Pizzas, crusts and other tasty things
MUMS Talk

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Scenario

You and your friend order a pizza. After waiting for awhile, it arrives. You notice that each piece is of slightly different size. But all is not lost. You notice that although every cut is off-centred, they all intersect at a point. Moreover, the angles formed by these cuts are the same.

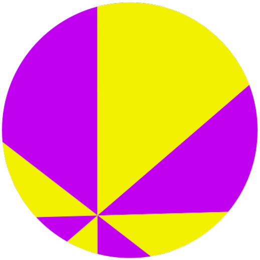


Figure: Courtesy of Wikipedia.

Scenario

The fight begins....

And that's when you realise that you need a new friend.

Suppose that you decide to share it equally by giving each person alternating slices. Will the two of you REALLY share the slices equally?

Question

A pizza is divided into $2N$ slices by means of N straight cuts. Every cut goes through the same point – say P – in the pizza which makes $2N$ "$\frac{\pi}{N}$ rad" angles at P. It is then shared by two individuals (Gray/White), who alternate slices. When does the total area of gray exceed that of the white slices?

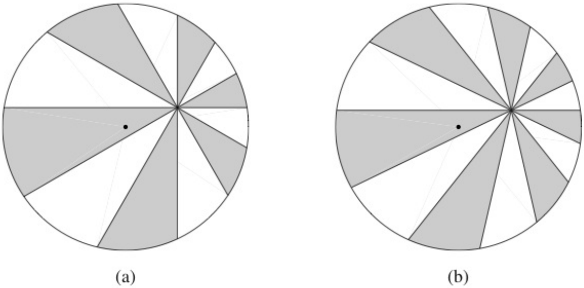


Figure: Mabry and Deiermann '09

Easy case

When $P*$ or O lies on one of the cuts.

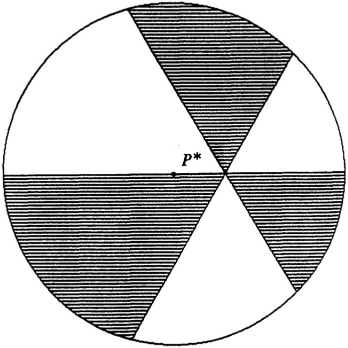
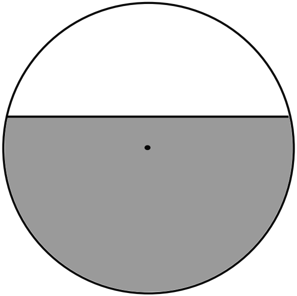


Figure: Mabry and Deiermann '95

$N = 1$

Not that hard....



$N = 2$

Gray gets more pizza: Proof by picture.

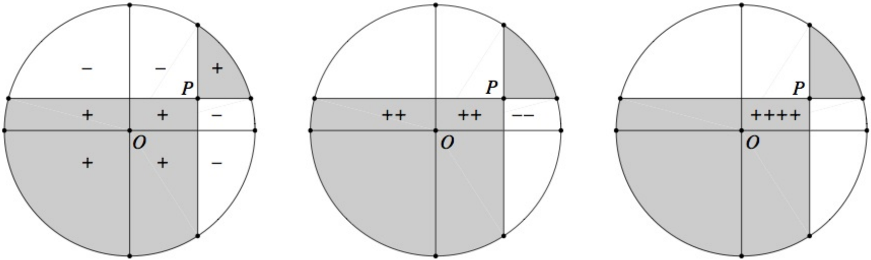


Figure: Mabry and Deiermann '09

$N = 3$

Gray gets more pizza: Proof is a little more involved...

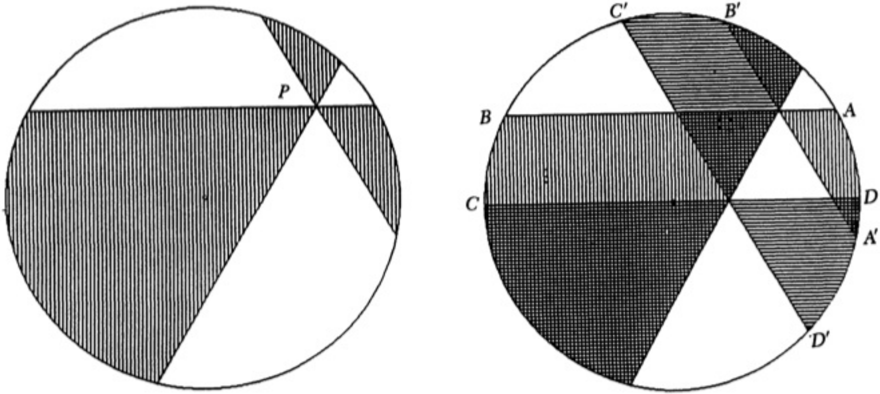


Figure: Mabry and Deiermann '95

$N = 4$

Both get the same: Proof by picture!

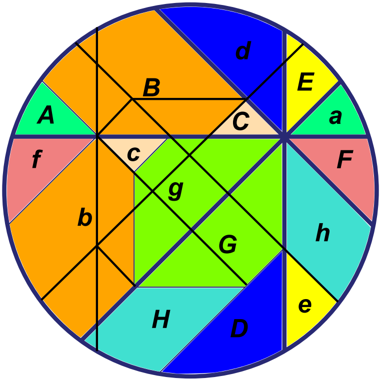


Figure: Courtesy of Wikipedia by Carter and Wagon '94

N even

What about other N? Say N is even.

Theorem (Pizza Theorem 1)

For $N \geq 4$ even, the total area of shaded regions is the same as that of white ones.

Let's prove it!

First Observation

Lemma

Consider the diagram below. Suppose that the radius of the circle is 1. Then

$$\sum_{i=0}^{2n-1} a_i^2 = 2n$$

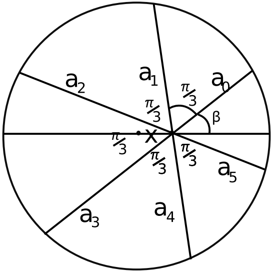
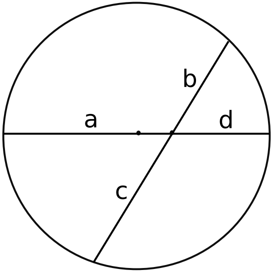


Figure: when $n = 3$

First Observation

Recall your high school geometry...



We were told that

$$ad = bc.$$

Replace a with $1+x$ and d with $1-x$, and you get: for all $0 \leq i \leq n-1$,

$$a_i a_{n+i} = (1+x)(1-x) = 1 - x^2.$$

Note that

$$\begin{aligned} a_i^2 + a_{n+i}^2 &= (a_i^2 + 2a_i a_{n+i} + a_{n+i}^2) - 2a_i a_{n+i} \\ &= (a_i + a_{n+i})^2 - 2(1-x^2). \end{aligned}$$

So,

$$\begin{aligned} \sum_{i=0}^{2n-1} a_i^2 &= \sum_{i=0}^{n-1} ((a_i + a_{n+i})^2 - 2(1-x^2)) \\ &= \left(\sum_{i=0}^{n-1} (a_i + a_{n+i})^2 \right) - 2n(1-x^2). \end{aligned}$$

So, the problem is reduced to working out the lengths of these chords!
Awesome!

The chord length

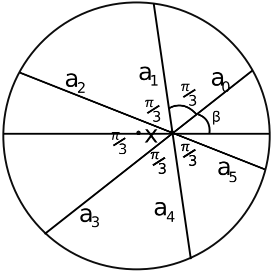
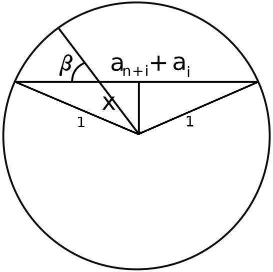


Figure: when $n = 3$

The chord length



A little bit of trigonometry, and we get

$$\begin{aligned} a_{n+i} + a_i &= 2\sqrt{1 - (x\sin(\beta))^2} \\ (a_{n+i} + a_i)^2 &= 4(1 - x^2\sin^2(\beta)). \end{aligned}$$

The chord length

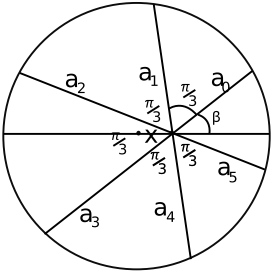


Figure: when $n = 3$

Observe that the angle change by π/n (not β) every time we increase i by 1. So, we can index our sum, and get

$$\sum_{i=0}^{n-1}(a_i + a_{n+i})^2 = \sum_{i=0}^{n-1} 4(1 - x^2\sin^2(\beta + \frac{i\pi}{n}))$$

The chord length

$$\begin{aligned}
\sum_{i=0}^{n-1}(a_i + a_{n+i})^2 &= \sum_{i=0}^{n-1} 4(1 - x^2 \sin^2(\beta + \frac{i\pi}{n})) \\
&= 4n - 4x^2 \sum_{i=0}^{n-1} \sin^2(\beta + \frac{i\pi}{n}) \\
&= 4n - 4x^2 \sum_{i=0}^{n-1} \frac{1 - \cos(2\beta + \frac{2i\pi}{n})}{2} \\
&= 4n - 2nx^2 + 2x^2 \sum_{i=0}^{n-1} \cos(2\beta + \frac{2i\pi}{n}) \\
&= 4n - 2nx^2.
\end{aligned}$$

Finally...

From our original sum

$$\sum_{i=0}^{2n-1} a_i^2 = \left(\sum_{i=0}^{n-1}(a_i + a_{n+i})^2\right) - 2n(1-x^2)$$
$$= 4n - 2nx^2 - 2n + 2nx^2$$
$$= 2n.$$

Magic!

Proof for $N = 2m > 2$

Recall from Calculus I, especially your polar coordinate. Let $R(\theta)$ be the distance from P to the circle making an angle θ to the x-axis. The total gray area is

$$\frac{1}{2}\int_0^{\frac{\pi}{N}} (R(\theta))^2 \mathrm{d}\theta + \frac{1}{2}\int_{\frac{2\pi}{N}}^{\frac{3\pi}{N}} (R(\theta))^2 \mathrm{d}\theta + \ldots + \frac{1}{2}\int_{\frac{(2N-2)\pi}{N}}^{\frac{(2N-1)\pi}{N}} (R(\theta))^2 \mathrm{d}\theta.$$

Reparametrised, and we get

$$\begin{aligned}
&= \frac{1}{2}\int_0^{\frac{\pi}{N}} (R(\theta))^2 + (R(\theta + \frac{2\pi}{N}))^2 \ldots + (R(\theta + \frac{(2N-2)\pi}{N}))^2 \mathrm{d}\theta \\
&= \frac{1}{2}\int_0^{\frac{\pi}{N}} \sum_{i=0}^{N-1} (R(\theta + \frac{2i\pi}{N}))^2 \mathrm{d}\theta \\
&= \frac{1}{2}\int_0^{\frac{\pi}{N}} N \mathrm{d}\theta \\
&= \frac{\pi}{2}.
\end{aligned}$$

N odd

For N odd, it is a lot harder. It wasn't solved until 2009!. Thanks to Rick Mabry and Paul Deiermann, we now have the following theorem:

General theorem

Cheese Pizza Theorem (Mabry, Deiermann '09)

For positive integer N, divide a pizza as usual. Let O be the centre of the pizza. Then the following is true.

- If $N \geq 4$ is even or O is on one of the cuts, then

$$Area_{gray} = Area_{white}.$$

- If O lies in the interior of a gray slice, and $N \equiv 3 \mod 4$ then

$$Area_{gray} > Area_{white}.$$

- If O lies in the interior of a gray slice, and $N \equiv 1 \mod 4$ then

$$Area_{gray} < Area_{white}.$$

Crust

Some people prefer to eat the crust than the topping. Imagine a nice sausage embedded inside the crust! We can visualise a crust as the difference between two concentric disks.

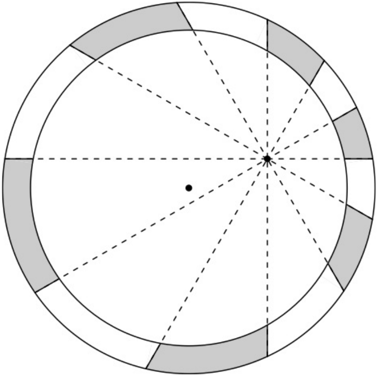


Figure: From Mabry, Deiermann '09

Crust

Again the proof of this is beyond the scope of this talk. But here is the theorem:

Thick Crust Theorem (Mabry, Deiermann '09)

For positive integer N, divide a pizza as usual. Let O be the centre of the pizza. Suppose that the centre of all cuts lies in the cheesy part of the pizza. Then the following is true.

- If N is even or O is on one of the cuts, then

$$Crust_{gray} = Crust_{white}.$$

- If O lies in the interior of a gray slice, and $N \geq 3$ is odd, then

$$Area_{gray} < Area_{white} \quad \Rightarrow \quad Crust_{gray} > Crust_{white}.$$

  and vice versa.

Other tasty things

Things you can try:

- **Calzones**: a 3D pizza stuffed with cheese. Think hemisphere or cone calzones.
- **Thin crust**: A thin crust is an arc on the circle. The result follows a similar pattern to the case of thick crust.

Some interesting questions:

- What if we have a square pizza or ellipsoidal shape?
- Can we apply this idea to cake (not necessary circular) cutting?

Moral of this story

If you want to eat more pizza,

- Make sure you pick the right initial slice;
- If you want to be absolutely sure of getting a fair share, order one for yourself.

Thank you for listening...

Thank you

I would like to express my gratitude to the following living things:

- Sam Chow. This talk could have been shorter without him.
- Those who contributed to my facebook status late at night.

More importantly, the following people for allowing me to plagiarise them.

- Rick Mabry and Paul Deiermann,
- J. D. E. Konhauser, D. Velleman, and S. Wagon,
- L. Carter and S. Wagon,
- Hirschhorn $\times 5$,
- L. J. Upton.