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SOME RESULTS ON RINGS WITH POLYNOMIAL IDENTITIES

by S. A. AMITSUR

A complete survey of recent results in this area should at least cover the following topics: 1) Group rings with polynomial identities. 2) Identities in rings with involutions. 3) Prime rings with generalized *p.i.* 4) Rings of polynomial and rational functions. 5) Tensor products. 6) Embedding in matrix rings. 7) Prime ideals and localizations, etc.

No attempt will be made to cover all topics, but some references will be given.

Let R be an algebra over a commutative ring Ω with a unit; R is said to satisfy a *p.i* $p[x] = \Sigma\alpha_{(i)}x_{i_1} \dots x_{i_2} = 0$ if $p[r_1, \dots] = 0$ for all substitution $x_i = r_i \in R$, and for simplicity we assume that some coefficient of a monomial of highest degree $\alpha_{(i)} = 1$.

1. **Tensor products.**

An old problem was: Does the matrix ring $M_n(R)$ satisfy a *p.i* if R satisfies, and more generally if R and S satisfy *p.i*, does $R \bigotimes_{\Omega} S$ satisfy a polynomial identity. The first part was solved rather simply in the affirmative by C. Procesi, L. Small [13] and consequently it follows that the endomorphism ring $\mathrm{Hom}_R(V, V)$ satisfies an identity if V is a finitely generated R-module and R satisfies a *p.i.* The second problem remains open in the general form though one can reduce the problem to nil rings. Liron-Vafne [5] have shown that by defining equivalence of two ring: $A \equiv B$ if they satisfy the same identities, then if also $C \equiv D$ then $A \otimes C = B \otimes D$. This in particular shows the invariance of the identities under scalar extension in the known cases.

2. **Embedding in matrix rings** (over commutative ring).

For simplicity assume henceforth that $\Omega = F$ is an infinite field. One of the earlier results was that semi-prime rings with a *p.i* of degree d can be embedded in $M_n(K)$ for some commutative ring K, $n \leq \frac{d}{2}$. An extension of this result that, a *p.i* is necessary and sufficient for a ring K to be embedable in a matrix ring over a commutative ring, was shown not to be valid by (Drazin and P. M. Cohn) and the problem remained whether the condition is that the ring shall satisfy exactly the identities of $M_n(F)$. This also is now known to be wrong. The first example was given in [2] by Amitsur, and a second example given by Small [11] will appear soon. Small's examples shows that one can have a finitely generated algebra with a nilpotent Jacobson's radical with the same identities as $M_n(F)$ and which cannot be embedded in matrices of finite order over any commutative ring.

These examples raise the problem: what characterizes subrings of matrix rings. If we try to attack the problem of finding all commutative rings K for which we have an embedding $R \hookrightarrow M_n(K)$, one gets readily to the following simple observation: for fixed $n \geq 1$, for every ring R there exists a commutative ring $S = S(R; n)$ and a homomorphism $\rho: R \to M_n(S)$ such that for every homomorphism $\sigma: R \to M_n(K)$ for any commutative ring K, there exists a unique (!) homomorphism η: $S \to K$ such that the following diagram is commutative:

$$\begin{tikzcd} R \arrow[r, "\rho"] \arrow[dr, "\sigma"'] & M_n(S) \arrow[d, "M_n(\eta)"] \\ & M_n(K) \end{tikzcd}$$

The ring S is uniquely determined up to isomorphism and so is ρ. With the aid of the ring S one can show that if R is a finitely generated and $P_1 \subseteq P_2 \subseteq \ldots \subseteq P_r \subseteq \ldots$ is a sequence of ideals such that each R/P_i can be embedded in some $M_n(K_i)$, then the sequence must be finite, which is a form of a Hilbert basis theorem in the commutative case.

A non trivial property of $\rho: R \to M_n(S)$ is that $\rho(R)S = M_n(S)$ if and only if all irreducible representations of R are of dimension $\geq n$. In view of the properties of S, and the fact that ρ is mono for azumaya-algebras R —, it follows that, S is a generic splitting ring of R. Note, that even for the simple case $R = M_n(F)$, $S \neq F(!)$.

This is the place to mention an interesting result of M. Artin [4], that an algebra R with a unit is an azumaya algebra if and only if R satisfies all identities of $M_n(F)$ and its irreducible representations are all of the same dimension n.

3. Prime ideals.

Rings with $p.i$ are a natural generalization of commutative rings, and one would expect to be able to extend our knowledge of the latter to rings with $p.i.$ This in fact was the leading thought in developing the theory of rings with $p.i.$ First steps in this direction have been done, like existence of rings of quotients, and the non commutative Hilbert Nullstellersats. A great push toward this goal has been given recently by a series of works of C. Procesi. The extensions to the non-commutative case are far from being trivial and usually hold in a restricted form.

Let K be commutative, $R = K[a_i, \ldots, a_m]$ be a finitely generated prime ring with a $p.i$ of degree $d = 2n$, then R satisfies the ascending (descending) chain condition on prime ideals if K does. R is a Jacobson ring (a Hilbert algebra) if K is such. If K has finite rank then R has also finite rank. If $K = F$ is a field and C is the center of the ring of quotients of R, then rank $R \leq$ tr. deg $C/F \leq (m - 1)n^2 + 1$, and the maximum of tr. deg C/F is obtained for the ring of m generic matrices.

All these results, and more which were not mentioned are well known theorems for commutative domains (the case $d = 2n = 2$), which indicate that commutative theory can be pushed into rings with $p.i.$ But beware of the pitfalls, like the structure of the nil radicals of $p.i$ rings, and localization at primes. A prime ring R has a classical ring of quotient $Q(R)$, and if P is a prime ideal, then $Q_p = \{ a^{-1}b \mid a \text{ regular mod } P \}$ is a well defined subring of R only if R/P and R have the same identities (Small [12]). This reveals that only these prime ideals are " good " for the non-commutative case and those for which R/P has lower identities introduce distortion in the theory.

More examples to support our arguments that rings with $p.i$ are the next step after commutative rings can be found in the papers of C. Procesi. Special interest lies in the ring of generic matrices.

4. **The ring of generic matrices** is the algebra $F[X] = F[X_1, \ldots, X_m]$ generated by m generic matrices $X_i = (\xi^i_{\lambda\mu})$, $\lambda, \mu = 1, 2, \ldots, m$; with $\{ \xi^i_{\lambda\mu} \}$, mn^2 commutative indeterminates. This ring should be the replacement of the ring of commutative polynomials in m variables and in fact $F[X]$ is isomorphic with the ring of polynomial functions in m variables with values in $M_n(F)$, or in any central simple algebra of dimension n^2 over a center containing F for that matter. Some interesting properties of this ring proved by Procesi [8] are: $F[X]$ is a domain (Amitsur) with an Ore ring of quotient $F(X)$ which is central simple division algebra of dimension n^2 and which is isomorphic to the ring of all rational functions in m-variables over $M_n(F)$. The tr. deg. of its center C over F is $(m-1)n^2 + 1$ (obtained independently by Kyrilov). The field of all rational functions $F(\xi)$ is the $\{ \xi^i_{\lambda\mu} \}$ is a splitting field, and

$$F[X]F(\xi) = M_n(F(\xi));$$

furthermore the center C has an algebraic extension $C(u)$ which is a pure transcendental extension of F, and $C(u)$ is a normal extension of C and its Galois group in the full symmetric group on n-elements. Procesi gives also the cross product form of the class of $F(X)$. Again some properties of $F[X_1, \ldots, X_m]$ are extensions of the properties of the ring $F[t_1, \ldots, t_m]$ in m-commutative polynomials—but some do not! Hilbert Hullstellersatz does hold, but $F[X]$ is not Noetherian, and even it does not satisfy chain conditions on two sided ideals, but it does satisfy chain conditions for ideals P such that $F[X]/P$ can be embedded in $M_n(K)$ for some commutative ring K.

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