure of the coverings which defines the uniform space P_α. In this situation the completion cP_α of the space P_α is the maximal among those subsets of the compactification uP on which every open covering of the structure Σ_α can be extended into an open covering [5, 12]. Thus we have a well defined map of the set of all uniform spaces P_α with the same proximity P into the set of all subsets of the compactification uP.

But this map is no one-to-one even for complete uniform spaces. On the set of all natural numbers one can define many (continuum!) uniform spaces where every such space is complete and proximity-discrete. Moreover on the set M of the cardinality of the continuum one can construct hypercontinuum complete uniform spaces with discrete proximity where every such space has the same set of all uniform-continuous real-valued functions. Thus the uniform space U can not be in general characterised by the group $C'(U)$ of all uniform continuous real-valued functions.

There is the minimal space among all uniform spaces U_α defined on the same proximity-space P and having the same group $C'(U_\alpha)$ (i.e. all these groups are naturally isomorphic). These minimal spaces are defined in another way by J. Isbell [13]. He calls these spaces weak-uniform. The natural map of the set of all weak-uniform spaces (defined on the same topological space X) into the set of all subgroups of the group $C'(X)$ of all real-valued continuous functions is one-to-one.

Let $C'(X)$ be the ring of all real-valued continuous functions on the topological space X and $C'(P)$ the group of all real-valued proximity-continuous functions on the proximity-space P.

QUESTION 3. Is it possible to characterize groups $C'(U_\alpha)$ where U_α are the uniform spaces on X (or on P) among all subgroups of the group $C'(X)$ (or corresponding of $C'(P)$) by their structural, algebraic and topological properties?

J. Isbell proved the Weierstrass-Stone's theorem for all these weak-uniform spaces U for which the groups $C'(U)$ are algebras having some composition-property. Let us have $P \subseteq S \subseteq uP$ where P is a some proximity-space, uP is its natural compactification and S—some Q-space in the sense of E. Hewitt. Then the group $C'(S)$ is an algebra and it has the composition-property.

QUESTION 4. Are only these groups $C'(S)$ algebras with the composition-property or not?

III. General metrics

At the last year there appeared a new theory of general metric spaces. It is clear that in every such case the distances between other points of a given metric space can be no real numbers—the elements of the real line E'—but elements of some its generalization. This generalization is a notion of a topological semifield given by T. Sarymsacov, V. Boltianski and M. Antonovski [14].

The topological semifield is defined as a commutative and associative ring in which some subset of its elements is chosen. They call these elements positive and this subset—cone. This cone should have a number of natural properties. It is easy to define the notion of completeness for topological semifields. The fundamental example of complete topological semifield is the product E^α of a finite or infinite number of real lines E^1, where topology is Tychonov topology and the operations are the following:

$$\{x_\alpha\}+\{y_\alpha\}=\{x_\alpha+y_\alpha\},\ \{x_\alpha\}\cdot\{y_\alpha\}=\{x_\alpha\cdot y_\alpha\}. \tag{5}$$

There are a natural ordering: $\{x_\alpha\}\leqslant\{y_\alpha\}$ if $x_\alpha\leqslant y_\alpha$ for every α. The cone of the E^α is a set of all elements $\{x_\alpha\}$ which $>\{o\}$. Thus every euclidean space E^n is a complete topological semifield. It was proved that every complete topological semifield is topological isomorph to some space E^α. By definition and by this proposition the distances $\varrho(x,y\}$ between other points of a given general metric space are the elements of some semifield E^α. Analogically we can define the general normed space: the norm $\|x\|$ of every point x from this space is a element of some semifield E^α. Of course the axioms are the usual axioms of real-valued metric or real-valued norm. The every semifield E^α is a general normed space because we can put that $\|\{x_\alpha\}\|=\{x_\alpha\}$. Thus it is a general metric space with distance $\varrho(\{x_\alpha\},\{y_\alpha\})=\{|x_\alpha-y_\alpha|\}$.

Every general metric space has a natural "metric" uniformity. Let ε be a some neighbourhood of the unity $\{1\}$ of E^α. Then we have the "ε-neighbourhood" $U_\varepsilon x$ for every point x of the general metric space R. This "ε-neighbourhood" is a set of all this points y of R for which $\varrho(x,y)\in\varepsilon$. The set Σ of all coverings of R having a refinement $\omega_\varepsilon=\{U_\varepsilon x, x\in R\}$ for some ε is a uniform structure. The (by other ways defined) notions of the compactness, of the full-boundness, of the completeness for general metric space are equivalent to corresponding notions for this metrical uniformity. It is easy to prove that every uniform space is metrizable by some general metric.

The three authors have proved the Kolmogoroff's theorem on a possibility of introducing the norm for linear topological spaces. They have generalized the Banach's theorem on open mappings on the general normed (and metric) spaces and the theory of abstract ergodic theorems of Eberlein [15].

Thus we see that this new theory of general metric (and normed) spaces is very convenient for analysis (more than uniform spaces and semiordered spaces of Kantorovitsch [18]).

Question 5. Is it possible to make the axiomatic of topological semifields more and essential simple than the given?

Question 6. Is it possible to prove the Banach's theorem on open mappings (see 12.2. from [16]) on the uniform spaces?

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