Actes, Congrès intern. Math., 1970. Tome 2, p. 73 à 77.

DIFFERENTIABLE ACTIONS OF COMPACT CONNECTED LIE GROUP ON R^n

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I. Introduction.

In this lecture, I shall summarize some joint work on differentiable actions (unpublished yet) with my brother Wu-yi Hsiang. Let $\Psi : G \times R^n \to R^n$ be a differentiable action of a compact connected Lie group G on R^n. Ψ defines a representation of G into Diff (R^n) which we also denote by $\Psi : G \to \mathrm{Diff}(R^n)$. For a fixed inner product structure on R^n, we have an inclusion $\mathrm{SO}(n) \subset \mathrm{Diff}(R^n)$. We say that Ψ is linear if, up to conjugacy in Diff (R^n), if factors through SO(n). Even though most actions are non-linear, we may still find many features of an action resembling a linear one. Therefore, we shall follow the following guiding principle in our study : *Compare the behaviour of general differentiable actions with that of linear ones.* At the end, I shall also discuss actions on homotopy spheres. Although the result summarized here are extensions of [$2^{\mathrm{I},\,\mathrm{II}}$], the proofs are actually independent of the previous works. We make use of the weight system of [9] and the group generated by differentiable reflections [6] as our new ingredients.

II. Geometric weight system and a fundamental fixed point theorem :

Let Ψ be a differentiable action of a compact connected Lie group G on a Q-acyclic manifold X. Let T be a fixed maximal torus of G and it follows from P.A. Smith theory [1] that the fixed point set of T, $F(T, X) = M$ is also Q-acyclic and consequently, connected. Hence, the local representation (2), $(\Psi|T)_x$, is independent of the choice of x. It is an invariant of Ψ, and shall denote it by $\Omega(\Psi)$. We may split the representation of T as a sum of 2-dim representations and some trivial representations. As usual, we write a non-trivail 2-dim representation of T as exp $(\pm\, 2i\alpha\pi)$ and identify the corresponding weights in $\Omega(\Psi)$ by $\pm\,\alpha$. We shall identify the trivial representations with the zero weights in $\Omega(\Psi)$ and denote the subset of non-zero weight in $\Omega(\Psi)$ by $\Omega'(\Psi)$. $\Omega(\Psi)$ is symmetric with respect to $W(G) = N(T)/T$ the Weyl group of G. The weights in $\Omega'(\Psi)$ appear in pairs $\pm\,\alpha$. $\Omega(\Psi)$ was first introduced in [9] for studying effective actions of Spin (m) on acyclic manifolds, and was used to determine the identity

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(1) Partially supported by NSF Grant GP-9452.

(2) After we give an invariant metric on X, the local representation $(\Psi|X)_x$ is just the induced action of T on the tangent space at x.

component of the principal isotropy subgroup(1) of a classical group acting on acyclic manifolds [2^{II}]. $\Omega(\Psi)$ is not a complete invariant of Ψ, but it is a rather good book-keeping device. Our first problem is to find possible patterns of $\Omega(\Psi)$, and then determine whether it resembles the weight of some linear representation of R^n. If G itself has a fixed point, $\Omega(\Psi)$ coïncides with the character of the local representation Φ of G at the fixed point. But unfortunately, G does not always have a fixed point [21], [9]. So we would like to find a large maximal rank subgroup K such that $\Psi|K$ has fixed points. For this purpose, let us introduce the following subsets of the root system $\Delta(G)$ of G relative to $\Omega(G)$: $\Sigma_j(\Psi)$ is the subset of α in $\Delta(G)$ such that the integral multiples of α in $\Omega'(\Psi)$ form exactly a j-string, i.e., $\pm\,\alpha$, $\pm\,2\alpha\,, \ldots , \pm\,j\alpha$, Note that most of $\Sigma_j(\Psi)$ are empty, $\Sigma_0(\Psi) = \alpha \in \Delta(G)\,, \alpha \in \Omega'(\Psi)$ and $\Sigma_1(\Psi)$ is the subset of α in $\Delta(\Psi)$ such that $\Omega'(\Psi)$ contains only one pair of integral multiples of α, $\pm\,\alpha$.

THEOREM 1. – *Let X be a Z_2-acyclic manifold*(2) *and Ψ be a differentiable action of a compact connected Lie group G on X. Then, there is a maximal rank subgroup K of G such that $F(K, X)$ is also Z_2-acyclic and*

$$\Delta(K) \supset \Sigma_0(\Psi) \cup \Sigma_1(\Psi) \cup \Sigma_3(\Psi).$$

Theorem 1 seems technical, but it is rather strong. As we shall see in the next section, it gives a strong hold of the principal isotropy subgroup of the action. The following results are also consequences of Theorem 1. (A) If Ψ has at most 3 types of orbits(3), then G has a fixed point. One may eventually classify actions on R^n with up to 3 types of orbits. (B) If the dimension of the orbit space of Ψ is less then or equal to 6, then G has a fixed point(4). Therefore, we shall call Theorem 1 the *fundamental fixed point theorem.* It was proved by a combination of weight system, the fixed point theorem of differentiable reflections and an analysis of SO(3) actions on Z_2-acyclic manifolds.

III. Determination of principal isotropy subgroups and a reduction theorem.

For a differentiable action Ψ of G on a manifold M, there is an absolute minimum among the conjugate classes of isotropy subgroups under the partial ordering by inclusion. Denite it by (H_Ψ). For $H_\Psi \in (H_\Psi)$, G/H_Ψ is called the principal orbit of Ψ. The Montgomery-Samelson-Yang theorem [13] [14] asserts that the union of all principal orbits in M is an everywhere dense open submanifold. From [2^{II}] [3], we see that (H_Ψ) has a strong influence on other isotropy subgroup classes and it is desirable to determine (H_Ψ). We say that (H_Ψ) is non-trivial if

(1) For the definition of principal isotropy subgroups, see § III.

(2) Z_2-acyclicity implies Q-acyclicity.

(3) I. e., there are at most three conjugate classes of isotropy subgroups. For results on actions with up to 2 types of orbits, see [I, Ch. XIV].

(4) Montgomery-Samelson-Yang had results for actions with the orbit space of the dim less or equal to 2 [15].

H_Ψ is not equal to the kernel of the representation $\Psi : G \to \mathrm{Diff}(M)$. For determining (H_Ψ) of a differentiable action Ψ of G on an acyclic manifold, it would be desirable, of course, if G had a fixed point whenever (H_Ψ) was non-trivial. Then the classification of (H_Ψ) would be reduced to the linear actions. Unfortunately, there are actions of F_4 on euclidean spaces with (Spin (5)) and (Spin (2)) as the principal isotropy subgroups without a fixed point. So, we can only expect the next best thing.

THEOREM 2. – *Let Ψ be a differentiable action of a simple compact connected Lie group G on R^n. Suppose that (H_Ψ) is non-trivial. Then, we have either*

(1) $F(G, R^n)$ *is* $\mathbf{Z}_2$*-acyclic,* or

(2) $G = F_4$, $(H_\Psi) = (\mathrm{Spin}\,(5))$ *and*

$$\Omega'(\Psi) = 2 \cdot \left\{\frac{1}{2}\,\theta_1 \pm \theta_2 \pm \theta_3 \pm \theta_4),\, \pm\theta_1, \pm\theta_2, \pm\theta_3, \pm\theta_4\right\}, \text{ or}$$

(3) $G = F_4$, $(H_\Psi) = (\mathrm{Spin}\,(2))$ *and*

$$\Omega'(\Psi) = 3 \cdot \left\{\frac{1}{2}\,(\pm\theta_1 \pm \theta_2 \pm \theta_3 \pm \theta_4),\, \pm\theta_1, \pm\theta_2, \pm\theta_3, \pm\theta_4\right\}.$$

In fact, the cases (2), (3) *do occur.*

We can extend the result of Theorem 2 to semi-simple connected compact Lie groups, but the statement becomes a little technical due to the possible normal factors of F_4-type. However, we can still show that for a differentiable action Ψ of a compact connected Lie group G on R^n, if (H_Ψ) is not-trivial, then there is a linear representation Φ such that $(H_\Phi) = (H_\Psi)$. If G is simple, it is a consequence of Theorem 2 that we may choose Φ such that $(H_\Phi) = (H_\Psi)$ and $\Omega'(\Phi) = \Omega'(\Psi)$. In any case, we complete the determination of principal isotropy subgroups of actions on R^n. (Cf. [2^{II}] [7] [12]). The basic reason why we can do this is because of the fundamental fixed point theorem (Theorem 1).

Of course, we recover all the regularity theorems of [$2^{\mathrm{I},\mathrm{II}}$] for euclidean spaces as we did in [2^{II}]. In fact, we have the following reduction theorem motivated by [2^{I}], [10], [11].

THEOREM 3. – *Let Ψ be a differentiable action of a compact connected Lie group G on R^n. Let H_Ψ be a fixed principal isotropy subgroup, i.e., a fixed element in (H_Ψ). Set $W(\Psi) = N(H_\Psi)/H_\Psi$. Then Ψ induces a differentiable action Φ of $W(\Psi)$ on $M = F(H_\Psi, R^n)$ and Φ determines Ψ.*

For example, if H_Ψ is a maximal torus of G, then M is $\mathbf{Z}$-acyclic and $W(\Psi) = W(G)$ acts on M as a group generated by differentiable reflections. Using [6], we have a complete understanding of this case. In fact, if G is a classical group, then it follows from Theorem 2 that either Ψ is a regular action in the sense of [2^{II}], [11] or the induced action Φ is generated by reflections. For this case, we also have a fairly good understanding by [$2^{\mathrm{I},\mathrm{II}}$] [6].

IV. Concluding Remarks.

When we started our work [2I], we made use of the dimension restriction of the total space relative to the group and the property of 'vanishing first Pontrjagin class' to nail down the identity components of the isotropy subgroups. We then applied P.A. Smith theory and a formula of Borel [I, pp. 175-179] to get the structure of isotropy subgroups. Under this approach, euclidean space and homotopy spheres are completely parallel. But now, we use weight system and the group generated by reflections as our tools. They depend on the fact that the fixed point set of the restriction of the action to a maximal torus is acyclic. The situation becomes somewhat different for these two cases. However, it seems to us that we still have all the parallel results if we use Borel's formula quoted above carefully and systematically. The interest in working out the homotopy sphere case is because of the existence of various differentiable structure on spheres. One expects to have more refined and interesting results on the 'degree of symmetry of spheres' [4] [5] [8] when the corresponding results for spheres are obtained.

Finally, let us pose two rather important problems from the present point of view :

PROBLEM 1. – *For a given differentiable action Ψ of G on R^n, is there a linear representation Φ such that $\Omega'(\Phi) = \Omega'(\Psi)$?*

PROBLEM 2. – *Classify all the differentiable actions of $SO(3)$, $Sp(1)$ on R^n and write down their weight system.*

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