Generalization of the Weierstrass excess formula, as deduced from the Hilbert-De Donder independence theorem

by JULES GÉHÉNIAU, Aspirant F.N.R.S. (*)

Abstract. – In this note, we deduce the Weierstrass formula, when extended to the case of n independent variables in m unknown functions, from the Hilbert-De Donder independence theorem.

1. Extremal equations. Let:

$$\mathcal{F} \equiv \mathcal{F}(x^i, y^\alpha, y_i^\alpha) \qquad i = 1, \ldots, n; \quad \alpha = 1, \ldots, m \tag{1}$$

be a function of n independent variables x^i, m functions y^α, and their first partial derivatives:

$$y_i^\alpha = \frac{dy^\alpha}{dx^i}. \tag{2}$$

The differential equations of the extremals of:

$$\delta \int \mathcal{F}\, d(x^1, \ldots, x^n) = 0 \tag{3}$$

are:

$$\frac{\delta \mathcal{F}}{\delta y^\alpha} = 0. \tag{4}$$

The canonical variables p_α^i that are conjugate to the y^α are given by:

$$p_\alpha^i = \frac{\partial \mathcal{F}}{\partial y_i^\alpha}. \tag{5}$$

These canonical variables are thus expressed, thanks to (5), as functions of the x^i, y^α, and y_j^α ($j = 1, \ldots, n$). We also write the Hamiltonian function:

$$\mathcal{H} = -\mathcal{F} + \sum_\alpha \sum_i p_\alpha^i y_i^\alpha . \tag{6}$$

If the Jacobian:

$$\left\| \frac{\partial^2 \mathcal{F}}{\partial y_i^\alpha \partial y_j^\beta} \right\| \neq 0, \qquad \begin{cases} \alpha, \beta = 1, \cdots, m \\ i, j = 1, \cdots, n \end{cases} \tag{7}$$

(*) Presented by Th. De Donder.

then, thanks to (5), the function $\mathcal{H}$ may be considered as being expressed as a function of the x^i, y^α, and p^i_α. Equations (4) then take the canonical form ([1]):

$$\frac{dy^\alpha}{dx^i} = \frac{\partial \mathcal{H}}{\partial p^i_\alpha}, \tag{8}$$

$$\sum_i \frac{dp^i_\alpha}{dx^i} = -\frac{\partial \mathcal{H}}{\partial y^\alpha}. \tag{9}$$

2. Jacobi's theorem extended to the case of *n* independent variables. Let:

$$V_{(i)} \equiv V_{(i)}(x^i, y^\alpha, C^{\alpha j}) \tag{10}$$

be a complete integral ([1]) of the partial differential equation:

$$\sum_i \frac{\partial V_{(i)}}{\partial x^i} + \mathcal{H}\left(x^i, y^\alpha, \frac{\partial V_{(i)}}{\partial y^\alpha}\right) = 0. \tag{11}$$

The $C^{\alpha j}$ are mn arbitrary constants. Set:

$$p^i_{*\alpha} \equiv \frac{\partial V_{(i)}}{\partial y^\alpha} \tag{12}$$

and consider the equations:

$$p^i_\alpha = p^i_{*\alpha}, \tag{13}$$

$$\frac{\partial V_{(i)}}{\partial C^{\alpha j}} = \mathcal{C}^i_{*\alpha j}. \tag{14}$$

The $\mathcal{C}^i_{*\alpha j}$ are new integration constants or functions of the $x^1, \ldots, x^n$ such that:

$$\sum_i \frac{d\mathcal{C}^i_{*\alpha j}}{dx^i} = 0.$$

It results from the extended Jacobi theorem ([1]) in the case of n independent variables that equations (13) and (14) represent an integral of equations (8) and (9). This signifies that from these mn^2 equations (14) one may deduce the m functions y^α of the $x^1, \ldots, x^n$, the $C^{\alpha i}$, and the other constants that were introduced by the $\mathcal{C}^i_{*\alpha j}$.

Take the total derivative of (14) with respect to x^i and sum over the index i. One may deduce from the equations thus obtained that:

([1]) TH. DE DONDER, *Théorie invariantive du Calcul des Variations*, 1930, Paris, Gauthier-Villars and Co. See § 32 especially.

$$\frac{dy^\alpha}{dx^i} = Y_i^\alpha(x^j, y^\beta, C^{\beta j}) . \tag{15}$$

We remark that the compatibility conditions:

$$\frac{dY_i^\alpha}{dx^j} = \frac{dY_j^\alpha}{dx^i} \tag{16}$$

will be satisfied. On the other hand, we remark that each of these Y_j^α will be well-defined at any point (x^j, y^β) that is taken in the $n + m$-dimensional domain considered.

3. Hilbert-De Donder independence theorem. – *a*) One knows that ([2]) the canonical equations (8), (9) admit the relative integral invariant:

$$f_{(i)} \equiv \sum_\alpha p_\alpha^i d'y^\alpha \qquad (d'x^j \equiv 0). \tag{17}$$

It then results, as Th. De Donder has proved ([3]), that the 1-uple integral form:

$$\overline{j'_{(i)}} \equiv \sum_\alpha p_{*\alpha}^i d'y^\alpha \tag{18}$$

is an integral invariant of the immediately integrable total differential equations:

$$dy^\alpha = \sum_i Y_i^\alpha dx^i . \tag{19}$$

b) In the cited work ([2]), it is likewise proved that in order for the n-uple form in the $m + n$-dimensional space of $x^1, \dots, x^\alpha, y^1, \dots, y^m$:

$$\overline{j} \equiv \sum_i \sum_\alpha p_{*\alpha}^i \, d(x^1, \cdots, x^{i-1}, y^\alpha, x^{i+1}, \cdots, x^n) + (\bar{\mathcal{F}} - \sum_i \sum_\alpha p_{*\alpha}^i Y_i^\alpha) \, d(x^1, \cdots, x^n) \tag{20}$$

to be an exact differential n-uple, it is necessary and sufficient that the $\overline{j'_{(i)}}$ be exact differential 1-uples.

One has set:

$$\bar{\mathcal{F}} \equiv \mathcal{F}(x^i, y^\alpha, Y_i^\alpha) . \tag{21}$$

([2]) Loc. cit., form. 216.

([3]) Loc. cit. Chap. VIII; see § 36 or: TH. DE DONDER, "Sur le théorème d'indépendence de Hilbert," C.R. Ac. Sc., Paris **156** (1913), pp. 609 and 868.

4. The Weierstrass excess function. – *a*)

THEOREM: The form $\overline{j'_{(i)}}$ is always an exact differential.

Proof. – Thanks to (12):

$$\overline{j'_{(i)}} = \sum_{\alpha} \frac{\partial V_{(i)}}{\partial y^{\alpha}} d'y^{\alpha} ; \tag{22}$$

thus:

$$\overline{j'_{(i)}} = d' V_{(i)}. \tag{23}$$

b) By virtue of this latter theorem and the independence theorem (*b*, § 3), the *n*-uple form (20) is an exact differential *n*-uple. One will thus have, upon letting F_n denote a closed *n*-uple manifold in the $n + m$-dimensional space of $x^1, \ldots, x^n, y^1, \ldots, y^m$:

$$\oint_{F_n} \overline{j} = 0. \tag{24}$$

c) Let φ_n be a portion of the extremal manifold defined by (14). That portion of the extremal manifold will be bounded by the closed n – 1-uple manifold φ_{n-1} . Let $\mathcal{V}_n$ denote an *arbitrary* *n*-uple manifold that passes through φ_{n-1} , and is analytically represented by the equations:

$$y^{\alpha} = y^{\alpha}(x^1, \ldots, x^n). \tag{25}$$

As we remarked at the end of § 2, the Y_i^{α} will have well-defined values at each point of $\mathcal{V}_n$. Here, relation (24) will give:

$$\int_{\mathcal{V}_n} \overline{j} + \int_{\varphi_n} \overline{j} = 0. \tag{26}$$

However, one immediately sees, upon referring to (20), that:

$$\int_{\varphi_n} \overline{j} = \int_{\varphi_n} \overline{\mathcal{F}}\, d(x^1, \cdots, x^n) ; \tag{27}$$

thus:

$$\int_{\mathcal{V}_n} \overline{j} + \int_{\varphi_n} \overline{\mathcal{F}}\, d(x^1, \cdots, x^n) = 0. \tag{28}$$

Finally, let:

$$\int_{\mathcal{V}_n} \mathcal{F}\, d(x^1, \cdots, x^n) \tag{29}$$

be an integral taken over $\mathcal{V}_n$. The function $\mathcal{F}$ is defined by (1); the derivatives that appear in it are obtained by differentiating the functions (25). We denote these derivatives by y_i^{α} . Set:

$$\Delta I \equiv \int_{\mathcal{V}_n} \mathcal{F}\, d(x^1, \cdots, x^n) + \int_{\varphi_n} \bar{\mathcal{F}}\, d(x^1, \cdots, x^n)\,. \tag{30}$$

By virtue of (28), one will have:

$$\Delta I = \int_{\mathcal{V}_n} [\mathcal{F}\, d(x^1, \cdots, x^n) - \bar{j}\,]\,, \tag{31}$$

or, furthermore, upon referring to (20):

$$\Delta I = \int_{\mathcal{V}_n} \mathcal{E}\, d(x^1, \cdots, x^n)\,, \tag{32}$$

with:

$$\mathcal{E} \equiv \mathcal{F} - \bar{\mathcal{F}} - \sum_i \sum_\alpha p^i_{*\alpha}(y^\alpha_i - Y^\alpha_i)\,.$$

This is the Weierstrass formula extended to the case of n independent variables x^i and m unknown functions y^α. We have assumed that the function $\mathcal{F}$ depends only upon the x^i, the y^α, and the first derivatives of the y^α with respect to the x^i. The general case, where the function $\mathcal{F}$ depends upon derivatives of arbitrary order, in addition, will be treated in the next article.

Institut belge de Recherches radioscientifiques.

Generalization of the Weierstrass excess formula, as deduced from the Hilbert-De Donder independence theorem

by JULES GÉHÉNIAU, Aspirant F.N.R.S.

(*Second communication*) (*)

Abstract. – By using the Hilbert-De Donder independent theorem, we extend the Weierstrass excess formula to the case of n variables x^i, m functiona y^α, and their derivatives up to arbitrary order. We remark that in the case where $m = 1$ – i.e., where there is only one function y – that extension is immediate.

1. Extremal equations. Let:

$$\mathcal{F} \equiv \mathcal{F}(x^i, y^\alpha, y^\alpha_{i_1\cdots i_k}) \qquad \left.\begin{array}{l} i, i_1, \cdots, i_k = 1, \cdots, n \\ \alpha = 1, \cdots, m \\ k = 1, \cdots, c \end{array}\right\} \tag{1}$$

be a function of n independent variables x^i, m functions y^α, and their partial derivatives:

$$y^\alpha_{i_1\cdots i_k} \equiv \frac{\partial^k y^\alpha}{\partial x^{i_1} \cdots \partial x^{i_k}}. \tag{2}$$

The extremal equations of:

$$\delta \int \mathcal{F}\, d(x^1, \ldots, x^n) = 0 \tag{3}$$

are:

$$\frac{\delta \mathcal{F}}{\delta y^\alpha} = 0. \tag{4}$$

$$\frac{\partial^k y^\alpha}{\partial x^{i_1} \cdots \partial x^{i_k}} = y^\alpha_{i_1\cdots i_k}. \tag{4$'$}$$

These equations possess the relative integral invariant (1):

$$j'_{(i)} \equiv \sum_\alpha \sum_{k=1}^{c} \sum_{i_1} \cdots \sum_{i_k} p_\alpha^{i i_1\cdots i_k} \delta' y^\alpha_{i_1\cdots i_k} \qquad (\delta' x^i \equiv 0) \tag{5}$$

where:

$$p_\alpha^{i i_1\cdots i_k} \equiv \frac{\delta \mathcal{F}}{\delta y^\alpha_{i i_1\cdots i_k}}. \tag{5$'$}$$

(*) Presented by Th. De Donder.

(1) TH. DE DONDER, *Théorie invariantive du Calcul des Variations*, Paris, Gauthier-Villars, 1930, pp. 40.

The Hamiltonian function is defined by:

$$\mathcal{H} = -\mathcal{F} + \sum_{\alpha}\sum_{k}\sum_{i_1}\cdots\sum_{i_k} p_{\alpha}^{i\,i_1\cdots i_k}\, y_{i_1\cdots i_k}^{\alpha} \ . \tag{6}$$

2. Two theorems of Th. De Donder ([2]). – *a*) *If equations* (4), (4′) *are satisfied by the mn functions:*

$$\bar{y}_i^{\alpha} \equiv \bar{y}_i^{\alpha}(x^j, y^{\beta}), \qquad i, j = 1, \dots, n; \quad \alpha, \beta = 1, \dots, m \tag{7}$$

one may deduce from the integral invariant $j'_{(i)}$ [see (5)], *a relative integral invariant* $\bar{j}'_{(i)}$ *of the immediately integrable total differential equations:*

$$dy^{\alpha} = \sum_i \bar{y}_i^{\alpha} dx^i \, . \tag{8}$$

The relative integral invariant $\bar{j}'_{(i)}$ may be written:

$$\bar{j}'_{(i)} \equiv \sum_{\alpha}\left[\bar{p}_{\alpha}^{i} + \sum_{\beta}\sum_{i=2}^{c}\sum_{i_2}\cdots\sum_{i_l}\bar{p}_{\beta}^{i\,i_1\cdots i_l}\,\frac{\partial\bar{y}_{i\,i_2\cdots i_l}^{\beta}}{\partial y^{\alpha}}\right] d'y^{\alpha} \, , \tag{9}$$

where:

$$d'x^i \equiv 0. \tag{10}$$

The horizontal bars indicate that we have replaced y_i^{α}, $y_{i_1\cdots i_k}^{\alpha}$ by the functions (7) and the ones that one derives from them; for example:

$$\bar{y}_{ij}^{\alpha} \equiv \frac{\partial\bar{y}_i^{\alpha}}{\partial x^j} + \sum_{\beta}\frac{\partial\bar{y}_i^{\alpha}}{\partial y^{\beta}}\bar{y}_j^{\beta} \, . \tag{11}$$

b) Set:

$$\bar{N}_{\alpha}^{i} \equiv \bar{p}_{\alpha}^{i} + \sum_{\beta}\sum_{k=1}^{c}\sum_{i_2}\cdots\sum_{i_k}\bar{p}_{\beta}^{i\,i_1\cdots i_k}\,\frac{\partial\bar{y}_{i_2\cdots i_k}^{\beta}}{\partial y^{\alpha}} \tag{12}$$

and consider the *n*-uple integral form $\bar{j}$ in the $(n + m)$-dimensional space of x^i, y^{α}:

$$\bar{j} \equiv \sum_{\alpha}\sum_{i}\bar{N}_{\alpha}^{i} d(x^1\cdots x^{i-1} y^{\alpha} x^{i+1}\cdots x^n) + \left(\bar{\mathcal{F}} - \sum_{i}\sum_{\alpha}\bar{N}_{\alpha}^{i}\bar{y}_i^{\alpha}\right) d(x^1\cdots x^n) \, . \tag{13}$$

([2]) Loc. cit. Chap. VIII; see § 36 or: TH. DE DONDER, "Sur le théorème d'indépendence de Hilbert," C.R. Ac. Sc., Paris **156** (1913), pp. 609 and 868, and *Théorie invariantive du Calcul des Variations,* chap. VIII.

In order for $\overline{j}$ to be an exact differential n-uple, it is necessary and sufficient that the $\overline{j}'_{(i)}$ be exact differential 1-*uples.*

In particular, if $m = 1$ then the expression (13) will *always* become an exact differential n-uple.

3. The generalized Weierstrass excess formula. – *a*) *Case where* $m = 1$. – Let F_n denote a closed n-uple manifold in the $(n + m)$-dimensional space of the x^i, y^α. It immediately results from the two theorems that we just recalled that in the case considered ($m = 1$):

$$\oint_{F_n} \overline{j} = 0. \tag{14}$$

By reasoning as in our preceding Note ([3]), one will arrive at the *generalized Weierstrass excess formula:*

$$\mathcal{E} = \mathcal{F} - \overline{\mathcal{F}} - \sum_\alpha \sum_i \overline{N}^i_\alpha (y^\alpha_i - \overline{y}^\alpha_i)\,. \tag{15}$$

b) *Case where m is arbitrary.* – Now, let ([4]):

$$V_{(i)} \equiv V_{(i)}(x^j, y^\alpha, y^\alpha_{i_2\cdots i_k}, C^{\alpha j}, C^{\alpha j}_{i_2\cdots i_k}) \tag{16}$$

be a *complete* integral of the generalized Jacobi partial differential equation:

$$\sum_i \frac{\partial V_{(i)}}{\partial x^i} + \mathcal{H}\left(x^j, y^\alpha, y^\alpha_{i_2\cdots i_k}, \frac{\partial V_{(i)}}{\partial y^\alpha}, \frac{\partial V_{(i)}}{\partial y^\alpha_{i_2\cdots i_k}} \right) = 0. \tag{17}$$

The $C^{\alpha j}$, $C^{\alpha j}_{i_2\cdots i_k}$ are $mn \sum_{k=1}^{c} D_n^{k-1}$ arbitrary constants; the symbol D_n^{k-1} represents the number of combinations with repetitions of the n elements $k-1$ with $k-1$. Set:

$$p^i_{*\alpha} \equiv \frac{\partial V_{(i)}}{\partial y^\alpha} \quad \text{and} \quad p^{i\,i_2\cdots i_l}_{*\alpha} \equiv \frac{\partial V_{(i)}}{\partial y^\alpha_{i_2\cdots i_l}} \qquad \text{with} \quad l = 2, \ldots, c \tag{18}$$

and consider the equations:

$$p^{i\,i_2\cdots i_k}_{*\alpha} = p^{i\,i_2\cdots i_k}_{\alpha} \tag{19}$$

$$\frac{\partial V_{(i)}}{\partial C^{\alpha j}_{i_2\cdots i_k}} = \mathcal{C}^{i\,i_2\cdots i_k}_{*\alpha j}\,, \qquad k = 1, 2, \ldots, c \tag{20}$$

([3]) *Bull. Ac. roy. Belg.*, t. XXI, pp. 385.
([4]) TH. DE DONDER, loc. cit., ch. VII.

where the $\mathcal{C}_{*\alpha j}^{i\,i_2\cdots i_k}$ are functions of the $x^1, \ldots, x^n$ that satisfy:

$$\sum_i \frac{\partial \mathcal{C}_{*\alpha j}^{i\,i_2\cdots i_k}}{\partial x^i} = 0. \tag{21}$$

Recall that the integral (16) will be called *complete* if one may deduce from equations (20) the $m\sum_{k=1}^{c} D_n^{k-1}$ functions $y_{i_2\cdots i_k}^{\alpha}$ ($k = 1, \ldots, c$), which satisfy the $mn^2 \sum_{k=1}^{c} D_n^{k-1}$ equations (20).

One will have, in addition:

$$\left\| \frac{\partial^2 V_{(i)}}{\partial C_{j_2\cdots j_k}^{\alpha j}\,\partial y_{i_2\cdots i_k}^{\beta}} \right\| \neq 0, \qquad \begin{array}{c} i, i_2, \cdots, i_k, j, j_2, \cdots, j_k = 1, \cdots, n \\ \alpha, \beta = 1, \cdots, m. \end{array} \tag{22}$$

It results from the generalized Jacobi theorem that equations (19), (20) represent a solution of equations (4), (4′). Let:

$$y_{*i_2\cdots i_k}^{\alpha} \equiv y_{*i_2\cdots i_k}^{\alpha}(x^j, C, \mathcal{C}^*) \tag{23}$$

be such a solution, where C and $\mathcal{C}^*$ stand for $C^{\beta j}$, $C_{i_2\cdots i_k}^{\beta j}$, and $\mathcal{C}_{*\alpha j}^{i\,i_2\cdots i_k}$.

We show explicitly how one may deduce mn functions (7) from (23) that satisfy (4), (4′). From (23), one infers that if $k = 1$ then:

$$y_*^{\alpha} \equiv y_*^{\alpha}(x^j, C, \mathcal{C}^*). \tag{24}$$

We remark that:

$$y_{*i_1 i_2\cdots i_k}^{\alpha} \equiv \frac{\partial^k y_*^{\alpha}}{\partial x^{i_2}\cdots\partial x^{i_k}} \tag{25}$$

and that if $k > 1$ then the derivations will be *permutable.*

Now, suppose that the C, $\mathcal{C}^*$ depend upon m parameters p^{β}, in such a way that one may write:

$$y^{\alpha} = \eta^{\alpha}(x^j, p^{\beta}), \tag{26}$$

with:

$$\left\| \frac{\partial y^{\alpha}}{\partial p^{\beta}} \right\| \neq 0. \tag{27}$$

Solve (26) with respect to the p^{β}; so:

$$p^{\beta} = \pi^{\beta}(x^j, y^{\alpha}). \tag{28}$$

Introduce these π^{β} into (23); we obtain the *field equations:*

$$\boxed{y^\alpha_{*i_2\cdots i_k} = \bar{y}^\alpha_{i_2\cdots i_k}(x^j, y^\beta)}, \qquad k = 1, \dots, c. \tag{29}$$

In particular, one will have for the *first* partial derivatives:

$$y^\alpha_{*i} = \bar{y}^\alpha_i(x^j, y^\beta), \qquad (i, j = 1, \dots, n;\ \alpha, \beta = 1, \dots, m). \tag{30}$$

These $\bar{y}^\alpha_i$ indeed represent a solution to (44); in other words, one will have:

$$\frac{\delta\bar{\mathcal{F}}}{\delta y^\alpha} \equiv 0 \tag{31}$$

identically in the x^j *and* y^β; the bar serves to reiterate that in the variational derivative one has replaced the $y^\alpha_i, \dots, y^\alpha_{i_1 i_2\cdots i_c}$ with their values expressed as functions of the $\bar{y}^\alpha_i(x^j, y^\beta)$ and their derivatives are also expressed in terms of the x^j and y^β ($j = 1, \dots, n$; $\beta = 1, \dots, m$).

Totally differentiate the two sides of (30) with respect to the x^j; thanks to (20) and (25), one will have:

$$\bar{y}^\alpha_{ij} = \frac{\partial\bar{y}^\alpha_i}{\partial x^j} + \sum_\beta \frac{\partial\bar{y}^\alpha_i}{\partial y^\beta}\bar{y}^\beta_j, \tag{32}$$

which is nothing but (11).

On the other hand, by virtue of (18) and (19):

$$\bar{p}^{i\,i_2\cdots i_k}_\alpha = \overline{\frac{\partial V_{(i)}}{\partial y^\alpha_{i_2\cdots i_k}}}. \tag{33}$$

Introduce (33) into (12). One then sees that:

$$\bar{N}^i_\alpha = \frac{\partial\bar{V}_{(i)}}{\partial y^\alpha}, \tag{34}$$

where $V_{(i)}$ is nothing but the $V_{(i)}$ that appears in the right-hand side of (16), and in which one has replaced the $y^\alpha_{i_2\cdots i_k}$ with the $\bar{y}^\alpha_{i_2\cdots i_k}$ that are given by (29).

Upon taking (34) and (9) into account, one will have:

$$\bar{j}' = d'\bar{V}_{(i)} \qquad (d'x^i \equiv 0). \tag{35}$$

From this, by virtue of the theorems recalled in paragraph 2, it results that the $\bar{j}$ given by (13) *is an exact differential n-uple in the* ($n + m$)*-dimensional space of the* x^i, y^α.

Upon preserving the notations of our preceding Note, we may write:

$$\bar{j} = [\bar{\mathcal{F}} + \sum_{\alpha}\sum_{i}(y_i^{\alpha} - \bar{y}_i^{\alpha})\bar{N}_{\alpha}^i]\,d(x^1 \cdots x^n)\,. \tag{36}$$

However, by virtue of (6), (17), and (34):

$$\bar{\mathcal{F}} - \sum_{\alpha}\sum_{i}\bar{y}_i^{\alpha}\bar{N}_{\alpha}^i = \sum_{i}\frac{\partial\bar{V}_{(i)}}{\partial x^i}; \tag{37}$$

thus, thanks to (34) and (37):

$$\bar{j} = \sum_{i}\frac{\partial\bar{V}_{(i)}}{\partial x^i}\cdot d(x^1, \dots, x^n), \tag{38}$$

where one has set:

$$\frac{d\bar{V}_{(i)}}{dx^i} = \frac{\partial\bar{V}_{(i)}}{\partial x^i} + \sum_{\alpha}\frac{\partial\bar{V}_{(i)}}{\partial y^{\alpha}}\,y_i^{\alpha}\,. \tag{39}$$

Finally, upon always preserving the same notations, and referring to our first Note, one will have:

$$\oint_{F_n}\bar{f} = 0, \tag{40}$$

as well as the generalized Weierstrass excess formula:

$$\boxed{\mathcal{E} = \mathcal{F} - \bar{\mathcal{F}} - \sum_{i}\sum_{\alpha}(y_i^{\alpha} - \bar{y}_i^{\alpha})\bar{N}_{\alpha}^i.} \tag{41}$$

Institut belge de Recherches radioscientifiques.

On the geodesic fields of the calculus of variations

By Th. H. J. LEPAGE ([*])

(*First communication.*)

1. When one seeks to extend the Legendre and Jacobi conditions, the Hilbert invariant integral, and the Weierstrass excess function to *multiple* integrals that depend upon *several* unknown functions one encounters difficulties of a special character that were indicated a long time ago in a memoir of *Clebsch* ([1]) that was dedicated to a generalization of the transformations that were performed by Jacobi on the second variation of integrals. *J. Hadamard* discussed these difficulties in two notes that were published 1902 and 1905 in the *Bulletin de la Société Mathématique de France*. Notably, he observed that the quadratic form that appears in the expression for the second variation is not necessarily positive definite in the case of a minimum, but it suffices that it become so when one combines with certain alternating bilinear forms in an arbitrary manner ([2]).

In the course of recent years, various authors have extended the Hilbert independence theorem and the theories of Weierstrass and Hamilton-Jacobi to the case that I just recalled. Now, it is curious to observe that the results that were obtained differ, although they all reduce to the classical results when one supposes that the number of unknown functions is equal to one. For example, whereas for De Donder ([3]), Gehéniau ([4]), and Hermann Weyl ([5]), the excess function is linear, for Caratheodory ([6]) the function E, which generalizes the classical Weierstrass function, is not, and the Legendre condition that is introduced by that author differs from the Legendre condition that was considered by the previous authors. Quite recently, H. Börner ([7]) has completed the results of C. Caratheodory on an important point. When applied to Cauchy's theory of characteristics it shows that any sufficiently small portion of an extremal may be incorporated

---

([*]) Presented by Th. De Donder.

([1]) A. CLEBSCH, Über die zweite Variation vielfacher Integrale. (*Journal de Crette*, 1859, 122-148).

([2]) J. HADAMARD. Sur une question du Calcul des Variations. (*Bull. Soc. Math. France*, 1902, pp. 253-256).

([3]) TH. DE DONDER, Sur le théoreme d'independence de Hilbert (*C.R. Acad. Sc. Paris,* t. 156, pp. 609-611, Feb. 24, 1913, pp. 868-870, Mar. 17, 1913); *Théorie invariantive du Calcul des Variations*, 2nd ed., Paris, Gauthier-Villars (1935). Book II, pp. 95-170, especially ch. X and ch. XI.

([4]) J. GEHÉNIAU, Généralisation de la formule d'excès de Weierstrass déduite du théorème d'indépendence d'Hilbert-De Donder. (*Bull. Acad. Roy. Belg.*, *Cl. des Sc.* XXI, 1935, pp. 385 and 504.)

([5]) HERMANN WEYL, Observations on Hilbert's Independence Theorem and Born's Quantization of Field Equations, pp. 505-508 (*Phys. Rev*., vol. 46, second series (1934); *Annals of Math.*, 36 (1935).)

([6]) C. CARATHEODORY, Über die geometrische Behandlung der Extrema die Doppelintegralen (*Verh. d. Schw. Natur. Ges.*, 1917); Über die Variationsrechnung bei mehrfachen Integralen, pp. 193-216. (*Acta Szeged*, 4 (1929)); *Variationsrechnung und partielle Differentialgleichungen erster Ordnung*. Teubner, Leipzig (1935).

([7]) HERMANN BÖRNER, Über die Extremalen and geodätische Felder in der Variationsrechnung der mehrfachen Integrale. (*Math. Ann.*, 112 (1936), pp. 187-220.)

(*einbetten*) in a geodesic field surrounding that extremal, at least when a certain quadratic form is positive definite (Legendre condition).

A notion that plays a role in the Caratheodory theory that is analogous to that of the *Weierstrass extremal field* for simple integrals is that of a *geodesic field* along a portion of an extremal manifold. However, contrary to the situation in the following two cases: 1) *one independent variable and several unknown functions,* 2) *several independent variables, but one unknown function*, a geodesic field is not, in general, comprised of a family of extremals that uniformly cover a certain region of space. In other words, *it is not always a field of extremals*. However, as Boerner has remarked, this fact has no importance because in order to achieve a theory of conditions for an extremum it suffices to understand geodesic fields.

On the other hand, treating the same problem, De Donder and H. Weyl introduced fields that likewise lead to Hamilton-Jacobi theory. Nevertheless, these latter fields are not geodesic fields in the Caratheodory ([8]) sense.

The reason for this difference obviously stems from the difficulty that was brought to light by Clebsch and Hadamard. In what follows, seeking to account for this difference, I am led to propose a *new definition* for the fundamental notion of *geodesic field*. This notion has the advantage of subsuming that of Caratheodory, as well as the fields of De Donder and H. Weyl, in the sense that one obtains the one or the other upon specifying, in a suitable manner, certain indeterminate elements A_{ij} that present themselves in the differential systems of a geodesic field.

In order to achieve this goal, I use the method of the calculus of symbolic differential forms ([9]); I think that they permit us to represent all of the theory in a simpler manner.

To fix ideas, and also to simplify the notations somewhat, I consider a double integral in the calculus of variations:

$$I(z_i) \equiv \iint f\left(x, y; z_1, \cdots, z_n; \frac{\partial z_1}{\partial x}, \cdots, \frac{\partial z_n}{\partial x}; \frac{\partial z_1}{\partial y}, \cdots, \frac{\partial z_n}{\partial y}\right) dx\, dy\,,$$

in which x, y are independent variables, $z_1, \ldots, z_n$ are unknown functions, and $n \geq 1$. However, the following method immediately extends to the case of a triple integral, i.e., to the case in which the independent variables number more than one.

2. First of all, consider the symbolic quadratic form:

$$\omega = f(x, y; z_1, \ldots, z_n; p_1, \ldots, p_n; q_1, \ldots, q_n)\, dx\, dy \tag{2.1}$$

([8]) Except obviously, in the two extreme cases that were pointed out above, for which the two theories coincide with the classical Hamilton-Jacobi theory.

([9]) É. CARTAN, *Leçons sur les invariants integraux*. Paris, Hermann, 1922. TH. DE DONDER, *Théorie des invariants integraux*. Paris, Gauthier-Villars, 1927. E. GOURSAT, *Leçons sur les probléme de Pfaff*. Paris, Hermann, 1922, especially, chap. III. E. KÄHLER, Einführung in die Theorie der System von Differentialgleichungen (*Hamburger mathematische Einzelschreften*. 16 Heft., Teubner, Leipzig, 1935).

in the $3n + 2$ *independent* variables x, y, z_i, p_i, q_i. We suppose that f is a function that is several-times-differentiable in a certain domain Δ of the space $(x, y; z_1, \ldots, z_n)$, and for all of the finite values that are attributed to the variables p_i, q_i.

We introduce the n linearly independent Pfaff forms:

$$\omega_i = dz_i - p_i\,dx - q_i\,dy, \qquad i = 1, 2, \ldots, n. \tag{2.2}$$

The total symbolic derivative of a form φ will be denoted by the notation $d\varphi$. Therefore, the differential of the form ω will be the cubic symbolic form:

$$\begin{aligned} d\omega &= f_{z_i}\,dz_i\,dx\,dy + f_{p_i}\,dp_i\,dx\,dy + f_{q_i}\,dq_i\,dx\,dy \\ &= f_{z_i}\,dx\,dy\,\omega_i + (f_{p_i}\,dp_i + f_{q_i}\,dq_i)\,dx\,dy\,. \end{aligned} \tag{2.3}$$ [10]

Similarly, for the Pfaff differential forms ω_i one will have:

$$d\omega_i = dx\,dp_i + dy\,dq_i \qquad i = 1, 2, \ldots, n. \tag{2.4}$$

Recall that the differential of the symbolic product $\omega_i\,\omega_j$ of the two linear forms ω_i and ω_j is the cubic form:

$$d(\omega_i\omega_j) = (d\omega_i)\cdot\omega_j - \omega_i\,(d\omega_j). \tag{2.5}$$

Now let:

$$\theta_i = X_i\,dx + Y_i\,dy + A_{ij}\omega_j\,, \qquad i = 1, 2, \ldots, n, \tag{2.6}$$

be n Pfaff forms, where X_i, Y_i, A_{ij} are differentiable functions of (x, y, z_i, p_i, q_i) that are chosen arbitrarily, moreover.

It is obvious that the quadratic form:

$$\Omega = f\,dx\,dy + \theta_i\,\omega_i\,, \tag{2.7}$$

is the most general form that is congruent to $f\,dx\,dy$ (modulo ω_i):

$$\Omega \equiv \omega (\text{mod } \omega_1, \ldots, \omega_n).$$

Furthermore, we may suppose that one has:

$$A_{ij} + A_{ji} = 0\,. \tag{2.8}$$

Given this, *determine all of the forms* Ω *that give rise to congruences:*

$$d\Omega \equiv 0 \qquad (\text{mod } \omega_1, \ldots, \omega_n). \tag{2.9}$$

([10]) To simplify the notation, one agrees to write $\alpha_i\,\beta_i$ for $\sum_{i=1}^{n}\alpha_i\beta_i$ and $F_{ij}\cdot\Phi_{kj}$ for $\sum_{j=1}^{n}F_{ij}\cdot\Phi_{kj}$.

In order to accomplish this, since the θ_i are differential forms we have, by virtue of (2.7):

(2.10) $$d\Omega \equiv df\, dx\, dy + (d\theta_i)\omega_i - \theta_i\,(d\omega_i),$$

which becomes, upon taking (2.6) into account:

(2.11) $$\begin{aligned} d\Omega = f\, dx\, dy &+ (dX_i\, dx + dY_i\, dy)\omega_i - (X_i\, dx + Y_i\, dy)\, d\omega_i \\ &+ d(A_{ij}\cdot\omega_j)\cdot\omega_i - A_{ij}\,\omega_j\cdot(d\omega_i). \end{aligned}$$

or, by virtue of (2.3) and (2.5):

(2.12) $$\begin{aligned} d\Omega = \; & f_{z_i} dx\, dy\, \omega_i + (f_{p_i} dp_i + f_{q_i} dq_i)\, dx\, dy \\ &+ (dX_i\, dx + dY_i\, dy)\omega_i - (X_i\, dx + Y_i\, dy)(dx\, dp_i + dy\, dq_i) \\ &+ d(A_{ij}\cdot\omega_j)\cdot\omega_i - A_{ij}\,\omega_j\cdot(d\omega_i). \end{aligned}$$

Observe that the set of terms that do not depend upon any of the factors $\omega_1, \ldots, \omega_n$ is:

$$(f_{p_i} dp_i + f_{q_i} dq_i)\, dx\, dy - (X_i dq_i - Y_i dp_i)\, dx\, dy\,.$$

This then implies the following remark: *Among all of the forms:*

$$\Omega \equiv \omega\,(\mathrm{mod}\ \omega_1, \ldots, \omega_n),$$

the ones for which one takes:

(2.13) $$X_i = f_{q_i}\,, \qquad Y_i = -\, f_{p_i}\,, \qquad i = 1, 2, \ldots, n,$$

with the remaining A_{ij} arbitrary, give rise to the congruence:

$$d\Omega \equiv 0\,(\mathrm{mod}\ \omega_1, \ldots, \omega_n).$$

Henceforth, we suppose that we have made this choice for the X_i, Y_i, and we further write:

(2.14) $$\Omega = f\, dx\, dy + (f_{q_i} dx - f_{p_i} dy)\omega_i + A\ \omega_i\ \omega_j\,.$$

We thus have:

(2.15) $$d\Omega = (f_{z_i} dx\, dy + df_{q_i} dx - df_{p_i} dy)\omega_i + dA_{ij}\cdot\omega_i\omega_j + A_{ij}\{(d\omega_i)\omega_j - \omega_i(d\omega_j)\}\,.$$

REMARK. – If $n = 1$ then the expression Ω does not contain any A_{ij} coefficient because all of the symbolic products $\omega_i\ \omega_j$ are identically null. *It is no longer true that the number of variables* z_i *is greater than one.* Here is an important fact that will show up in all of what follows: in the case $n = 1$, and only in this case, the form Ω that is congruent to $f\, dx\, dy$ (modulo $\omega_1, \ldots, \omega_n$) is unique. It is the form:

(2.16) $$\Omega = f(x, y, z, p, q)\, dx\, dy + (f_q\, dx - f_p\, dy)(dz - p\, dx - q\, dy).$$

3. *The rank of the form* Ω. If $n = 1$ then the rank ([11]) of the form Ω is always two because one has, upon assuming that $f \neq 0$:

$$\Omega = [f\,dx + f_p\,(dz - p\,dx - q\,dy)] \cdot [f\,dy + f_q\,(dz - p\,dx - q\,dy)]. \tag{3.1}$$

On the contrary, if $n > 1$ then the rank of the form Ω is no longer necessarily of rank two, unless, as we have seen, we attribute certain particular values to the particular coefficients A_{ij} .

One knows that a quadratic form of rank two has a symbolic square that is zero, and conversely. Furthermore, the values of A_{ij} for which the corresponding form Ω is of rank two are solutions to the symbolic equation:

$$\Omega^2 = 0, \tag{3.2}$$

i.e.:

$$\tfrac{1}{2}\Omega^2 = f\,\theta_i\,\omega_i\,dx\,dy + \theta_i\,\theta_j\,\omega_i\,\omega_j = 0. \tag{3.3}$$

Upon taking (2.6) and (2.13) into account, developing the products in the left-hand side of equation (3.3), following the rules of symbolic manipulation, and finally annulling all of the coefficients of distinct terms – i.e., the coefficients of the monomials:

$$dx\,dy\,\omega_i\,\omega_j, \quad dx\,\omega_i\,\omega_j\,\omega_k, \quad dy\,\omega_i\,\omega_j\,\omega_k, \quad \omega_i\,\omega_j\,\omega_k\,\omega_s, \quad i, j, k, s = 1, 2, \ldots, n \tag{3.4}$$

we obtain:

$$A_{ij} = \frac{1}{f}\begin{bmatrix} f_{p_i} & f_{p_j} \\ f_{q_i} & f_{q_j} \end{bmatrix}, \qquad i, j = 1, 2, \ldots, n. \tag{3.5}$$

Hence, we have the proposition: *Among all of the forms* $\Omega \equiv \omega$ (modulo $\omega_1, \ldots, \omega_n$) *such that* $d\Omega \equiv 0$ (modulo $\omega_1, \ldots, \omega_n$) *there exists one and only one of them whose rank is two* ([12]). *It has the form:*

$$\Omega^* = f\,dx\,dy + (f_{q_i}dx - f_{p_i}dy)\omega + \frac{1}{f}\begin{bmatrix} f_{p_i} & f_{p_j} \\ f_{q_i} & f_{q_j} \end{bmatrix}\omega_i\omega_j\,. \tag{3.6}$$

Therefore, the rank r – which is necessarily even – of all of the other forms Ω – *in particular, those for which all of the* A_{ij} *are null* – is at least four:

$$4 \leq r \leq 3n + 2.$$

The form Ω^*, being of rank two, is the symbolic product of two linearly independent Pfaff forms. Moreover, one has, as one shows by a simple verification:

([11]) The *rank* of a symbolic form is defined to be its rank as an ordinary algebraic form. For example, see E. GOURSAT, *loc. cit.*, chap. III.

([12]) The minimum rank for a quadratic form that is not identically null.

(3.7) $$\Omega^* = \frac{1}{f}[f\,dx + f_{p_i}\cdot\omega_i]\cdot[f\,dy + f_{q_i}\cdot\omega_j].$$

One observes that there is an analogy that presents this form as the form (3.1) when $n = 1$.

4. All of the foregoing may be immediately extended to the case of a form of degree $m \geq 2$:

(4.1) $$\omega = f(x_1, \ldots, x_m; z_i; p_{1i}, p_{2i}, \ldots, p_{mi})\,dx_1 \ldots dx_m, \qquad i = 1, 2, \ldots, n.$$

One sets, as above:

(4.2) $$\omega_i = dz_i - p_{\alpha i}\,dx_\alpha, \qquad i = 1, 2, \ldots, n, \quad \alpha = 1, 2, \ldots, m.$$

Among all of the forms $\Omega \equiv \omega$ (modulo $\omega_1, \ldots, \omega_n$) there exists one of minimum rank m such that one has:

(4.3) $$d\Omega \equiv 0 \pmod{\omega_1, \ldots, \omega_n}.$$

It is the form:

(4.4) $$\Omega^* = \frac{1}{f^{m-1}}(f\,dx_1 + f_{1i_1}\omega_{i_1})\cdots(f\,dx_m + f_{mi_m}\cdot\omega_{i_m}),$$

in which $f_{\alpha i}$ denotes the partial derivative of f with respect to $p_{\alpha i}$.

5. In the preceding section, we assumed that x, y, p_i, q_i are independent variables, and we denoted a partial derivative with respect to the variables z_i, p_i, or q_i by an index z_i, p_i, or q_i. Now, we shall consider the p_i, q_i to be $2n$ differentiable functions of the independent variables $(x, y, z_1, \ldots, z_m)$, and in order that there be no fear of ambiguity the partial derivatives with respect to the variables x, y, z_i will now be exclusively represented by the symbols:

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z_i}.$$

With this convention, we have, letting F denote a function of x, y, p_i, q_i:

(5.1) $$\begin{cases} \dfrac{\partial F}{\partial x} = F_x + F_{p_i}\dfrac{\partial p_i}{\partial x} + F_{q_i}\dfrac{\partial q_i}{\partial x}, \\ \dfrac{\partial F}{\partial y} = F_y + F_{p_i}\dfrac{\partial p_i}{\partial y} + F_{q_i}\dfrac{\partial q_i}{\partial y}, \\ \dfrac{\partial F}{\partial z} = F_z + F_{p_i}\dfrac{\partial p_i}{\partial z} + F_{q_i}\dfrac{\partial q_i}{\partial z}, \quad i, k = 1, 2, \ldots, n. \end{cases}$$

At each point of a domain Δ in the space $(x, y, z_1, \ldots, z_n)$ the functions $p_i(x, y, z_1, \ldots, z_n)$, $q_i(x, y, z_1, \ldots, z_n)$ define an *element* (x, y, p_i, q_i) *of a field* $[p_i\ q_i]$. *We let* $[\Omega]$, $[\omega_i]$, $[f]$, *etc., denote the quantities that the expressions* Ω, ω_i, f *become when one evaluates them in the field* $[p_i\ q_i]$; *i.e., when one replaces the* $[p_i\ q_i]$ *with their values as functions of the* $(x, y, z_1, \ldots, z_n)$.

The field is called *integrable* if the Pfaff system:

$$[\Omega_i] = 0, \qquad i = 1, 2, \ldots, n \tag{5.2}$$

is completely integrable. In order for this to be true, it is necessary and sufficient that one has (Frobenius theorem):

$$[\omega_1 \cdot \omega_2 \cdot \ldots \cdot \omega_n\,(d\omega_i)] = 0, \quad i = 1, 2, \ldots, n, \tag{5.3}$$

which may also be written:

$$\frac{\partial p_i}{\partial y} + \frac{\partial p_i}{\partial z_k} q_k = \frac{\partial q_i}{\partial x} + \frac{\partial q_i}{\partial z_k} p_k, \quad (i = 1, 2, \ldots, n). \tag{5.4}$$

6. *The expression for* $d[\Omega]$. Suppose we are given a vector field $p_i = p_i(x, y, z_1, \ldots, z_n)$, $q_i = q_i(x, y, z_1, \ldots, z_n)$, which may or may not be integrable, and let us propose to calculate the expression for the symbolic differential of the form $[\Omega]$. It is a cubic form in the differentials dx, dy, dz_i that we may represent by a development of the following type:

$$d[\Omega] = \Omega_i[dx\ dy\ \omega_i] + \Omega_{ijx}[\omega_i\ \omega_j\ dx] + \Omega_{ijy}[\omega_i\ \omega_j\ dy] + \Omega_{ijk}[\omega_i\ \omega_j\ \omega_k], \tag{6.1}$$

in which Ω_i denotes the coefficient of the term in $[dx\ dy\ \omega_i]$ in the development of $d[\Omega]$, and similarly, Ω_{ijx} denotes the coefficient of the monomial $[\omega_i\ \omega_j\ dx]$, etc.

In order to calculate all of these coefficients, it suffices to develop the symbolic differential of the form $[\Omega]$, while taking (2.15) and (5.1) into account and eliminating the differentials dz_i in the result by means of the relations:

$$[\omega_i] = dz_i - p_i(x, y, z_1, \ldots, z_n)\, dx - q_i(x, y, z_1, \ldots, z_n)\, dy, \qquad i = 1, 2, \ldots, n. \tag{6.2}$$

Moreover, one has:

(6.3) $d[\Omega] =$

$$\left\{\begin{aligned} &\left\{[f_{z_i}]\,dx\,dy + \left[\frac{\partial f_{q_i}}{\partial y}\right] dy\,dx + \left[\frac{\partial f_{q_i}}{\partial z_j}\right] dz_j\,dx - \left[\frac{\partial f_{p_i}}{\partial y}\right] dx\,dy - \left[\frac{\partial f_{p_i}}{\partial z_j}\right] dz_j dy\right\}\cdot[\omega_i] \\ &\quad + \left(\frac{\partial A_{ij}}{\partial x}[dx\,\omega_i\omega_j] + \frac{\partial A_{ij}}{\partial y}[dy\,\omega_i\omega_j] + \frac{\partial A_{ij}}{\partial z}[dz\,\omega_i\omega_j]\right) + [A_{ij}]\cdot[(d\omega_i)\omega_j - \omega_i(d\omega_j)]. \end{aligned}\right.$$

However:

(6.4) $[(d\omega_i)\,\omega_j - \omega_i\,(d\omega_j)] =$

$$\left\{dx\left(\frac{\partial p_i}{\partial y}dy+\frac{\partial p_i}{\partial z_k}dz_k\right)+dy\left(\frac{\partial q_i}{\partial x}dx+\frac{\partial q_i}{\partial z_k}dz_k\right)\right\}[\omega_j]-[\omega_i]\left\{dx\left(\frac{\partial p_i}{\partial y}dy+\frac{\partial p_i}{\partial z_k}dz_k\right)+dy\left(\frac{\partial q_i}{\partial x}dx+\frac{\partial q_i}{\partial z_k}dz_k\right)\right\}$$

.

It then results from this that one has:

(6.5)

$$\begin{cases} d[\Omega]= \\ \qquad \left[f_{z_i}-\frac{\partial f_{q_i}}{\partial y}-\frac{\partial f_{p_i}}{\partial x}-\frac{\partial f_{q_i}}{\partial z_j}q_j-\frac{\partial f_{p_i}}{\partial z_j}p_j+A_{ij}\left\{\frac{\partial p_j}{\partial y}-\frac{\partial q_j}{\partial x}-\frac{\partial p_j}{\partial z_k}q_k-\frac{\partial q_j}{\partial z_k}p_k\right\}\right]\cdot[dx\,dy\,\omega_i] \\ \qquad +\left[\frac{\partial f_{q_i}}{\partial z_j}-\frac{\partial f_{q_j}}{\partial z_i}-\frac{\partial A_{ij}}{\partial x}+p_k\frac{\partial A_{ij}}{\partial z_k}+A_{kj}\frac{\partial p_k}{\partial z_i}+A_{ik}\frac{\partial p_k}{\partial z_j}\right]\cdot[dx\;\omega_i\,\omega_j] \\ \qquad +\left[\frac{\partial f_{p_j}}{\partial z_j}-\frac{\partial f_{p_j}}{\partial z_i}A_{ij}+\frac{\partial A_{ij}}{\partial y}+q_k\frac{\partial A_{ij}}{\partial z_k}+A_{kj}\frac{\partial q_k}{\partial z_i}+A_{ik}\frac{\partial q_k}{\partial z_j}\right]\cdot[dy\;\omega_i\,\omega_j] \\ \qquad +\left[\frac{\partial A_{ij}}{\partial z_k}+\frac{\partial A_{jk}}{\partial z_i}+\frac{\partial A_{ki}}{\partial z_j}\right]\cdot[\omega_i\omega_j\omega_k]. \end{cases}$$

Upon comparing (6.1) and (6.5), one immediately obtains the expressions for Ω_i, Ω_{ijx} , etc:

$$(6.6)\qquad \begin{cases} \Omega_i = f_{z_i}-\left(\frac{\partial f_{p_i}}{\partial x}+\frac{\partial f_{p_i}}{\partial z_j}p_j\right)-\left(\frac{\partial f_{p_i}}{\partial x}+\frac{\partial f_{p_i}}{\partial z_j}p_j\right)+A_{ji}\left(\frac{\partial p_i}{\partial y}-\frac{\partial q_j}{\partial x}+\frac{\partial p_j}{\partial z_k}q_k-\frac{\partial q_j}{\partial z_k}p_k\right) \\ \Omega_{ijx} = \cdots \end{cases}$$

Observe that if the field $[p_i\ q_i]$ is integrable then the expressions Ω_i simplify, since, by virtue of conditions (5.4) the coefficients A_{ij} are null in that event. Observe further that if $n = 1$ then all of the coefficients Ω_{ijx}, Ω_{ijy}, Ω_{ijz} are null, and the coefficients Ω_i reduce to only one, namely:

$$(6.7)\qquad f_{z_i}-\left(\frac{\partial f_{p_i}}{\partial x}+\frac{\partial f_{p_i}}{\partial z_j}p_j\right)-\left(\frac{\partial f_{p_i}}{\partial x}+\frac{\partial f_{p_i}}{\partial z_j}p_j\right),$$

since all of the A_{ij} are null (remark in sec. 2).

7. *Geodesic field relative to the form* Ω.

DEFINITION: *Any field* $[p_i\ q_i]$, *whether integrable or not, for which the form* $[\Omega]$ *is an exact (symbolic) total differential – i.e., for which one has:*

$$(7.1)\qquad d[\Omega] = 0,$$

will be called a *geodesic field relative to the form* Ω.

By virtue of the results of the preceding section, we see that $d[\Omega]$ is null if:

$$\Omega_i = \Omega_{ijx} = \Omega_{ijy} = \Omega_{ijz} = 0, \tag{7.2}$$

and conversely. Therefore, *a field is geodesic relative to a form* Ω, *which corresponds to an arbitrary choice of* A_{ij}, *moreover, if the functions* p_i q_i *of the field* $[p_i\ q_i]$ *verify conditions* (7.2), *and conversely.*

A geodesic field will be called *integrable* if, along with conditions (7.2), conditions (5.4) are satisfied.

One easily assures oneself that a field that is geodesic for a certain form Ω ([13]) is not geodesic for *all* forms Ω. Indeed, consider, for example, the two forms $\Omega(0)$ and $\Omega(\lambda)$, the first of which is obtained by annulling all of the A_{ij}, and the second of which is obtained by attributing a value $\lambda \neq 0$ to the A_{ij}. If the field is geodesic for both $\Omega(0)$ and $\Omega(\lambda)$ then the form:

$$[\Omega(\lambda) - \Omega(0)] = \lambda_{ij}[\omega_i\ \omega_j], \qquad \lambda_{ij} = \lambda$$

is an exact total differential, and conversely. However, it is obvious that one may always choose a λ that would make the latter form not be an exact total differential.

The following propositions, which we limit ourselves to merely stating, result immediately from known results on differential forms:

I. *For any field that is geodesic relative to* Ω *the hypersurface integral:*

$$\iint_\Sigma [\Omega], \tag{7.3}$$

when extended to any closed regular manifold Σ, *is null, and conversely.*

From this, in the following sections we shall deduce the extension of the Hilbert invariant integral theorem and its various consequences: the Weierstrass E functions, the Legendre condition, and the Hamilton-Jacobi equation.

II. *The associated system to* $[\Omega]$ *is completely integrable*, and one has:

$$[\Omega] = dS_1\, dS_2 + dS_3\, dS_4 + \ldots + dS_{2g-1} \cdot dS_{2\gamma}, \tag{7.4}$$

2γ being the order of the associated system, and the S_i denoting 2γ differentiable functions in the x, y, z_1, …, z_n that are mutually distinct. The number 2γ is called the *class* of the form $[\Omega]$.

III. In particular, for every field that is geodesic relative to the form Ω^* of rank two one has:

([13]) Thus, for certain choices of the functions A_{ij}.

(7.5)
$$\begin{cases} \gamma = 1 \\ [\Omega^*] = dS_1 \cdot dS_2. \end{cases}$$

The system associated to $[\Omega^*]$ is the system of two Pfaff equations:

(7.6)
$$\begin{cases} [f]dx + [f_{p_i}\omega_i] = 0, \\ [f]dy + [f_{q_i}\omega_i] = 0. \end{cases}$$

IV. The converse of proposition II is true: In other words, any field $[p_i\ q_i]$ that gives rise to the identity (7.4) is geodesic. From this, there results another way of writing the differential equations of a geodesic field relative to a form Ω.

Indeed, taking into account the identities:

(7.7)
$$dS_i = (S_{ix} + S_{iz_j}p_j)dx + (S_{iy} + S_{iz_j}q_j)dy + S_{iz_j}\omega_j, \qquad i = 1, 2, \dots, n,$$

the expression in the right-hand side of (7.4) may be written:

(7.8)
$$\sum_k dS_k \cdot dS_{k+1} = \sum_k \left\{ \Delta_k dx\, dy + \frac{\partial \Delta_k}{\partial q_i}[dx\, \omega_i] - \frac{\partial \Delta_k}{\partial p_i}[dy\, \omega_i] \right\} + \sum_k \begin{bmatrix} S_{kz_i} & S_{kz_j} \\ S_{k+1z_i} & S_{k+1z_j} \end{bmatrix} [\omega_i \omega_j],$$

in which $\sum_k$ indicates a summation over $k = 1, 3, 5, \dots, 2\gamma - 1$, and one sets:

(7.9)
$$\Delta_k = \begin{bmatrix} S_{kx} + S_{kz_j}p_j & S_{ky} + S_{kz_j}q_j \\ S_{k+1\,x} + S_{k+1\,z_j}p_j & S_{k+1\,x} + S_{k+1\,z_j}q_j \end{bmatrix}.$$

Upon taking the preceding remark into account, along with (7.4), (7.8), and the expression (2.14) for [Ω], one gets:

(7.10)
$$\begin{cases} [f] = \sum_k \Delta_k \equiv \Delta \\ [f_{p_i}] = \dfrac{\partial \Delta}{\partial p_i}, \qquad [f_{q_i}] = \dfrac{\partial \Delta}{\partial q_i}, \\ [A_{ij}] = \sum_k \begin{bmatrix} S_{kz_i} & S_{kz_j} \\ S_{k+1\,z_i} & S_{k+1\,z_i} \end{bmatrix} \qquad k = 1, 3, 5, \cdots, 2\gamma - 1. \end{cases}$$

This system is entirely equivalent to the differential system (7.2) of a geodesic field relative to Ω.

In particular, if the field is geodesic relative to the form Ω^* of rank two then the preceding system simplifies beautifully because, on the one hand $\gamma = 1$, and, on the other hand, one has (sec. 3.5):

$$[A_{ij}] = \frac{1}{[f]}\begin{bmatrix} f_{p_i} & f_{p_j} \\ f_{q_i} & f_{q_j} \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} \dfrac{\partial\Delta}{\partial p_i} & \dfrac{\partial\Delta}{\partial p_j} \\ \dfrac{\partial\Delta}{\partial q_i} & \dfrac{\partial\Delta}{\partial q_j} \end{bmatrix} = \begin{bmatrix} S_{1z_i} & S_{1z_j} \\ S_{2z_i} & S_{2z_j} \end{bmatrix};$$

so that the equations of a field relative to the form Ω^* are:

$$(7.11) \qquad \begin{cases} [f] = \begin{bmatrix} S_{1x} + S_{1z_j} p_j & S_{1y} + S_{1z_j} q_j \\ S_{2x} + S_{2z_j} p_j & S_{2x} + S_{2z_j} p_j \end{bmatrix} \equiv \Delta, \\ [f_{p_i}] = \dfrac{\partial\Delta}{\partial p_i}, \quad [f_{q_i}] = \dfrac{\partial\Delta}{\partial q_i}, \quad i = 1, 2, \cdots, n. \end{cases}$$

(*To be continued.*)

Mayer fields in the calculus of variations for multiple integrals

by ROBERT DEBEVER, Licensee in physical sciences ([*])

1. One knows the objective of the Weierstrass-Hilbert method and the role of the independent integral ([1]).

By systematically appealing to the algorithm of symbolic differential forms, Lepage ([2]) has developed a theory of the independent integral for multiple integrals and the notion of geodesic field that is necessarily introduced in the Weierstrass-Hilbert method.

It appears that there exists, in general, an infinitude of independent integrals, and by that fact itself, an infinitude of excess functions that are each attached to a particular differential form. Lepage then characterized, in a simple manner, the differential forms that correspond to the excess functions that were already known to De Donder-Weyl, on the one hand, and Carathéodory, on the other.

The principal problem in the theory of geodesic fields consists in proving the existence of a field, relative to a form, that "incorporates" a given extremal. This problem was studied by Weyl ([3]) and Boerner ([4]) for the fields that related to the forms of De Donder and Carathéodory, respectively.

The method that was developed by Lepage permits us to establish the existence of Mayer fields for multiple integrals. We may then approach the problem of incorporation in a different spirit from that of the preceding articles.

2. Suppose that a problem in the calculus of variations has been posed relative to the *n*-uple integral:

$$I = \int_n \mathcal{F}(x^i, y^\alpha, y_i^\alpha)\, dx^1 \cdots dx^\mu \qquad \begin{cases} i = 1, \cdots, n, \\ \alpha = 1, \cdots, m. \end{cases} \tag{2.1}$$

Suppose that there locally exists an m-parameter family of extremals of class C_2 of the problem (2.1):

$$y^\alpha = y^\alpha(x^1, \ldots, x^n, \lambda^1, \ldots, \lambda^m), \tag{2.2}$$

such that:

$$\frac{\partial(y^1 \cdots y^m)}{\partial(\lambda^1 \cdots \lambda^m)} \neq 0. \tag{2.3}$$

We are now in a position to define a *velocity field* ([5]).

([*]) Presented by Th. De Donder.

([1]) HILBERT, D. *Gesammelte Abhandlungen*, ed. Springer (1935), vol. III, pp. 35-55.

([2]) LEPAGE, TH., "Sur les champs géodésiques du calcul des variations," Bull. de l'Acad. Roy. de Belg., Cl. de Sc. (1936), pp. 716-729 and 1036-1046.

([3]) WEYL, H., "Geodesic fields," Ann. of Math., **36** (1935), pp. 607-629.

([4]) BOERNER, H., "Über die Extremalen und geodätischen Felder in der Variationsrechnung der mehrfachen Integrale," Math. Annalen **112** (1936), pp. 187-220.

([5]) DE DONDER, TH., *Théorie invariantive du calcul des variations*, Gauthier-Villars, new ed., 1935), pp. 137.

Indeed, we may solve (2.2) with respect to the λ^β; we obtain m uniform functions of x^i, y^α :

$$\lambda^\beta = \bar{\lambda}^\beta(x, y)\,. \tag{2.4}$$

A velocity field will be defined by the mn functions of the x, y:

$$\bar{y}_i^\alpha(x, y) = \left(\frac{\partial y^\alpha}{\partial x^i}\right)_{\lambda^\beta = \bar{\lambda}^\beta(x,y)} \tag{2.5}$$

that one obtains by replacing the λ^β with their values (2.4) in the partial derivatives of the y^α with respect to the x^i that are given in (2.2).

3. Consider the differential form:

$$\Omega_n - \mathcal{F} dx^1 \cdots dx^m - (-1)^i \mathcal{F}_{y_i^\alpha} \omega^\alpha d(i) - (-1)^{i+j} A_{\alpha\beta}^{ij} \omega^\alpha \omega^\beta d(i, j)\,, \tag{3.1}$$

where the $A_{\alpha\beta}^{ij}$ are undetermined functions of the x, y ([6]) and we have set:

$$\omega^\alpha \equiv dy^\alpha - y_i^\alpha dx^i\,.$$

We propose to see whether it is possible to choose these functions in such a manner that the form Ω_n becomes an exact differential in the variables x, y when one replaces the arguments y_i^α that appear in $\mathcal{F}$ and $\mathcal{F}_{y_i^\alpha}$ with functions of the x, y that are defined by (2.5).

When one studies the question in its full generality one considers a differential form (3.12) that contains the terms:

$$A_{\alpha_1 \cdots \alpha_k}^{i_1 \cdots i_k} \omega^{\alpha_1} \cdots \omega^{\alpha_k} d(i_1 \cdots i_k)$$

with $k = 1, 2, \ldots, n$, but it suffices here to take $k = 2$ and annul all of the A with $k > 2$ identically.

We shall perform the calculations upon substituting the variables x, λ with the variables x, y using formula (2.4). It is then equivalent to demanding that (3.1) be an exact differential in x, λ when one substitutes the functions (2.2) for the arguments and the partial derivatives of these functions with respect to the x for the y_i^α.

The form Ω_n will be written in terms of the variables x, λ as:

([6]) Unless indicated to the contrary, $(\sum_i \sum_\beta \cdots)$, the summations of the type $A_{\alpha\beta}^{ij} a_{ij}^{\alpha\beta}$ are extended over all simple combinations of pairs of m letters and n letters taken from $\alpha\beta$ and ij, respectively.

$$[\Omega_n] = B\, dx^1 \,\dots\, dx^n - (-1)^i\, B^i_\alpha\, d\lambda^\alpha\, d(i) - (-1)^{i+j}\, B^{ij}_{\alpha\beta}\; dx^\alpha\, dx^\beta\, d(i, j), \tag{3.2}$$

where:

$$B \equiv [\mathcal{F}], \tag{a_1}$$

$$B^i_\alpha \equiv [\mathcal{F}y^\alpha_i]\frac{\partial y^\gamma}{\partial \lambda^\alpha}, \tag{a_2}$$

$$B^{ij}_{\alpha\beta} \equiv [A^{ij}_{\gamma\delta}]\frac{\partial(y^\gamma, y^\delta)}{\partial(\lambda^\alpha, \lambda^\beta)}, \tag{a_3}$$

the symbols between brackets being the functions of the x, λ that one obtains by the substitutions that we spoke of.

The equations of the problem:

$$d[\Omega_n] = 0, \tag{3.3}$$

is equivalent to the following system:

$$\frac{\partial B}{\partial \lambda^\alpha} = \frac{\partial B^i_\alpha}{\partial x^i}, \tag{α_1}$$

$$\frac{\partial B^{ij}_{\alpha\beta}}{\partial x^j} = \frac{\partial B^i_\alpha}{\partial \lambda^\beta} - \frac{\partial B^i_\beta}{\partial \lambda^\alpha}, \tag{α_2}$$

$$\frac{\partial B^{ij}_{\alpha\beta}}{\partial \lambda^\gamma} + \frac{\partial B^{ij}_{\beta\gamma}}{\partial \lambda^\alpha} + \frac{\partial B^{ij}_{\gamma\alpha}}{\partial \lambda^\beta} = 0. \tag{α_3}$$

We now study this system of partial differential equations in the unknown functions $B^{ij}_{\alpha\beta}$. The equations (α_1) are satisfied identically; indeed, they express that the functions (2.2) are extremals.

Let $(\lambda^\alpha, \lambda^\beta)$ denote the right-hand sides of the equations (α_2); they are unknown functions of x, λ:

$$(\lambda^\alpha, \lambda^\beta)_i = \frac{\partial[\mathcal{F}y^\gamma_i]}{\partial \lambda^\alpha}\frac{\partial y^\gamma}{\partial \lambda^\beta} - \frac{\partial[\mathcal{F}y^\gamma_i]}{\partial \lambda^\beta}\frac{\partial y^\gamma}{\partial \lambda^\alpha}. \tag{3.4}$$

By virtue of the identities (α_1), they enjoy the following property:

$$\sum_i \frac{\partial(\lambda^\alpha, \lambda^\beta)}{\partial x^i} = 0, \tag{3.5}$$

a property that will permit us to determine solutions to the system (α_2), (α_3) by quadratures. It suffices to take:

$$B^{ij}_{\alpha\beta} = \frac{1}{n}\left\{\int (\lambda^\alpha, \lambda^\beta)_j\, dx^i - \int (\lambda^\alpha, \lambda^\beta)_i\, dx^j\right\}. \tag{3.6}$$

One verifies immediately, upon taking (3.5) into account, that these are indeed solutions.

To find the corresponding values of the $[A_{ij}^{\alpha\beta}]$, one must then solve the Cramer system (α_3). It ultimately suffices to replace the λ in the result with their values (2.4) in order to obtain the A as functions of the x, y and thus achieve the determination of a form Ω_n that becomes an exact differential when one substitutes the velocity field (2.5) for the y_i^α. One then says that the field (2.5) is *geodesic* for the corresponding differential form and that the extremal field (2.2) is a *Mayer field.* We thus see that *being given a family of extremals that uniformly covers a certain region of space, it is always possible to determine a form for which it constitutes a Mayer field for a multiple integral.*

REMARKS:

1. In general, a given family does not constitute a Mayer field for a given form (when the A are given), in particular, for the De Donder-Weyl form, where all of the A are identically null, or for the Carathéodory form:

$$A_{\alpha\beta}^{ij} = \frac{1}{\mathcal{F}}\begin{vmatrix} \mathcal{F}y_i^\alpha & \mathcal{F}y_i^\beta \\ \mathcal{F}y_j^\alpha & \mathcal{F}y_j^\beta \end{vmatrix},$$

where the relations (α_2) will not be satisfied.

2. If $m = 1$and $n > 1$, the functions A and the expressions (3.4) are identically null; in this case, it ensues that any one-parameter family of extremals that uniforms covers a certain region of space constitutes a Mayer field.

3. Contrary to what happens for multiples integrals, for simple integrals ($n = 1$ and $m > 1$) one must impose some conditions on the family (2.2). Since all of the A are null, it is necessary that the expressions (3.4):

$$(\lambda^\alpha, \lambda^\beta) = \frac{\partial[\mathcal{F}y^\gamma]}{\partial\lambda^\alpha}\frac{\partial y^\gamma}{\partial\lambda^\beta} - \frac{\partial[\mathcal{F}y^\gamma]}{\partial\lambda^\beta}\frac{\partial y^\gamma}{\partial\lambda^\alpha} \tag{3.7}$$

are identically null. This is the well-known condition that one imposes on the "Lagrange brackets" (3.7). Observe further that the property (3.5) expresses here the property that the Lagrange brackets must preserve a constant value along any extremal.

4. Instead of starting with a given form, as Weyl and Boerner did, and searching for a geodesic form that "incorporates" a given extremal, it suffices for us to possess a family of extremals such that (2.2) includes it.

Indeed, let there be an extremal with equations:

$$\overset{0}{y}{}^\alpha = \overset{0}{y}{}^\alpha(x^1, \cdots, x^n) \tag{4.1}$$

and a family (2.2) that includes (4.1) for $\lambda = \overset{0}{\lambda}$:

$$\overset{0}{y}{}^{\alpha} = y^{\alpha}(x^1, \cdots, x^n, \overset{0}{\lambda}{}^i, \cdots, \overset{0}{\lambda}{}^n) . \tag{4.2}$$

The extremal (4.2) will then be "incorporated" into the velocity field (2.5); indeed, one has identically in x:

$$\bar{y}_i^{\alpha}(x, \overset{0}{y}) \equiv \frac{\partial \overset{0}{y}{}^{\alpha}}{\partial x^i} ,$$

and furthermore we know that we may always find a form for which (2.5) is geodesic.

Remark 3.1 shows us that the velocity field (2.5) will not be, in general, geodesic in the sense of either De Donder or Carathéodory.

5. Suppose that, upon starting with the extremal (4.1), we have constructed a family (2.2) and a form for which (2.5) is geodesic. We are then in possession of an independent integral, and we may easily find ([7]) the value of the corresponding excess function:

$$\mathcal{S}(t, y, y_i^{\alpha}, \bar{y}_i^{\alpha}) = \mathcal{F} - \bar{\mathcal{F}} - \bar{\mathcal{F}}_{y_i^{\alpha}}(y_i^{\alpha} - \bar{y}_i^{\alpha}) - A_{\alpha\beta}^{ij} \begin{vmatrix} y_i^{\alpha} - \bar{y}_i^{\alpha} & y_i^{\beta} - \bar{y}_i^{\beta} \\ y_j^{\alpha} - \bar{y}_j^{\alpha} & y_j^{\beta} - \bar{y}_j^{\beta} \end{vmatrix}, \tag{5.1}$$

in which the barred symbols indicate that we have replaced the y_i^{α} with the $\bar{y}_i^{\alpha}$ in the field (2.5).

We pass over the actual statement of the sufficient condition for a strong minimum that is given by the function $\mathcal{S}$ to conclude by giving the value of the quadratic form that one obtains upon limiting the development of the function $\mathcal{S}$ in powers of:

$$u_i^{\alpha} \equiv y_i^{\alpha} - \bar{y}_i^{\alpha}$$

to the quadratic terms.

Calculated on the extremal:

$$y^{\alpha} = \overset{0}{y}{}^{\alpha} \qquad \text{so} \qquad \bar{y}_i^{\alpha} \equiv \frac{\partial \overset{0}{y}{}^{\alpha}}{\partial x^i} ,$$

it has the following value:

$$2Q_0 = \sum_{\alpha} \sum_{\beta} \sum_{i} \sum_{j} (\overset{0}{\mathcal{F}}_{y_i^{\alpha} y_j^{\beta}} - \overset{0}{A}{}_{\alpha\beta}^{ij}) u_i^{\alpha} u_j^{\beta} . \tag{5.2}$$

([7]) LEPAGE, TH., "Sur les champs géodésiques du calcul des variations," Bull. de l'Acad. roy. de Belgique, Cl. des Sc. **9** (1936), pp. 1039.

Observe that the $\overset{0}{A}$ are obtained simply by making $\lambda = \overset{0}{\lambda}$ in the result of the solution to the system (α_3).

It suffices that this quadratic form be positive-definite for a given extremal to realize a weak minimum.

The infinitesimal contact transformations of the variational calculus. [1])

By ERNST HÖLDER in Leipzig

With 1 figure

1. In the following, I would like speak on the implications that the concept of a one-parameter group of contact transformations, as well as their infinitesimal transformations, has in the calculus of variations – and also for the multiple extremal integrals with many desired functions. For one-dimensional extremal integrals, the relation to the geometry of contact transformations – which is already implicit in Hamilton's [2]) optical works – are well-known, if they are, however, perhaps not always sufficiently discussed in the textbooks.

Lie [3]), without referring to Hamilton, has stated several times that the simplest example of a one-parameter group of contact transformations was given by the wave motions, and that the group property of all dilatations was intimately connected with Huygens's principle. In a similar way, the images of a surface under an *arbitrary one-parameter group of contact transformations* can be regarded as originating in a wave process in a permanent regime that satisfies Huygens's principle of ray optics. An initial wave surface Σ_0, which after a time Θ becomes a certain wave surface $\Sigma = T_\Theta \Sigma_0$ (by means of a contact transformation) has, at the time $\Theta + \Theta'$, the position $T_{\Theta+\Theta'} \Sigma_0 = T_{\Theta'} \Sigma$, which originates from the new initial location Σ after the time Θ': $T_{\Theta+\Theta'} = T_\Theta T_{\Theta'}$; the time Θ is the canonical parameter.

The *partial differential equation* of first order for the wave process is obtained from the assumption that the infinitesimal contact transformation, by way of its Lie characteristic function, (essentially) gives the normal velocity of the wave for each direction of the wave normal at every point. If one goes a distance from the origin that is equal to the normal velocity at a certain point for variable normal direction, as well as the plane that it is normal to it, then this envelops a *point* structure: the *ray surface* at the point considered. From this, one obtains, by a similar reduction of 1 to $\delta\Theta$ in the time increment $\delta\Theta$, the "*elementary wave*" that is produced at each of the individual points of the surface elements and, as they vary, gives the envelope of the infinitesimally close wave surface. With this envelope construction (which is likewise also valid for finite contact transformations), one has outlined the scope of *Huygens's principle.*

---

[1]) A somewhat extended version of a presentation that was given to the Baden-Baden meeting of the D. M. V. (Sept., 1938).

[2]) W. R. Hamilton, Third Supplement to an Essay on the Theory of Systems of Rays (1832). In particular, articles 2, 26, Math. Papers I, Cambridge 1931 (on this, cf., also the remarks of the eds. A. W. Conway and J. L. Synge, pp. XXI, 189), as well in the survey edited by G. Prange: Über W. R. Hamiltons Abhandlungen zur Strahlenoptik, Leipzig 1933, as well as the footnote on this, in particular, pp. 168, et seq. cf., Prange, Nova Acta (?) Acad. (?) (1923), No. 1. Enzykl. d. math. Wiss. IV, 12 and 13, No. 13.

[3]) Cf., Lie and Scheffers, Geometrie der Berührungstransformationen, Bd. I, Leipzig 1896, pp. 966. (?), as well as Lie, Die infinitesimalen Berührungstransformationen der Optik, Ges. Abh., Bd. 6, pp. 615-617.

By means of this wave picture, the notion of a one-parameter group of transformations resolves to a "particle picture." This double aspect represents, in Hamilton's theory, a bridge across the dualism of Huygens's wave theory and Newton's emission theory that led Hamilton to make the transition from applying his method to optics to applying it to mechanics, and which was the stimulus for Schrödinger [4]) a hundred years later that led up to the new physical synthesis of wave mechanics.

In this particle picture one focuses on the *paths* of the individual surface elements under the transformations of the group, which are the *rays*, optically speaking. They lead from the contact point of the elementary wave to the envelope and are given by certain *ordinary differential equations* whose right-hand side is derived from Lie's characteristic function of the infinitesimal contact transformation.

There now exists the fundamental connection that the paths of the group are, at the same time, *extremals* (minimals) of a variational problem – the one in which the indicatrix is given by the ray surface: The rays satisfy *Fermat's principle* of shortest time. Correspondingly, in mechanics the paths satisfy the principle of least action (in the Jacobi form) when the energy constant is fixed.

I would like to briefly derive this connection anew on the basis of the very penetrating examination of Vessiot [5]) (which is independent of the optical aspects), simply from Lie's notion of a one-parameter group of contact transformations. Thus, I will use the inhomogeneous formulation by singling out an axis, as opposed to the most commonly used homogeneous representation that is often suitable in the beginning – particularly, when one goes to the multi-dimensional variational calculus.

By singling out a t-axis, we thus consider transformations of a space of coordinates (t, x^i) that take the *surface element* (t, x^i, P^i) to another surface element, and that take an n-dimensional *union* of surface elements $dt + P_i\,dx_i = 0$ into another such union. The position coordinates P_i are thus $-\partial t / \partial x_i = P_i$.

We now treat a *one-parameter group* $\mathfrak{G}$ *of contact transformations:*

$$(1)\qquad t' = g(t, x_j, P_j, \Theta),\qquad x_i' = g_i(t, x_j, P_j, \Theta),\qquad P_i' = h_i(t, x_j, P_j, \Theta),$$

$$\frac{\partial(t', x_i', P_i')}{\partial(t, x_j, P_j)} \neq 0.$$

This has the function $F(t, x_j, P_j) \neq 0$ as the *Lie characteristic function of the infinitesimal transformation;* it makes $F\ \delta\Theta$ the infinitesimal displacement of the surface element in

---

[4]) E. Schrödinger, Abhandlungen zur Wellenmechanik, 2nd ed., Leipzig 1928, pp. 489 et seq.

[5]) E. Vessiot. a) Sur l'interpretation des transformations de contact infinitésimales, Bull. Soc. math. de France **34** (1906), pp. 320-269. Vessiot also treated a time-varying medium, b) Essai sur la propagation par ondes, Annales de l'Éc. Normale sup. (3) **26** (1909), pp. 405-448. For the corresponding questions for the Lagrange problem, cf., Vessiot, c) Sur la théorie des multiplicités et le Calcul de Variations, Bull. Soc. math. de France **40** (1912), pp. 68 to 139; d) Sur la propagation par ondes et sur le problème de Mayer, Journal de Math. (6) **9** (1913), pp. 39-76.

Further representations are given for the case of an n-dimensional ray surface by T. Levi-Civita and U. Amaldi, Lezzioni di Meccanica razionale II, pp. 456-469 (Bologna 1927), L. P. Eisenhart, Continuous groups of tansfromations (Princeton 1933), p. 263-273 and G. Juvet, Mécanique analytique et mécanique ondulatoire, Mémorial Sci. Math. Fasc. **83** (Paris 1937).

the direction of the t-axis if Θ is the canonical parameter of the group. The paths of the group:

(2) $$t = g(t^0, x_j^0, P_j^0, \Theta), \quad x_i = g_i(t^0, x_j^0, P_j^0, \Theta), \quad P_i = h_i(t^0, x_j^0, P_j^0, \Theta),$$

obey the differential relations [6]):

(3) $$dt + P_i\, dx_i = F\, d\Theta + G_h\, dc_h, \qquad \frac{\partial G_h}{\partial \Theta} = F_t \cdot G_h,$$

in which c_h means an arbitrary parameter upon which the initial values t^0, x_j^0, P_j^0 depend; perhaps one can set $c_h = x_j^0$ and fix x_j^0 and P_j^0. Just like t, x_i, P_i, F and G_h then depend upon Θ, c_1, c_2, … One then has:

(4) $$\frac{\partial F}{\partial \Theta} = F_t \cdot F.$$

Conversely, a $2n$-parameter family:

(5) $$t = t(c_1, c_2, \ldots, c_{2n}; \Theta), \quad x_i = x_i(c_1, c_2, \ldots c_{2n}; \Theta), \quad P_i = P_i(c_1, c_2, \ldots c_{2n}; \Theta),$$

with:

(6) $$\frac{\partial(x_i, P_i)}{\partial(c_1, \cdots, c_{2n})} \neq 0$$

is characterized by the differential relation $(3)_1$, along with $(3)_2$, as the family of paths of a one-parameter group of contact transformations.

---

[6]) From the system of differential equations for the paths:

(3a) $$\begin{cases} \dfrac{dt}{d\Theta} = & F - P_i F_{P_i} & = -\Phi \\ \dfrac{dx_i}{d\Theta} = & F_{P_i} & = \Pi_i \\ \dfrac{dP_i}{d\Theta} = & -F_{x_i} + P_i F_t & \end{cases}$$

that are associated with the infinitesimal contact transformation and the canonical parameter Θ, it follows that there is agreement between the coefficients of $d\Theta$ on the left-hand and right-hand sides of the differential relation $(3)_1$, which thus defines the quantities G_h; in order to do this, one then calculates the derivative $(3)_2$.

Herglotz, in particular, treated the differential relations (3) in his seminar on continuum mechanics, Göttingen 1925/26. – There, one will also find the basic facts of ray optics derived from the second-order differential equations of continuum mechanics. He also treats the general case of variable regimes, which leads into the Mayer problem; cf., Vessiot, loc. cit. [5]) b) The specialization to permanent regimes produced the ordinary variational problem in homogeneous form. Heglotz has treated a one-parameter group of contact transformations in the plane in his seminar on differential equations, Göttingen Summer 1928, in which the paths were treated as extremals in a variational problem, and are denoted by the same independent variable x as the transversals in inhomogeneous form.

In order to go from the group of contact transformations to the associated family of canonical transformations, one writes:

$$\begin{aligned} \frac{P_i}{F} &= \pi_i \\ -\frac{1}{F} &= \varphi(t, x_j, \pi_j); \end{aligned} \tag{7}$$

from the first equations, under the assumption that $\Phi \neq 0$, the P_i may be represented as expressions in the new variables (impulses) π_i [7]), which will then be substituted in $-F^{-1}$. When one substitutes Θ for t by means of $(5)_1$ and substitutes in $(5)_{2,3}$, under the same assumption that $\Phi \neq 0$, formula (5) now gives the family:

$$x_i = \xi_i(c_1, \ldots, c_{2n}; t), \quad \pi_i = \eta_i(c_1, \ldots, c_{2n}; t), \tag{8}$$

with:

$$\frac{\partial(x_i, \pi_i)}{\partial(c_1, \cdots, c_{2n})} = \frac{\partial(x_i, \pi_i)}{\partial(x_j, P_j)} \frac{\partial(x_i, P_j)}{\partial(c_1, \cdots, c_{2n})} \neq 0, \tag{9}$$

for which, (3), after dividing by F, yields:

[7]) They are, in fact:

$$\frac{\partial \pi_i}{\partial P_j} = \frac{1}{F^2}(\delta_{ij} F - P_i \Pi_j), \quad \det\left(\frac{\partial \pi_i}{\partial P_j}\right) = \frac{F^{n-1}}{F^{2n}}(F - P_i \Pi_i) = -\frac{\Phi}{F^{n+1}}.$$

I then compute the differential:

$$d\varphi = \frac{1}{F^2}\left[F_t dt + F_{x_i} dx_i + F_{P_i}\left(F d\frac{P_i}{F} - F P_i d\frac{1}{F}\right)\right],$$

i.e.:

$$-\Phi\, d\varphi = (F - P_i \Pi_i)\, d\varphi = \frac{F_t}{F^{2n}} dt + \frac{F_{x_i}}{F} dx_i + \Pi_i d\pi_i .$$

Combining this with the Legendre transformation (13) gives:

$$p_i = -\frac{\Pi_i}{\Phi}, \qquad f = \frac{1}{F} - \frac{\Pi_i}{\Phi}\frac{P_i}{F} = -\frac{1}{\Phi},$$

hence, the birational involutory contact transformation $(F, P_i, \Pi_i) \rightarrow (f, p_i, \pi_i)$:

$$f = -\frac{1}{\Phi}, \qquad \varphi = -\frac{1}{F}, \qquad p_i = -\frac{\Pi_i}{\Phi}, \qquad \pi_i = \frac{P_i}{F}, \tag{7a}$$

that Haar presented (in another connection: Über adjungierte Variationsprobleme und adjungierte Extremalflächen. Math. Ann. **100** (1928), pp. 487 et seq.) and Carathéodory, loc. cit. [11]) d) pp. 194 et seq. has used in a definitive formulation of his generalized Legendre transformation; we shall discuss this in no. 2. The formulas with one independent variable that one subsequently needs are naturally much easier to prove.

$$(10)\qquad -\varphi\, dt + \pi_i\, dx_i = d\Theta + C_h\, dc_h\,, \qquad \frac{\partial C_h}{\partial t} = 0.$$

However, the differential relation (10) characterizes (8), with (9), as the family of solutions of the canonical system:

$$(11)\qquad \frac{dx_i}{dt} = \varphi_{\pi_i}\,, \qquad \frac{d\pi_i}{dt} = -\varphi_{x_i}$$

with the Hamilton function $\varphi(t, x_i, \pi_i)$.

With no further restrictions, the family of canonical transformations is then also given by:

$$x_i = x_i(x_j^0, \pi_j^0; t)\,, \qquad \pi_i = \pi_i(x_j^0, \pi_j^0; t)\,.$$

With this, we have the bridge to the variational problem:

$$(12)\qquad \int f\, dt = \min \quad \text{for the curve } x_i = x_i(t)$$

(for given endpoints), whose extremals are the paths of the group. Its basic Lagrange function $f(t, x_i, p_i)$, with $p_i = dx_i\,/dt$, goes over, in a well-known way, by using the Legendre transformation:

$$(13)\qquad p_i = \varphi_{\pi_i}\,, \qquad f = -\varphi + p_i\,\pi_i\,,$$

to the Hamilton function φ and thus to the Lie characteristic function F [8]).

The value of the extremal integral along a path segment is equal to the associated canonical parameter increment Θ.

Our representation allows us to immediately recognize that, conversely, the entire path of the variational problem can also be obtained from the family of canonical transformations as one runs through the one-parameter group of contact transformations. The transformation of the desired functions is now (under the assumption that $f \cdot \varphi \neq 0$), from (7):

$$(14)\qquad \begin{cases} P_i = -\dfrac{\pi_i}{\varphi} \\[2ex] F = -\dfrac{1}{\varphi}, \end{cases}$$

[8]) Here, we restrict ourselves the case in which the Hessian determinant satisfies:

$$(13a)\qquad \left|\varphi_{\pi_i \pi_j}\right| = \frac{F^{n+2}}{(-\Phi)^{n+2}} \left|F_{P_i P_j}\right| \neq 0.$$

In other cases, one is led to a Lagrange problem, cf., Vessiot, loc. cit. [5]), pp. 81, 107, as well as more recently in the textbook of Carathéodory, loc. cit. [11]), a), pp. 354 et seq., and also Boerner [11]), pp. 201, second formula from the top, where the first two factors on the right must be $(fF)^{2(n+\mu)-n\mu}$.

in which (similar to $(7)_1$ in rem. [7])), under the assumption that $f \neq 0$, the first equation $(14)_1$ can be solved for the π_i (on this, cf., also Carathéodory, loc. cit. [11]), pp. 358) and its expressions in t, x_i, P_i can be substituted in $(14)_2$; as the independent variable, one introduces $\int f\,dt = \Theta$ along the extremal. If (8) and the differential relations (10), as well as (9), are true for this situation then the differential relation (3), as well as (6), follows for (5), which characterizes (5) as the path of a one-parameter group of contact transformations.

In the variational calculus, one says that a surface element (t, x_i, P_i) intersects its path (with the line element (t, x_i, p_i)) *transversally*. With the addition of the impulse π_i the transversality is expressed by (7), ((14, resp.).

If one then takes an initial surface (union) M_n^0 and subjects it to the contact transformation T_Θ of the group $\mathfrak{G}$ then on any image surface (union) M_n the canonical parameter will, in a certain neighborhood, describe a function of position [9]):

$$\Theta = S(t, x_i). \tag{15}$$

The family of ∞^1 M_n: $S(t, x_i) = \Theta = \text{const.}$ is called a *geodetic field*; it intersects the paths *transversally* (and together with them defines a complete figure in the sense of Carathéodory).

For $S(t, x_i)$, one has the partial differential equation [10]):

---

[9]) This is true under the assumption that $F \neq 0$, which we have already made. $F = 0$ represents another first-order partial differential equation, namely:

$$F\left(t, x_i, -\frac{\partial t}{\partial x_i}\right) = 0,$$

for *one* surface $t = t(x_i)$ in the same t, x_i space by which it is determined that it includes the surface element with $F = 0$ that lies on an $n-1$-dimensional manifold. This surface has the property that its surface elements are displaced into *themselves* under the one-parameter group of contact transformations, so any surface element with $F = 0$ will be displaced to an infinitely close element that is *united* with it on the characteristic strip that is determined by the initial element. Cf., S. Lie, Ges. Abh. IV, pp. 287, as well as pp. 591; VI, pp. 636, as well as footnote pp. 905; furthermore, see the footnotes of Engels in Bd. III, pp. 615, and Theorie der Transformationsgruppen II, pp. 256 (Leipzig 1890). In recent representations, in the construction of the integral surface as the characteristic strip, it is mostly not emphasized that it can be described by a one-parameter group of contact transformations on the entire space of integral elements.

I remark that the paths that appear here (as anomalous line elements) are *boundary curves*, which can be either minima or maxima of the variational problem. Cf., Vessiot, loc. cit. [5]) c), pp. 69, as well as Carathéodory, loc. cit [11]) a), pp. 283.

Different formal considerations are presented for this case by M. Herzberger, Theory of transversal curves and the connections between the calculus of variations and the theory of partial differential equations. Proc. Nat. Acad. Sciences **24** (1938), pp. 466-473.

[10]) On M_n, one has:

$$S_t\,dt + S_{x_i}\,dx_i = 0, \qquad dt + P_i\,dx_i = 0. \tag{15a}$$

Furthermore, one has:

$$S(-\Phi) + S_{x_i}\Pi_i = 1. \tag{15b}$$

(16) $$S_t F = 1.$$

The equation:

(17) $$S_{x_i} = P_i\, S_t$$

then exhibits P_i as an expression in the derivatives of S. By means of (7), this also makes:

(18) $$S_t + \varphi = 0,$$

with:

(19) $$\pi_i = S_{x_i}\,,$$

which is the first-order differential equation of Hamilton-Jacobi.

The extremal integral over an *arbitrary* comparison curve that runs through the geodetic field is:

(20) $$\int f\, dt = \Theta + \int \mathcal{E}\, dt,$$

where Θ is the difference between the S-value at the endpoint of the arc and at the starting point. If the $\mathcal{E}$-function > 0 here then one obtains the minimizing property of the extremals (paths).

We have derived the complete connection between the one-parameter group of contact transformations and the variational problem in a somewhat different manner from that of Vessiot, and in the (inhomogeneous) formulation throughout, which represents a one-dimensional case of the general formulas discussed by Carathéodory for multi-dimensional variational calculus. In the stated special case, we added the interpretation of Carathéodory's F as the Lie characteristic function.

In the new representation that Carathéodory [11]) gave in his textbook on the variational calculus, as well as in his Geometrische Optik, for the form of variational calculus – I am speaking, at the moment, of a line integral – will, in any case, from the outset, be regarded as a certain embodiment of both the principles of Fermat and Huygens; thus, the selfsame origin in the group viewpoint is not completely realized here. The representation – without the apparatus of the contact transformations – will therefore be briefly unsurpassed, and, what is extremely important beyond the didactic advantage, it is suitable for the generalization to multiple extremal integrals (with many unknown functions) that Carathéodory has based his theory on.

2. If we now consider a variational problem for a multiple integral:

(21) $$\int f\, dt_1 \ldots dt_\mu = \text{min.},$$

From (15a), one deduces (17), and then from (15b), by means of $(3a)_1$, also (16).

[11]) C. Carathéodory, a) Variationsrechnung und partielle Differentialgleichungen erster Ordnung. Leipzig 1935. b) Geometrische Optik, Erg. d. Math. IV_5. Berlin 1937. Cf., above all, also C. Carathéodory, c) Les transformations canoniques de glissment et leur application a l'optique géométriques, Rom. Linc. Rend. (6) **12** $(1930)_2$, pp. 353-360, in particular, pp. 357 et seq. Die mehrdimensionale Variationsrechnung bei mehrfacher Integralen, Acta Szeged **4** (1928-29), pp. 193-216. Cf., also the representation of H. Boerner, Über die Extremalen und geodätischen Felder in der Variationsrechnung der mehrfachen Integrale. Math. Annalen **112** (1936), pp. 187-220.

in order to define the basic function:

$$f = f(t_\alpha, x_i, p_{i\alpha})$$

for a μ-dimensional surface:

$$x_i = x_i(t_\alpha), \tag{22}$$

that lies in the space $R_{n+\mu}$ of the variables t_α, x_i ($\alpha = 1, \ldots, \mu$; $i = 1, \ldots, n$), while we define $p_{i\alpha} = \partial x_i / \partial t_\alpha$ to be its surface element. This is to be integrated over a region G_t in the t-space, and the comparison functions shall be given on the boundary of G_t. Let the desired extremal surface be $\mathcal{E}_\mu$: $x_i = x_i(t_\alpha)$.

Carathéodory now takes a family of n-dimensional surfaces that depend upon μ parameters $\Theta_1, \ldots, \Theta_\mu$ thus:

$$\infty^\mu M_n: \qquad S_\alpha(t_\beta, x_j) = \Theta_\alpha = \text{const.} \tag{23}$$

(which will then be the family of surfaces that are transversal to the geodetic field) and, with the help of the basic function f, converts to an *equivalent* $f - \Delta$, which is associated with the same extremal surface $\mathcal{E}_\mu$. Therefore, the integral over Δ must depend only upon the boundary of the comparison surface segment; Carathéodory defines Δ to be the determinant:

$$\Delta = \left|\frac{\partial S_\alpha}{\partial t_\beta}\right| = |\, S_{i\alpha} + S_{i\alpha}\, p_{i\beta} \,| = \Delta(t_\alpha, x_i, p_{i\alpha}), \tag{24}$$

$$S_{\alpha\beta} = S_{\alpha t_\beta}, \qquad S_{i\alpha} = S_{\alpha x_i}.$$

The family of M_n shall now be chosen in such a way that at *one* particular point (t_α, x_i) the difference $f - \Delta$, which is regarded as a function of the $p_{i\alpha}$, possesses a null:

$$f - \Delta \geq 0; \tag{25}$$

thus, the equality symbol shall obtain for a certain surface element (t_α, x_i, $p_{i\alpha}$), which will "*transversally intersect the geodetic family* (23) *at the point in question.*"

A family that is geodetic at *any* point of a certain region in the space $R_{n+\mu}$ is called a *geodetic field*. That is the fundamental notion that Carathéodory introduced. The family that is geodetic at one point is only an auxiliary construction that I introduce in order to later on realize the covariance of the notion of transversality simply and independently of the (yet to be constructed) geodetic field.

The analytical condition for the family (23) to be geodetic at a point is obtained by the same considerations that Carathéodory has applied to the geodetic field, if they indeed always relate to just *one* point. We write M_n in the form $t_\alpha = t_\alpha(x_i; \Theta_\beta)$ and set:

$$-\frac{\partial t_\alpha}{\partial x_i} = P_{i\alpha}, \qquad S_{\alpha i} = S_{\alpha\rho}\, P_{i\rho}, \tag{26}$$

in other words, such that it expresses, in the event that the family (23) in (t_α , x_i) is geodetic, the surface element (t_α , x_i , $P_{i\alpha}$) in terms of only the transversally intersecting (t_α , x_i , $p_{i\alpha}$) (in term of only t_α , x_i , $\pi_{i\alpha}$, resp., where $\pi_{i\alpha} = f_{p_{i\alpha}}$) [12]):

$$(27) \qquad P_{i\alpha} = \frac{\bar{a}_{\alpha\beta}}{a}\pi_{i\beta} .$$

In this condition for the μ-dimensional surface element (t_α , x_i , $p_{i\alpha}$) to be transversally intersected by the n-dimensional surface element (t_α , x_i , $P_{i\alpha}$), one similarly defines the Hamilton function to be:

$$(28) \qquad \varphi(t_\alpha , x_i , p_{i\alpha}) = - f + p_{i\alpha}\, \pi_{i\alpha} \quad \text{with } \pi_{i\alpha} = f_{p_{i\alpha}} ,$$

$$(29) \qquad - a_{\alpha\beta} = - f\, \delta_{\alpha\beta} + p_{i\alpha}\, \pi_{i\beta}, \qquad a = \det(a_{\alpha\beta})$$

and $\bar{a}_{\alpha\beta}$ is the algebraic complement of $a_{\alpha\beta}$ in ($a_{\alpha\beta}$).

Carathéodory then introduced a construction of F that is similar to the one-dimensional case, namely:

$$(30) \qquad F = \frac{f^{\mu-1}}{a},$$

which proves to be a function $F(t_\alpha , x_i , P_{i\alpha})$, when one expresses the $p_{i\alpha}$ in terms of the $\pi_{i\alpha}$, and these, in turn, in terms of the $P_{i\alpha}$, by means of the transversality condition (27).

One then has the further condition [12]):

$$(31) \qquad |\, S_{\alpha\beta} \,| \cdot F = 1$$

---

[12]) From the property (23), it follows for the minimum, with the introduction of the algebraic complement $\bar{c}_{\alpha\beta}$ to $c_{\alpha\beta} \equiv S_{\alpha\beta} - S_{\alpha i}\, p_{i\beta}$, that:

$$f = \Delta \equiv |\, c_{\alpha\beta} \,|, \qquad \pi_{i\alpha} \equiv f_{p_{i\alpha}} = \Delta_{p_{i\alpha}} ,$$

from which:

$$\pi_{i\alpha} = S_{\rho i}\; \bar{c}_{\rho\beta}$$

$$a_{\alpha\beta} = f\, \delta_{\alpha\beta} - p_{i\alpha}\, \pi_{i\beta} = c_{\rho\alpha}\bar{c}_{\rho\beta} - p_{i\alpha}S_{\rho i}\bar{c}_{\rho\beta} = S_{\alpha\beta}\bar{c}_{\rho\beta} .$$

If $P_{i\alpha}$ were introduced by way of $(26)_2$ then this would make:

$$\pi_{i\beta} = P_{i\alpha}\, a_{\alpha\beta} ;$$

moreover, one has $a = |\, S_{\alpha\beta} \,|\, f^{\mu-1}$, hence, with F, (30), one also has $|\, S_{\alpha\beta} \,|\, F = 1$. By generalizing the considerations that pertained to (14) one sees: Equations (27) are soluble in terms of the $\pi_{i\beta}$; the expressions for the $\pi_{i\beta}$ in terms of the t_α , x_i , $P_{i\alpha}$ will be substituted in (30) on the right. Cf., Boerner, loc. cit. [11]), pp. 200 et seq. We shall not require the detailed formulation of the Carathéodory transformation here.

for the geodetic field. Thus, by way of:

$$(32) \qquad S_{\alpha\rho} P_{i\rho} = S_{\alpha i} ,$$

the $P_{i\alpha}$ are expressed in terms of the partial derivatives $S_{\alpha\beta}$, $S_{\alpha i}$ of the S_α (under the assumption that $| S_{\alpha\beta} | \neq 0$) and substituted into F. One thus obtain *one* first-order partial differential equation for the S_α ; this characterizes the geodetic field [13]).

With the help of the geodetic field, Carathéodory presented the "Legendre condition" and the "Weierstrass $\mathcal{E}$-function" for multiple integrals; they do not appear as one would presume. The most important thing is the fact that the functions S_α of the geodetic field drop out: All that remains are the $p_{i\alpha}$ or the $P_{i\alpha}$. Nonetheless, it is important to the construction of the theory, as well as the establishment of a strong minimum (for a positive $\mathcal{E}$-function), for a given extremal $\mathcal{E}_\mu$ to be embedded in a geodetic field that transversally intersects it.

Before I go into that, I remark that above all the notion of the geodetic field, and likewise that of being transversally intersected, is originally defined by (25) in manner that is *independent of the choice of variables* – that is only meaningful relative to the extremal integral that was given *a priori*.

We transform this to new independent variables $\overline{t}_\alpha$, which are functions of t_α and x_i :

$$(33) \qquad \overline{t}_\alpha = T_\alpha(t_\beta, x_i) \quad \text{with} \quad | T_{\alpha\beta} | \neq 0, \qquad T_{\alpha\beta} = \frac{\partial T_\alpha}{\partial t_\beta};$$

the x_i will remain the same. This transformation is arranged such that a comparison surface (lying in the neighborhood of the extremal in question) $x_i = x_i(t_\alpha)$ *intersects* the μ-parameter family of n-dimensional coordinate manifolds $\tilde{t}_\alpha$ = const. in such a way that for the assembled function:

$$(34) \qquad \tilde{t}_\alpha = T_\alpha(t_\beta, x_j(t_\beta))$$

the functional determinant is:

$$(34)_1 \qquad d = \frac{d(\tilde{t}_\alpha)}{d(t_\beta)} = | d_{\alpha\beta} | > 0, \qquad d_{\alpha\beta} = \frac{\partial T_\alpha}{\partial t_\beta} + \frac{\partial T_\alpha}{\partial x_j} p_{j\beta} = \frac{d\tilde{t}_\alpha}{dt_\beta}.$$

If one solves (34) for t_β then one obtains the following for the surface:

---

[13]) In addition, the indicated consideration shows: To given numerical values $P_{i\alpha}$ one can always very easily determine a *geodetic family at the point* (t_α , x_i), which has an M_n that goes through (t_α , x_i) with precisely the position $P_{i\alpha}$: One chooses the $S_{\alpha\beta}$ at (t_α , x_i) arbitrarily, except that the determinant satisfies:

$$| S_{\alpha\beta} | = \frac{1}{F(t_\alpha, x_i, P_{i\alpha})},$$

and then determines the $S_{\alpha i}$ by means of (32) at the point (t_α , x_i). The functions $S_\alpha (t_\beta , x_i)$ must then have the computed first derivatives $S_{\alpha\beta}$, $S_{\alpha i}$ only at (t_α , x_i).

(35) $$x_i = \tilde{X}_i(\tilde{t}_\beta), \qquad \tilde{p}_{i\alpha} = \frac{d\tilde{t}_\alpha}{dt_\beta},$$

and from the identity $x_i = \tilde{X}_i(T_\alpha(t_\beta, x_j(t_\beta)))$, it further follows that:

(36) $$p_{i\alpha} = \tilde{p}_{i\gamma} d_{\gamma\alpha},$$

where $d_{\gamma\alpha}$ depends only upon t_β, x_j, $p_{i\alpha}$, such that, with the algebraic complement $\bar{d}_{\alpha\beta}$ in the determinant d, the new expressions:

(37) $$\tilde{p}_{i\beta} = \frac{\bar{d}_{\beta\alpha}}{d}\, p_{i\alpha}$$

will be expressed in terms of only t_β, x_j, $p_{i\alpha}$.

Moreover, it follows from the required invariance of the extremal integrals that were given *a priori*, i.e., from the demand that $f\, dt_1 \dots dt_\mu = \tilde{f}\, d\tilde{t}_1 \cdots d\tilde{t}_\mu$, which gives the transformation character of the basic function $f = \tilde{f} \cdot d$, hence:

(38) $$\tilde{f} = \tfrac{1}{d} f,$$

where on the right-hand side the t_β are expressed in terms of the $\tilde{t}_\alpha$ and the x_i, $p_{i\alpha}$ in terms of the $\tilde{t}_\alpha$, x_i, $\tilde{p}_{i\alpha}$ – simply by switching the roles of t_α and $\tilde{t}_\alpha$.

The "path of the independent integral" $\int d\Theta_1 \dots d\Theta_\mu$ also allows one to convert the $\tilde{t}_\alpha$, where the invariant integral is represented by (35). If the conversion equations for the family (23) read:

(39) $$\tilde{S}_\alpha(\tilde{t}_\beta, x_j) = \Theta_\alpha$$

then one will have:

(40) $$\int d\Theta_1 \dots d\Theta_\mu = \int \tilde{\Delta}\, d\tilde{t}_1 \cdots d\tilde{t}_\mu \quad \text{with} \quad \tilde{\Delta} = \left|\frac{d\tilde{S}_\alpha}{d\tilde{t}_\beta}\right| = \left|\tilde{S}_{\alpha\beta} + \tilde{S}_{\alpha i}\tilde{p}_{i\beta}\right|;$$

since, on the other hand, this integral is:

(41) $$\int d\Theta_1 \dots d\Theta_\mu = \int \Delta\, dt_1 \dots dt_\mu = \int \tilde{\Delta} \cdot \tfrac{1}{d}\, d\tilde{t}_1 \cdots d\tilde{t}_\mu,$$

one then has:

(42) $$\tilde{\Delta} = \tfrac{1}{d}\Delta,$$

which one can also verify directly quite easily.

Under the transition to the new variables, one also merely multiplies the left-hand side of the fundamental relation (25) by $1/d > 0$; one has:

$$\tilde{f} - \tilde{\Delta} = \tfrac{1}{d}(f - \Delta) \geq 0 \tag{43}$$

when and only when $f - \Delta \geq 0$ is true, resp.: The geodetic field retains the property that the same is true for a family that is geodetic at a point, and thus an n-dimensional surface element that is transversal to a μ-dimensional surface element also remains transversal after the coordinate transformation – relative to transformed basic function of our (*a priori* given) extremal integral. The position coordinates and the equations of the family of surfaces (23) are naturally to be converted, but the analytic relations (27), (30), (31), (32), which are obvious consequences of the fundamental inequality (25), are *covariant*: They have the old form with regard to the unconverted basic function $\tilde{f}$.

I further remark that in the recent work of Finsler and Cartan [14]) such invariance considerations are presented in terms of the geometry of a space whose metric is based on the multiple extremal integral (with only one unknown function).

3. All that remains is the problem of embedding a given extremal $\mathcal{E}_\mu$ (at least in the small) in a geodetic field that intersects it transversally. Boerner [15]) has given a construction in the spaces of Carathéodory's theory. In conclusion, I would like to show how, when one is given an infinitesimal contact transformation of a family of $n + 1$-dimensional manifolds – which must only be transversal to $\mathcal{E}_\mu$ – the production of a geodetic field that is transversal to $\mathcal{E}_\mu$ can lead back to the aforementioned construction of line integrals.

For the construction of a geodetic field, one must solve only *one* first-order partial differential equation (31) for *one* function S_1, in the event that $S_{\alpha'}$, $\alpha' = 2, \ldots, \mu$ is given arbitrarily; thus the $P_{i\alpha}$ in F are to be replaced with their expressions in terms of the first derivatives of the S_α that one computes from (32). However, it is, above all, necessary for the field to be transversal to the given extremals $\mathcal{E}_\mu$. Boerner [16]) thus takes the functions $S_{\alpha'}(t_\beta, x_j)$ in such a way that the:

$$\infty^{\mu-1}\, R_{n+1}: \qquad S_{\alpha'}(t_\beta, x_j) = \Theta_{\alpha'} = \text{const.} \tag{44}$$

is transversal to $\mathcal{E}_\mu$, i.e., it includes the transversal n-directions $\bar{P}_{i\alpha}$ that are transversal to the surface element $p_{i\alpha}$ of $\mathcal{E}_\mu$.

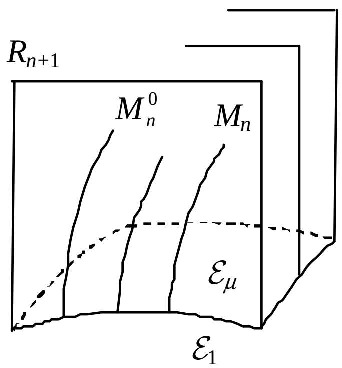
---

[14]) Cf., E. Cartan, Les espaces de Finsler, Actualités scient. et ind. no. 79. Paris 1934; Les espaces métriques fondés sur la notion d'aire, id. no. 72, Paris 1933.

[15]) H. Boerner, loc. cit. [11]), pp. 203-213. On the basis of another definition of the geodetic field, H. Weyl gave a field construction in: Geodesic fields in the calculus of variations for multiple integrals, Annals of Math. **36** (1935), pp. 607-629. Th.-H.-J. Lepage considered the two definitions within a unified viewpoint in: Sur les champs géodesiques du calcul des variations, Bull. Acad. Roy. Belg. (5) **22** (1936), pp. 716-729 and pp. 1036-1046. Boerner has recently explained how the Carathéodory theory is indicated within this general Ansatz (talk at the Marburger Colloquium, Feb., 1939).

[16]) Cf., loc. cit. [15]), pp. 209, footnote 23.

However, we immediately convert this R_{n+1} to (n+1-dimensional) *coordinate planes* $S_{\alpha'} \equiv t_{\alpha'} = \Theta_{\alpha'}$ by the introduction of new independent variables, which are again denoted by $t_{\alpha'}$ [17]); $t_1 = t$ can remain true [18]).



A family of ∞^1 M_n must be completely contained in each R_{n+1} for the n-dimensional surfaces M_n of the geodetic field to be constructed. An M_n therefore has the n-direction $P_{i1} = P_i$, $P_{i\alpha'} = 0$.

I now allow the M_n in R_{n+1} to go over to each other under a one-parameter group of contact transformations whose infinitesimal contact transformation has the following Lie characteristic function:

$$F^0(t, x_i, P_i) = F(t, \Theta_2, \ldots, \Theta_\mu, x_i, P_i, 0, \ldots, 0), \tag{45}$$

i.e., the Carathéodory function that is specialized to $t_{\alpha'} = \Theta_{\alpha'} =$ const., $P_{i\alpha'} = 0$. Thus, let a surface be chosen in each R_{n+1} to be the initial surface M_n^0, and which is transversal to the (one-dimensional) intersection curve $\mathcal{E}_1$ of $\mathcal{E}_\mu$ with R_{n+1} – relative to F^0 in R_{n+1}.

First, the totality of all ∞^μ M_n (in all R_{n+1}) defines a geodetic field in any case. If M_n has the canonical parameter $\Theta_1 = \Theta = S(t, x_i) = S_1(t, \Theta_2, \ldots, \Theta_\mu, x_i)$ under the group F^0 then the original partial differential equation (16) is valid for the function of position $S(t, x_i)$ on R_{n+1}, which is still independent of the parameter $\Theta_{\alpha'}$, only with F^0 instead of F, hence:

$$S_i F^0 = 1 \qquad \text{with} \qquad S_t P_i = S_{x_i}. \tag{46}$$

If one then again introduces the quantities with the indices $t = t_1$, $\Theta_{\alpha'} = t_{\alpha'}$, $S = S_1$, $P_i = P_{i1}$, and observes the special form of the $S_{\alpha'}$, by means of which (32) gives $0 = P_{i\alpha'}$, then

---

[17]) The fact that the properties of the transformation $\tilde{t}_1 = t_1$, $\tilde{t}_{\alpha'} = S_{\alpha'}(t_\beta, x_i)$ that were required above in (33) and $(34)_1$ are satisfied can be gathered from the previously-cited footnote in Boerner's work: Let $g_{11} \neq 0$ (possibly achieved by a suitable transformation that is produced at *one* starting point of the extremal $\bar{P}_{i\alpha} = 0$), and then assume that the $s_{\alpha'}(t_\beta)$ are independent of t_1.

[18]) One can seek to carry out the suitable transformation $x_i = \dot{x}_i(t) + \bar{x}_i$, $\pi_{i\alpha} = \dot{\pi}_{i\alpha}(t) + \bar{\pi}_{i\alpha}$ that brought about a great simplification in Weyl, loc. cit. [15]) in the spaces of Carathéodory's theory, as well – with the intention of applying the contact transformation with the necessary foresight to convert the new Lagrange function f^*, which *vanishes* along the initial extremal $x_i = \dot{x}_i(t_\beta)$, $\pi_{i\alpha} = \dot{\pi}_{i\alpha}(t_\beta)$, along with its first derivatives, and convert the likewise-obtained new Hamilton function φ^* into $\bar{f} = f^* + 1$, $\bar{\varphi} = \varphi^* - 1$.

One then arrives at certain surfaces $\sigma_\alpha(t, \bar{x}) =$ const., which do not, however, yield the transversal surfaces of the original problem in the original space simply by conversion; these are, moreover, other surfaces that are given by $S_\alpha(t_\beta, x_j) =$ const., where $S_\alpha(t_\beta, x_j) = S_\alpha(t_\beta, \dot{x}_j(t)) + \dot{\pi}_{i\alpha}(t)\bar{x}_i + \sigma_\alpha(t_\beta, \bar{x}_j)$. – Furthermore, it seems to me that in Weyl, pp. 621, one must add the (negative Hamilton function $H = -\varphi^*$ in formula (35) and the sum on the right-hand side of the foregoing one.

one recognizes, with no further assumptions [19]), that one can write the formula (46), just as well as the differential equation (31) in $R_{n+\mu}$.

Now, we still have to show that this geodetic field intersects the given extremal $\mathcal{E}_\mu$. As one then realizes, that already suffices in order to prove that all M_n in R_{n+1} are transversal to the intersection $\mathcal{E}_1$ (of $\mathcal{E}_\mu$ with R_{n+1}) – relative to F^0, which is therefore the curve $\mathcal{E}_1$ defined by the total evolution of a surface element of under the group F^0.

Above all, one has $\bar{P}_{i\alpha} = 0$ for the n-direction that is transversal to $\mathcal{E}_\mu$, since, by assumption, it indeed lies in a coordinate plane R_{n+1}. Hence, the system of equations (32) must be satisfied for $\alpha' = 2, \ldots, \mu$ by the special functions $S_{\alpha'} \equiv t_{\alpha'}$.

From (27), one then also concludes [20]) $\pi_{i\alpha} = 0$ on $\mathcal{E}_\mu$.

The Euler partial differential equations for $\mathcal{E}_\mu$, which are written canonically as:

$$(47) \qquad \frac{dx_i}{dt_\alpha} = \varphi_{\pi_{i\alpha}}, \qquad \frac{d\pi_{i\alpha}}{dt_\alpha} = -\varphi_{x_i}$$

with the Hamilton function (28), yield, since $\pi_{i\alpha'} = 0$, a canonical system ordinary differential equations with independent variables $t_1 = t$ for x_i, and the canonically conjugate impulse $\pi_{i1} = \pi_i$:

$$(48) \qquad \frac{dx_i}{dt} = \varphi^0_{\pi_i}, \qquad \frac{d\pi_i}{dt} = -\varphi^0_{x_i};$$

thus, the Hamilton function is:

$$(49) \qquad \varphi_0(t, x_i, \pi_i) = \varphi(t, \Theta_2, \ldots, \Theta_\mu, x_i, \pi_i, 0, \ldots, 0)$$

which the general Hamilton function (28), when specialized to $\pi_{i\alpha'} = 0$.

---

[19]) Under certain assumptions relative to F that guarantee the differentiability of S_1 with respect to the parameters $\Theta_{\alpha'}$.

[20]) From (29), (27), under the assumption that:

$$g_{\alpha\beta} = \delta_{\alpha\beta} + P_{i\alpha}\, p_{i\beta} = \frac{1}{a}(a\,\delta_{\alpha\beta} + \bar{a}_{\alpha\gamma}\pi_{i\gamma}\, p_{i\beta}) = \frac{1}{a}\bar{a}_{\alpha\beta}\, f ,$$

the relation follows:

$$f\, P_{i\alpha} = g_{\alpha\beta}\, \pi_{i\beta} .$$

For $P_{i\alpha'} = 0$, one has $g_{\alpha'\beta} = \delta_{\alpha'\beta}$ and $\pi_{i\alpha'} = 0$.

Moreover, from $\pi_{i\beta} = P_{i\alpha}\, a_{\rho\beta}$, one obtains:

$$\pi_{i1} = P_{i1}\, a_{11} = P_{i1}\,(f - p_{i1}\, \pi_{i1}) = P_{i1}(-\varphi).$$

Thus, one has:

$$-\varphi = \frac{1}{F} = \frac{a}{f^{\mu-1}} \qquad \text{since} \qquad a = |\,a_{\alpha\beta}\,| = \begin{vmatrix} -\varphi & 0 & \cdots & 0 \\ * & f & \cdots & 0 \\ * & 0 & \cdots & f \end{vmatrix}, \qquad \text{from (29).}$$

On the grounds of the formal remarks that were made in footnote [20]) (that for $P_{i\alpha'} = 0$, $\pi_{i\alpha'} = 0$, the Carathéodory formulas (27), (30) go over to the corresponding one-dimensional formula (14)), the Hamilton function φ^0 is associated with the Lie function $F^0 = -1/\varphi^0$, in the sense of the first section, i.e., $\mathcal{E}_1$ is a path, relative to F^0, and indeed consists of those surface elements of M_n^0 that, by construction, intersect $\mathcal{E}^1$ transversally.

Under the transformations of the group F^0 in R_{n+1}, the image of this initial element, which is displaced along $\mathcal{E}_1$, is always transversal to $\mathcal{E}_1$. The formula:

$$\pi_i = \frac{P_i}{F^0}, \tag{50}$$

which is valid on any $\mathcal{E}_1$ and expresses this transversality relative F^0 in R_{n+1}, is the full content of Carathéodory's transversality condition (27), since $\mathcal{E}_\mu$ possesses the location $\pi_{i1} = \pi_i$, $\pi_{i\alpha'} = 0$, and the M_n (lying in R_{n+1}) possess the location $P_{i1} = P_i$, $P_{i\alpha'} = 0$. From footnote [20]), due to the unique solubility of the same, the surface element $P_{i\alpha}$ is therefore transversal to the surface elements $p_{i\alpha}$ of $\mathcal{E}_\mu$ along $\mathcal{E}_\mu$ in the space of all variables $P_{i\alpha} = \bar{P}_{i\alpha}$.

The property of the μ-parameter family (M_n) that is thus proved in order to construct a geodetic field – viz., that the extremal $\mathcal{E}_\mu$ intersects it transversally – does not depend upon the variables used, but has an invariant meaning for the variational problem.

(Received on 2/5/39)

Calculus of variations from Stokes's theorem

By

Hermann Boerner in Munich
Presently with the Weather Service for the Reich

In the following, the most important formulas for the calculus of variations for simple integrals will be presented in a novel way, namely, as a consequence of the use of the *calculus of alternating differential forms.*

One of the results of this calculus, which represents, in a certain sense, the natural generalization of the integral calculus to several variables, is the fact that all of the general theorems of partial integration – the theorems of Gauss, Stokes, etc. – can be summarized in on simple formula:

$$\int \omega = \int d\omega,$$

where the left-hand side involves a differential form and the right-hand side, its differential, resp., which is integrated over a manifold and its boundary, resp.; this is the generalized "Stokes theorem."

Partial integration is used in the calculus of variation in various places: in the derivation of Euler's differential equation, in the Hilbert independent integral. These otherwise distinct topics – one of which is associated with the classical calculus of variations of the 18th Century, and the other, to the modern theory of Weierstrass – appear here, as it were, in a unified formalism. From it, one obtains a particularly simple derivation of the transversality condition, and Legendre's necessary condition may also be effortlessly arrived at.

These developments take on a special meaning by the fact that they make possible a much more convenient derivation for multiple integrals than the one that has usually been employed up to now. The Belgian *Lepage* first made use of this calculus in order to present a field theory with the aid of the Hilbert integral that summarized the various theories that existed up to that point in time. In a work that appeared at the same time as the present one on multiple integrals [1]), in which I had hoped to further clarify a mystery, I also set down on paper the consequences of the use of this calculus in all aspects of the calculus of variations there.

It seems expedient to me to first present a naturally complicated treatment of this relatively straightforward topic, in which precisely everything appears that will later be generalized. However, it might also be of interest to the reader for its own sake, and be useful in giving those who are not familiar with the domain a quick orientation; therefore, nothing will be assumed.

---

[1]) H. Boerner, Über die Legendresche Bedingung und die Feltheorien in der Variationsrechnung der mehrfacher Integrale, Math. Zeit. **46** (1940), pp. 720-742.

Since the calculus of alternating differential forms is not yet widely known, in the first section I have summarized everything that we will be busy with later [2]).

1. Rules of calculation for differential forms. The summation sign will be omitted. Simple sums, e.g., over i, go from 1 to n. For a double sum over i, j, the indices i and j range from 1 to n independently of each other; if anything else is summed over then the summation sign will be used: e.g., $\sum\limits_{i<j}$.

For *differential forms* $a_i\, dy_i$, etc., whose coefficients are functions of the spatial coordinates y_i, the coefficients shall be skew-symmetric and the differentials anticommutative:

$$dy_i\, dy_j = -\, dy_j\, dy_i\,.$$

For example:

$$\omega = \sum_{i<j} a_{ij}\, dy_i\, dy_j = \tfrac{1}{2} a_{ij}\, dy_i\, dy_j\,. \tag{1.1}$$

is a differential form of degree two; here, one has $a_{ij} = -\, a_{ji}$. The coefficients ("scalars") are interchangeable with the differential forms. Forms of degree one are called *Pfaff forms.*

The *product* of two forms will be defined by juxtaposition, and is independent of the order of the factors, in general.

The *differential* of a scalar a is the form:

$$da = \frac{\partial a}{\partial y_i} dy_i\,.$$

In order to define the differential of any form, one writes the coefficients to the left and puts a d in front of them. E.g., one has:

$$d(a_i\, dy_i) = da_i\, dy_i = \frac{\partial a_i}{\partial y_j} dy_j dy_i = \sum_{i<j}\left(\frac{\partial a_i}{\partial y_j} - \frac{\partial a_j}{\partial y_i}\right) dy_i dy_j\,.$$

The *integration* of a Pfaff form over a curve is well known. In order to integrate the form (1.1) of degree two over a two-dimensional piece $\mathfrak{F}$ of a surface, one relates $\mathfrak{F}$ (possibly piecewise) to the parameters u, v (in those terms, $\mathfrak{F}$ will be *oriented* as follows: the positive sense of traversal goes from the direction of u to v); one then has:

$$\int_{\mathfrak{F}} \omega = \int_{\mathfrak{F}} \sum_{i<j} a_{ij} dy_i dy_j = \int \sum_{i<j} a_{ij} \frac{\partial(y_i, y_j)}{\partial(u,v)} du\, dv\,, \qquad [3])$$

---

[2]) One finds a thorough presentation and proofs in E. Goursat, Leçons sur le probleme de Pfaff, Paris 1922, Chap. III. Incidentally, on pp. 21, et seq., of this book one already finds a hint of the calculus of variations, but only in a particular, and basically trivial, variational problem.

where the right-hand side is an ordinary integral over the region in the (u, v)-plane that corresponds to $\mathfrak{F}$. Thus, the value of the integral is independent of the choice of parameters; its sign depends upon the orientation.

One has *Stokes's theorem*, which we will need only for a Pfaff form ω:

$$\int_{\mathfrak{R}} \omega = \int_{\mathfrak{F}} d\omega ;$$

here, $\mathfrak{R}$ means the boundary of the two-dimensional surface piece $\mathfrak{F}$, when traversed in the positive sense. The boundary curve may therefore have corners and the surface piece may have kinks; in order to avoid such jumps, we assume in what follows that all tangents and tangent planes exist and are continuous.

We would ultimately like to make use of the well-known theorem that the integral of a *complete differential*, i.e., a form ω such that $d\omega = 0$, is "path-independent," i.e., (in the case of degree one) it only depends upon the starting point and the end point of the integration curve. Indeed, this follows from Stokes's theorem, or the even simpler well-known fact that a complete differential (of degree one) is the differential of a scalar. For higher dimensions, corresponding statements are valid.

2. The Euler equations. One deals with the integral:

(2.1) $$J_C = \int f(t, x_i, \dot{x}_i)\, dt$$

over a curve C: $x_i(t)$ in the $(n + 1)$-dimensional (t, x_i)-space $\mathfrak{R}_{n+1}$; the integral shall, perhaps, give a minimum for given endpoints P_1, P_2 of C. The basic function $f(t, x_i, p_i)$ shall be continuous, at least in all of the derivatives that appear in what follows, which are essentially the first derivatives of f and f_{p_i}.

We consider the form:

$$\omega = f(t, x_i, p_i)\, dt$$

in the $(2n + 1)$-dimensional (t, x_i, p_i)-space $\mathfrak{R}_{2n+1}$, and along with it, the n Pfaff forms:

$$\omega_i = dx_i - p_i\, dt .$$

Instead of (2.1), we take the integral $\int \omega$ in $\mathfrak{R}_{2n+1}$ to be extended over an integral curve of the "Pfaff system" $\omega_i = 0$, which one obtains when one takes any function of t for x_i and sets $p_i(t) = \dot{x}_i$. Indeed, one always has:

---

[3]) Due to the summation convention, one can also write the right-hand side as $\int a_{ij} \frac{\partial y_i}{\partial u} \frac{\partial y_j}{\partial v} du\, dv$; one observes that from (1.1), one thus has:

$$a_{ij}\, dy_i\, dy_j = 2a_{ij} \frac{\partial y_i}{\partial u} \frac{\partial y_j}{\partial v} du\, dv,$$

with the factor 2.

$$J_C = \int_{\mathfrak{C}} \omega ,$$

if $\mathfrak{C}$ is the curve in $\mathfrak{R}_{2n+1}$ that corresponds to C.

In the classical calculus of variations one considers a curve C that solves the problem and a nearby comparison curve $\bar{C} : \bar{x}_i(t)$, and embeds them both in a one-parameter family C_θ: $x_i(\theta, t)$, that makes, say, C correspond to $\theta = 0$ and $\bar{C}$ to $\theta = \varepsilon$. In $\mathfrak{R}_{2n+1}$, this gives us a two-dimensional surface piece:

$$\mathfrak{F};\ x_i = x_i(\theta, t), \qquad p_i = \frac{\partial x_i(\vartheta, t)}{\partial t},$$

that is referred to the parameters θ, t, and is therefore oriented. Its boundary consists of the curves $\mathfrak{C}$ and $\bar{\mathfrak{C}}$ that correspond to C and $\bar{C}$, and two further curve segments (corresponding to the end values $t = t_1$ and $t = t_2$), along which, one has:

$$dt = 0 \quad \text{and} \quad \omega_i = 0. \tag{2.2}$$

With consideration for the orientation, and due to (2.2), Stokes's theorem gives:

$$\int_{\bar{\mathfrak{C}}} \omega - \int_{\mathfrak{C}} \omega = \operatorname{sgn} \varepsilon \int_{\mathfrak{F}} d\omega . \tag{2.3}$$

If a curve C solves the problem then this difference must be positive for an arbitrary $\bar{C}$, and, in particular, for an arbitrary sign of ε, as long as $\bar{C}$ is sufficiently close to C; i.e., ε is sufficiently small.

Obviously, one does not change the problem and its solution when one replaces ω with another form Ω with $\Omega \equiv \omega (\text{mod } \omega_i)$ in all of the computations, i.e.:

$$\Omega = \omega + A_i(t, x_j, p_j)\ \omega_i .$$

For the calculations in terms of the parameters θ, t, one has:

$$\omega_i = \frac{\partial x_i}{\partial \vartheta} d\vartheta . \text{ [4]} \tag{2.4}$$

It will therefore be convenient to choose A_i in such a manner that:

$$d\Omega \equiv 0\ (\omega_i) .$$

For this, one must set $A_i = f_{p_i}$; one will then have:

---

[4]) Hence, $\omega_i = \delta x_i$ along the curve C ($\theta = 0$) in the classical variational calculus.

(2.5) $$\Omega = f\,dt + f_{p_i}\omega_i = (f - p_i\, f_{p_i})\,dt + f_{p_i}\,dx_i$$

and:

$$d\Omega = (df_{p_i} - f_{x_i}\,dt)\omega_i\,.$$

Now we can compute the "variation" (2.3); due to (2.2), one gets:

$$\begin{aligned} J_{\bar{C}} - J_C &= \int_{\bar{\mathfrak{C}}}\omega - \int_{\mathfrak{C}}\omega = \int_{\bar{\mathfrak{C}}}\Omega - \int_{\mathfrak{C}}\Omega = \operatorname{sgn}\varepsilon \int_{\mathfrak{F}} d\Omega \\ &= \operatorname{sgn}\varepsilon \int_{\mathfrak{F}}\left(df_{p_i} - f_{x_i}\,dt\right)\omega_i = \operatorname{sgn}\varepsilon \int\left(f_{x_i} - \frac{df_{p_i}}{dt}\right)\frac{\partial x_i}{\partial\vartheta}\,d\vartheta dt \\ &= \int_0^{\varepsilon} d\vartheta\left\{\int_{t_1}^{t_2}\left(f_{x_i} - \frac{df_{p_i}}{dt}\right)\frac{\partial x_i}{\partial\vartheta}\,dt\right\}\,. \end{aligned}$$

Here, *d/dt* denotes the partial derivative with respect to the *parameter t*, or, what amounts to the same thing, the derivative along the family of curves.

For sufficiently small ε, the last expression has the same sign as the contents of the curly brackets when $\theta = 0$, as long as ε is positive; for negative ε, it has the opposite sign. Thus, a minimum can occur when these quantities vanish for arbitrary variations δx_i, from which one concludes the existence of the *Euler equations*:

(2.6) $$f_{x_i} - \frac{df_{p_i}}{dt} = 0$$

in a well-known way. The solutions of these differential equations are called *extremals* [5]).

3. Transversality. It is possible that the starting point P_1 is not fixed, but moves on an arbitrary manifold. The starting point of our family $C_\theta : x_i(\theta, t)$ of comparison curves then lies on an oriented curve:

$$S\colon t = t(\theta), \qquad x_i = x_i\,(\theta) = x_i(\theta, t(\theta)).$$

The equations of the previous section are then modified as follows:

$$\int_{\bar{\mathfrak{C}}}\Omega - \int_{\mathfrak{C}}\Omega + \operatorname{sgn}\varepsilon\int_{\mathfrak{S}}\Omega = \operatorname{sgn}\varepsilon\int_{\mathfrak{F}} d\Omega\,.$$

Now, one has:

$$\int_{\mathfrak{S}}\Omega = \operatorname{sgn}\varepsilon\int_0^{\varepsilon}\left\{(f - p_i\, f_{p_i})t' + f_{p_i}x_i'\right\}d\vartheta\,,$$

and for a small ε this has the sign of:

[5]) This is the usual terminology. *Carathéodory* defined extremals otherwise; see Variationsrechnung und partielle Differentialgleichungen erster Ordnung. Leipzig-Berlin 1935, pp. 190 et seq.

$$(f - p_i f_{p_i}) t'(0) + f_{p_i} x_i'(0).$$

One next concludes the existence of the Euler equations, since families of curves with fixed endpoints, hence $t' = x_i' = 0$, are indeed also permissible, and then the relation:

$$(f - p_i f_{p_i}) t'(0) + f_{p_i} x_i'(0) = 0,$$

because otherwise the new terms that appear would change sign with ε for small ε. Since $t'(0)$, $x_i'(0)$ are the components of an arbitrary tangent vector to the manifold on which the starting point must lie, the solution curve C must satisfy the following *transversality condition*: The vector with the components:

$$f - p_i f_{p_i}, \qquad f_{p_i} \tag{3.1}$$

must be perpendicular to its starting point on this manifold. One then says: the curve will cut the manifold transversally.

4. Legendre's necessary condition [6]). We consider a line element (t^0, x_i^0, p_i^0) of a curve C, in whose neighborhood the tangent to the curve is continuous (i.e., C shall have no corners at the location t^0). For this line element, let the quadratic form:

$$f_{p_i p_j} u_i u_j \tag{4.1}$$

be either positive definite or positive semi-definite; there are thus numbers ρ_i with the square-sum 1, such that:

$$f_{p_i p_j} \rho_i \rho_j = -k < 0 .$$

We then construct a comparison curve $\bar{C}$ that agrees with C for $| t - t^0 | \geq \tau$, and for $| t - t^0 | \leq \tau$ it is described by:

$$\bar{x}_i(t) = x_i(t) + \varepsilon \rho_i(\tau \pm t \mp t^0),$$

in which, as in the following, the upper sign is to be taken for $t \leq t^0$ and the lower one for $t \geq t^0$.

We embed it in the family:

$$C_\theta : x_i(\theta, t) = x_i(t) + \theta \rho_i(\tau \pm t \mp t^0), \qquad 0 \leq \theta \leq \varepsilon;$$

hence:

$$p_i(q, t) = p_i(t) \pm \theta \rho_i .$$

---

[6]) The following proof is only a minor modification of the proof that one finds in Carathéodory's book that was cited in [6]) on pp. 193, et seq., which is shortened somewhat by the application of our method.

Let $\varepsilon > 0$ and $\tau_0 > 0$ be chosen so small that for all of the values of the arguments of f for the C_θ in the interval $|t - t^0| \leq \tau < \tau_0$, the subsequent derivatives lie under a fixed limit and:

$$f_{p_i p_j} \rho_i \rho_j \leq -\frac{k}{2}.$$

Let the surface piece that is determined by the family $\mathfrak{C}_\theta$ in $\mathfrak{R}_{2n+1}$ be denoted by $\mathfrak{F}$. We then obtain:

$$\begin{aligned} J_{\bar{C}} - J_C &= \int_{\mathfrak{F}} d\omega = \int_{\mathfrak{F}} (f_{x_i} dx_i + f_{p_i} dp_i)\, dt \\ &= \int_{\mathfrak{F}} \rho_i \left\{ f_{x_i} (\tau \pm t \mp t^0) \pm f_{p_i} \right\} d\vartheta\, dt. \end{aligned}$$

I assert that this expression is negative for a sufficiently small τ.

In fact, there is a number $M_1 > 0$ that is independent of t such that the value of the first summand is smaller than:

$$M_1\, \varepsilon\, \tau^2 .$$

In the second summand, we set:

$$\begin{aligned} &f_{p_i}(t, x_j(\vartheta, t), p_j(\vartheta, t)) = \\ &f_{p_i}(t, x_j(\vartheta, t), p_j(\vartheta, t)) - f_{p_i}(t, x_j(\vartheta, t), p_j(0, t)) + f_{p_i}(t, x_j(\vartheta, t), p_j(\vartheta, t)) \\ &- f_{p_i}(t^0, x_j^0, p_j^0) + f_{p_i}(t^0, x_j^0, p_j^0). \end{aligned}$$

The last summand obviously contributes zero to the integral. The preceding difference goes uniformly to zero with τ in θ. The associated integral will thus be estimated by:

$$M_2(\tau)\, \varepsilon\, \tau,$$

where $\lim_{\tau \to 0} M_2 = 0$. For the first difference we finally write, from the mean value theorem:

$$\pm \overline{f}_{p_i p_j} \vartheta \rho_i ,$$

where the circumflex denotes a certain associated mean value in our value domain. In the corresponding part of the integral there is always a + sign. Its value is thus negative and its absolute value greater than:

$$\tfrac{1}{2} \varepsilon^2\, \tau\, k .$$

One sees that by a suitable choice of τ one can achieve:

$$J_{\bar{C}} - J_C < 0.$$

For any line element, a solution curve therefore gives:

$$f_{p_i p_j} u_i u_j \geq 0 \,.$$

One calls a line element *regular* if the quadratic form is positive definite, *singular* when it is positive semi-definite, and in all other cases, *irregular*. (If one would also like to treat maximum problems then one calls only the line element with an indefinite form irregular and distinguishes positive- and negative-regular (-singular, resp.) elements.) The solutions of the minimum problem include no irregular elements.

5. In order to obtain a *field* of curves, we consider an (n + 1)-dimensional "surface" $p_i = p_i(t, x_j)$ in $\mathfrak{R}_{2n+1}$. By integrating the system of equations:

$$\dot{x}_i = p_i(t, x_j)$$

one finds integral curves of the Pfaff system $\omega_i = 0$ that lie on this surface. The corresponding curves in $\mathfrak{R}_{n+1}$ define a "field," i.e., they cover a piece of this space simply and completely.

The replacement of the p_i in a function $F(t, x_j, p_j)$ by functions of t and x_j will be, when necessary, suggested by square brackets. The derivative of such a function [F] on $\mathfrak{R}_{n+1}$ will be denoted by the round ∂ and the derivative along the field curves (in accord with § 2) by the plain d:

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x_i} p_i \,.$$

6. In § 2, the differential of Ω was considered. The Euler equations resulted, in which this differential was assumed to vanish along a curve, as it were. One arrived at sufficient conditions for a minimum when one considered fields that made this differential vanish in a region of $\mathfrak{R}_{n+1}$. One called a field a *geodesic field* [7]) when $d[\Omega] = 0$, i.e., when [Ω] is a complete differential. One has:

$$\begin{aligned} d[\Omega] &= (df_{p_i} - f_{x_i} dt)\omega_i \\ &= \left\{ \frac{\partial f_{p_i}}{\partial t} dt + \frac{\partial f_{p_i}}{\partial x_j} (\omega_j + p_j dt) - f_{x_i} dt \right\} \omega_i \\ &= \left(\frac{df_{p_i}}{dt} - f_{x_i} \right) dt\, \omega_i + \sum_{i<j} \left(\frac{\partial f_{p_j}}{\partial x_i} - \frac{\partial f_{p_i}}{\partial x_j} \right) \omega_i \omega_j \,. \end{aligned}$$

We must therefore expect that the field curves satisfy the equations (2.6), hence, they are extremals, and that the (n – 1)n/2 equations:

---

[7]) Naturally, the notion has been known since *Weierstrass*. The name originated with *Carathéodory*, but the simple definition that is given here is due to *Lepage*.

(6.1) $$\frac{\partial f_{p_j}}{\partial x_i} - \frac{\partial f_{p_i}}{\partial x_j} = 0$$

are valid.

If C is now a field extremal and $\bar{C}$ is any comparison curve with the same endpoints that extends entirely within the field then one has:

$$J_C = \int_{\bar{C}} \omega = \int_C [\Omega] = \int_{\bar{C}} [\Omega],$$

and thus, when one substitutes from (2.5):

$$J_{\bar{C}} - J_C = \int_{\bar{C}} \mathcal{E}(t, x_i, p_i, \bar{p}_i)\, dt,$$

where p_i are the field functions, $\bar{p}_i$ are the line elements of $\bar{C}$, and:

$$\mathcal{E}(t, x_i, p_i, \bar{p}_i) = f(t, x_j, \bar{p}_j) - f(t, x_j, p_j) - (\bar{p}_i - p_i)\; f_{p_i}(t, x_j, p_j)$$

is the *Weierstrass excess function.*

It is sufficient for a *strong* minimum (comparison curves in a sufficiently small neighborhood with arbitrary directions) that one can embed all of the extremals C in a geodesic field [8]), and that for the line element (t, x_i, p_i) of the field and all $\bar{p}_i$ one has:

(6.2) $$\mathcal{E}(t, x_i, p_i, \bar{p}_i) > 0\,.$$

For a *weak* minimum (comparison curves have the same direction in a sufficiently small neighborhood), the same is true for any line element of the field for the $\bar{p}_i$ in a certain neighborhood of p_i. (6.2) is the *Weierstrass condition.*

Since the development of $\mathcal{E}$ in powers of $\bar{p}_i - p_i$ begins with $\frac{1}{2} f_{p_i p_j} (\bar{p}_i - p_i)(\bar{p}_j - p_j)$ (in the case where only the second derivatives of f with respect to the p_i exist, one uses Taylor's theorem) one can replace the condition (6.2) for the weak minimum with the condition that the quadratic form (4.1) be positive definite. This is "Legendre's sufficient condition."

7. The meaning of conditions (6.1) becomes clear when one writes down the fact that $[\Omega]$ is a complete differential in the form:

(7.1) $$[\Omega] = dS,$$

[8]) The possibility of embedding a sufficiently small piece of an extremal in a field follows from the way that one derives the Hamilton-Jacobi differential equation in the theory of first order partial differential equations; cf., chapter 3 in the textbook of Carathéodory that was mentioned in rem. [5]). The question of the possibility of embedding the *entire* curve will be answered in the theory of the second variation.

where $S(t, x_i)$ is a function on $\mathfrak{R}_{n+1}$. One has:

$$dS = \frac{\partial S}{\partial t}dt + \frac{\partial S}{\partial x_i}dx_i = \frac{dS}{dt}dt + \frac{\partial S}{\partial x_i}\omega_i\,,$$

and when one equates this with (2.5), one writes either:

$$(7.2) \qquad [f_{p_i}] = \frac{\partial S}{\partial x_i}, \qquad [f] = \frac{dS}{dt}$$

or:

$$(7.3) \qquad [f_{p_i}] = \frac{\partial S}{\partial x_i}, \qquad [f - p_j\ f_{p_j}] = \frac{\partial S}{\partial t}.$$

Carathéodory first placed the equations (7.2) at the forefront of the calculus of variations. The geometric meaning of (7.3) shows the comparison with (3.1): *Extremals of the field will be cut transversally by the n-dimensional surfaces S = const.* Furthermore, one sees that the equations (2.6) and (6.1), taken together are nothing but the integrability conditions for the *n*-dimensional surface elements that are transversal to the field curves to fit together into a one-parameter family of surfaces S = const. in a certain way [9]).

8. The relation:

$$(8.1) \qquad \int_{\bar{C}}[\Omega] = \int_C[\Omega]$$

is valid not only for two arbitrary curves that run through the geodesic field whose starting points and endpoints agree. Indeed, one has:

$$\int_C[\Omega] = \int_C dS = S_2 - S_1\,,$$

where S_1 and S_2 mean the values of the function S that belong to the starting point and the endpoint of C, and for the validity of (8.1), it thus suffices for the starting points and the endpoints of C and $\bar{C}$ to lie on the same surface S = const. The theorem that is implied by this, that *for a geodesic field the surface S* = const. ("geodesic equidistants") *on the field extremals pieces give equal values for the integral* J_C (equal "geodesic length"), is a

[9]) If one would expect only that the transversals were surface elements that belonged to a family S = const. then the derivatives of S would only be proportional to the left-hand side of (7.3). The "normalization" of the gradients, which is somewhat expected by the last of equations (7.3) (the others are then satisfied automatically) may generally be achieved by a transformation $S' = \varphi(S)$ only for a point of each surface. This last equation, as one must expect in addition, is, when one carries out the Legendre transformation, nothing but the Hamilton-Jacobi partial differential equation of the problem. Example: The orthogonal trajectories of an arbitrary family of surfaces define a geodesic field for the variational problem of shortest length with the basic function $f = \sqrt{1 + p_i\,p_i}$ (for which transversal = orthogonal) when and only when the length of the gradients is constant on each surface; the orthogonal trajectories are thus straight lines then and only then. (For this, confer the theorem of § 8.)

generalization of a well-known theorem about geodesic lines on surfaces and normal congruences, and they will thus be referred to as *Kneser transversals* [10]). For Carathéodory, an extremal field, together with its geodesic transversals, hence, the geometric structure for which the Kneser theorem is valid, and therefore gives an answer to all of the questions that arise from the variational problem, is called a "*complete figure.*"

9. Furthermore, one again finds the transversality condition of § 3 here effortlessly. We consider the simplest case: Let the starting point P_1 move on an n-dimensional surface, while the endpoint P_2 is fixed. We take the surface to be the surface $S = S_1$ of a geodesic field; the point P_2 may lie in the field and the Weierstrass condition (6.2) is satisfied. It is then clear that of all of the curves that run through the field, only the field extremal that goes through P_2 is a solution of the problem. In particular, an extremal that runs through the field that connects any point of the surface $S = S_1$ with P_2 can certainly not be a solution when it does not intersect this surface transversally.

A system of sufficient conditions for this case then reads: Euler equations, transversality, possibility of the field construction, and the Legendre condition for the weak minimum, the Weierstrass condition for the strong minimum.

If the starting point moves on a surface of lower dimension then one lays an n-surface over it and proceeds in precisely the same way. The behavior becomes complicated when both endpoints are variable. Here, let us suggest only the simplest case: P_1 shall lie on an n-surface, P_2, on any surface F, and the conditions mentioned above are satisfied; moreover, let $f > 0$. Obviously, a minimum actually occurs when the aforementioned field construction gives a surface F_2 in the neighborhood of P_2 that lies completely on *one* side of the surface S = const. that goes through P_2, and indeed on the opposite one, through P_1.

(Received on 1 May 1940.)

---

[10]) A. Kneser, Lehrbuch der Variationsrechnung, pp. 46. Braunschweig 1900.

On the Legendre condition and field theory in the calculus of variations for multiple integrals

By

Hermann Boerner in Munich
Presently at Weather Service for the Reich

In the calculus of variations for multiple integrals – several independent *and* dependent variables – one runs into noteworthy difficulties in the presentation of the Legendre condition, which *Hadamard* was the first to point out in his study of the works of *Clebsch* [1]). It takes the form of being given, not just *one*, but arbitrarily many sufficient conditions (although it is sufficient to satisfy any one of them), whereas the necessity of any of them is indeed implicit, but means nothing more. Surprisingly, *Carathéodory's* generalization in the form of his theory of "geodesic fields" [2]) thus led to a completely well-defined sufficient Legendre condition that (with the ≥ sign) indeed seems to be necessary, in a certain sense, from which the possibility of embedding a given extremal in a field was proven by me [3]). One finds formulas for field theory that are completely different from the (relatively complicated) ones of Carathéodory written down in the book by *De Donder* in a purely formal way [4]), and the same theory was developed by *Weyl*, who also proved the possibility of embedding in a field [5]); this theory leads to another, likewise uniquely determined, sufficient Legendre condition. However, the Weierstrass $\mathcal{E}$-functions are different in the two theories.

The Belgian *Lepage* has brought both theories under one roof (and into agreement with Hadamard's Legendre condition), by regarding them as both special cases in an entire family of theories that depend upon a large number of arbitrary functions [6]).

In the following, I will show that one is inevitably led to one theory in particular – namely, that of Carathéodory – when one places the additional demand on the field theory that it be applicable to all problems, and therefore to the ones with *moving*

---

[1]) J. Hadamard, Sur un question de calcul des variations. Bull. Soc. math. de France **30** (1902), pp. 253-256; Sur quelques questions de calcul des variations. *ibid*. **33** (1905), pp. 73-80.

[2]) C. Carathéodory, Über die Variationsrechnung bei mehrfachen Integralen. Acta Szeged **4** (1928/29), pp. 193-216.

[3]) H. Boerner, Über die Extremalen und geodätischen Felder in der Variationsrechnung der mehrfachen Integrale. Math. Ann. **112** (1936), pp. 187-220. A very interesting group-theoretical treatment of the proof was recently given by *E. Hölder:* Die infinitesimalen Berührungstransformation der Variationsrechnung, Jber. D. M. V. **49** (1939), pp. 162-178 – cf., *infra*, rem. [25]).

[4]) Th. De Donder, Théorie invariantive du calcul des variations, 10 treatises in the Bull. Acad. Roy. Belg. v.'s. **15** (1929) and **16** (1930). New edition, Paris 1935.

[5]) H. Weyl, Geodesic fields in the calculus of variations for multiple integrals. Annals of Math. **36** (1935), pp. 607-629.

[6]) Th.-H.-J. Lepage, Sur les champs géodesiques du calcul des variations. Bull. Acad. Roy. Belg. v. **22** (1936), pp. 716-729, 1036-1046.

boundaries [7]). In a truly elementary example I will show that the sufficient condition of Hadamard or Weyl is thus included, in fact.

Lepage devoted himself in his work to the calculus of alternating differential forms – whose use is, in fact, suggested when one is concerned with integrals that are "path-independent." However, the calculus is not only of service in that context, it shows, in addition, that one can arrive at the entire calculus of variations for multiple integrals in the most convenient way.

In the first chapter, I will show the necessary conditions. I derive the Euler equations, and obtain the transversality condition that one can apparently find nowhere, at present, and which is essential for the further consequences. I also give the first complete proof of Hadamard's necessary condition [8]).

The second chapter is then dedicated to the promised development of the field theory, and indeed we operate on arbitrarily many variables from the outset (Lepage confined himself to two).

In a simultaneously-appearing work I have likewise treated the calculus of variations for simple integrals by the calculus of differential forms, and thus brought a number of things into one unified formalism [9]). It includes precisely everything that was previously generalized, and while it is also sensibly independent thereof there is much that becomes easily understandable that one has read about in the realm of that which is well-known.

First chapter

The necessary conditions

1. Rules of computation and theorems on differential forms. Greek indices run from 1 to μ, Latin ones from 1 to n. The summation sign will be omitted. Summations over several different indices are assumed to be independent of each other, as usual, except when a summation sign is present that would demand otherwise.

One finds the rules of computation for alternating differential forms $a_i\, dy_i$, etc., whose coefficients are functions on an n-space (y_i) in "*St.*" previously. The coefficients will be appropriately called "skew-symmetric." For degree three one has, e.g.,:

$$a_{ijk} = -\, a_{ikj} = a_{jki} = -\, a_{jik} = a_{kij} = -\, a_{kji}\ ,$$

and one sums only over triples of numbers, and writes:

$$\omega = \sum_{i<j<k} a_{ijk} dy_i dy_j dy_k \quad (= \tfrac{1}{6} a_{ijk}\, dy_i\, dy_j\, dy_k).$$

---

[7]) This result was already shown by me in a talk that I gave in February 1939 in Frankfurt and Marburg under the title of "Probleme der Variationsrechnung in mehreren Veränderlichen."

[8]) Hadamard showed this only for three independent and three dependent variables in his book "La propagation des ondes," Paris 1903, on pp. 253.

[9]) H. Boerner, Variationsrechnung as dem Stokeschen Satz, Math. Zeit. **46** (1940), pp. 709-719. Denoted by "*St.*," in the sequel.

In the construction of the *integral* $\int_{\mathcal{F}} \omega$ of a form ω of degree p over a piece $\mathcal{F}$ of a p-surface one must consider its *orientation*. One can regard any sufficiently small piece of $\mathcal{F}$ in terms of parameters $u_1, \ldots, u_p$, and thus as a region G in the "$(u_1, \ldots, u_p)$-plane." An orientation is nothing but a *class of parameter representations:* two parameter representations $u_1, \ldots, u_p$ and $v_1, \ldots, v_p$ provide the same orientation (i.e., belong to the same class) when the functional determinant $\frac{\partial(u_1,\cdots,u_p)}{\partial(v_1,\cdots,v_p)}$ is positive, and have opposite orientations when it is negative. Thus, there are precisely two orientations. We shall assume that all surfaces are orientable in the sequel, i.e., that they shall be covered by parameter representations like roofing tiles, in such a way that neighboring parameter representations in the same part of the surface are provided with the same orientation. The sign of the integral, which will be constructed as the ordinary integral of:

$$dy_{i_1} dy_{i_2} \cdots dy_{i_p} = \frac{\partial(y_{i_1},\cdots,y_{i_p})}{\partial(u_1,\cdots,u_p)} du_1\, du_2 \ldots du_p$$

over G, thus depends on the orientation, though its value is independent of the choice of parameter representation.

One has *Stokes's theorem:*

$$\int_{\mathcal{R}} \omega = \int_{\mathcal{F}} d\omega,$$

where $\mathcal{R}$ is the (p-dimensional) boundary of the (here, (p+1)-dimensional) piece of the surface $\mathcal{F}$, and one suitably orients $\mathcal{R}$ and $\mathcal{F}$ with respect to each other. From now, one understands the following: One considers $\mathcal{F}$ in the neighborhood of a piece of $\mathcal{R}$ in terms of parameters $u_1, \ldots, u_p$ in such a way that u_1 is constant on $\mathcal{R}$ and is smaller on the interior of $\mathcal{F}$; the $u_1, \ldots, u_{p+1}$ then produce the $u_2, \ldots, u_{p+1}$ for a suitable orientation of $\mathcal{F}$ ($\mathcal{R}$, resp.).

The surfaces that figure in Stokes theorem may have "kinks": i.e., they might decompose into regions on whose boundaries the tangent planes "bend." In the sequel, we shall assume that all manifolds possess continuous tangent planes with no such kinks.

From Stokes's theorem, one concludes that the integral of a "*complete differential*," i.e., a form ω with $d\omega = 0$, is independent of the path: its value depends upon only the boundary of surface portion over which it is integrated.

By the *rank* of a differential form, one understands that to mean the smallest number of Pfaff forms (i.e., forms of degree one) by which one can represent it. The smallest possible rank of a form of degree p is obviously p, so it is therefore a product of linear forms:

$$\omega = \omega_1\, \omega_2 \ldots \omega_p\,.$$

By the *class* of a form, one understands that to mean the smallest number of variables (as arguments of the coefficients and under the d sign), with whose help the form can be represented by coordinate transformations. In general, the class will be larger than the rank. We thus have the theorem:

For a complete differential the rank and class agree [10]).

2. The Euler equations. We consider the μ-fold integral:

$$J_E = \int_G f\left(t_\alpha, x_i, \frac{\partial x_i}{\partial t_\alpha}\right)(dt) \text{ }^{11})$$

to be taken over a μ-surface E: $x_i = x_i(t_\alpha)$ in the $(\mu+n)$-space $\mathfrak{R}_{\mu+n}$ of the variables t_α, x_i. The basic function $f(t_\alpha, x_i, p_{i\alpha})$ shall be continuous in a certain region of $\mathfrak{R}_{\mu+n}$ and for all $p_{i\alpha}$, at least up to its first derivatives and the first derivatives of the $f_{p_{i\alpha}}$.

In order to derive necessary conditions for the minimum, when given a fixed boundary, we consider a comparison surface $\bar{E}$: $x_i = \bar{x}_i(t_\alpha)$, where the functions $\bar{x}_i(t_\alpha)$ are defined in the same region G of the "t_α-plane" as the $x_i(t_\alpha)$ and agree with them on the boundary of G. If $\bar{E}$ is a solution of the problem then the "variation" $J_{\bar{E}} - J_E$ will be positive for an arbitrary $\bar{E}$ that is sufficiently close

We embed E and $\bar{E}$ in a one-parameter family of comparison surfaces E_θ that all possess the same boundary:

(2.1) $\qquad E_\theta$: $x_i = x_i(\theta, t_\alpha)$; $\qquad x_i(0, t_\alpha) \equiv x_i(t_\alpha)$, $\qquad x_i(\varepsilon, t_\alpha) = \bar{x}_i(t_\alpha)$.

In the $(\mu + n + \mu n)$-space $\mathfrak{R}_{\mu+n+\mu n}$ of variables t_α, x_i, $p_{i\alpha}$ we consider the μ-degree form:

$$\omega = f(t_\alpha, x_i, p_{i\alpha})\, dt.$$

When one lets $\mathfrak{E}_\theta$ denote the surfaces in this space that correspond to the E_θ by the relations $p_{i\alpha} = \partial x_i / \partial t_\alpha$ then one has:

$$J_{E_\vartheta} = \int_{\mathfrak{E}_\vartheta} \omega\,.$$

Such surfaces in $\mathfrak{R}_{\mu+n+\mu n}$ are, when one introduces the n Pfaff forms:

---

[10]) *E. Goursat*, Leçons sur le problème de Pfaff, Paris 1922, pp. 135. One generally finds approximations to the notions of "rank" and "class" there on pp. 126-139. *W. Maak* has given a modern definition of the surface integral for very general surfaces and without the use of a parameter representation by the method of integral geometry: Oberflächenintegral und Stokes-Formel im gewöhnlichen Raume. Math Annalen **116** (1939), pp. 574-597. For the purposes of variational calculus, where one always has suitable parameter representations on hand and must make corresponding restricting assumptions on the surfaces, the reduction to the volume integral that is given in the text will generally suffice.

[11]) Here, we are already using the abbreviation (dt) for the product $dt_1 \dots dt_\mu$ that we shall later see makes the overview of computing with differential forms much easier. $(dt)_\alpha$ means $dt_1 \dots dt_{\alpha-1}\, dt_{\alpha+1} \dots dt_\mu$, and $(dt)_{\alpha\beta}$ is defined analogously.

$$\omega_i = dx_i - p_{i\alpha}\, dt_\alpha\,,$$

integral manifolds of the "Pfaff system" $\omega_i = 0$. Since we shall only integrate over surfaces, we can replace ω with any form that is congruent to it (mod ω_i):

$$\Omega = f\, dt_1 \ldots dt_\mu + A_{i\alpha}\, dt_1 \ldots dt_{\alpha-1}\; \omega_i\, dt_{\alpha+1} \ldots dt_\mu + \\ + \sum_{i<j,\alpha<\beta} A_{i\alpha,\, j\beta} dt_1 \cdots dt_{\alpha-1}\omega_i dt_{\alpha+1} \cdots dt_{\beta-1}\omega_j dt_{\beta+1} \cdots dt_\mu + \cdots$$

As in "*St.*", we determine Ω in such a way that:

$$d\Omega \equiv 0 \quad (\omega_i).$$

In this, one must replace only $A_{i\alpha} = f_{p_{i\alpha}}$; *All of the other functions* $A_{i\alpha,\, j\beta}$ *remain arbitrary*, and may chosen to be fixed, anyhow. We thus have:

(2.2) $$\Omega = f\, dt_1 \ldots dt_\mu + f_{p_{i\alpha}}\, dt_1 \ldots dt_{\alpha-1}\; \omega_i\, dt_{\alpha+1} \ldots dt_\mu + \\ + \sum_{i<j,\alpha<\beta} A_{i\alpha,\, j\beta} dt_1 \cdots dt_{\alpha-1}\omega_i dt_{\alpha+1} \cdots dt_{\beta-1}\omega_j dt_{\beta+1} \cdots dt_\mu + \cdots$$ [12]).

From Stokes's theorem, one now has:

(2.3) $$J_{\bar{E}} - J_E = \int_{\bar{\mathfrak{E}}} \Omega - \int_{\mathfrak{E}} \Omega = \operatorname{sgn} \varepsilon \int_{\mathfrak{F}} d\Omega\,,$$

since one has $\Omega = 0$ on the rest of $\mathfrak{F}$. Here, $\mathfrak{F}$ is the $(\mu+1)$-surface that is composed of the points of all of the $\mathfrak{E}_\theta$, and (2.3) is correct when we establish its orientation by the choice of parameters $t_1, \ldots, t_\mu$ for $\mathfrak{E}$ and $\bar{\mathfrak{E}}$ and $\theta, t_1, \ldots, t_\mu$ for $\mathfrak{F}$. The calculation of $d\Omega$ is simplified when we compute in exactly these parameters. One then has:

$$\omega_i = \frac{\partial x_i}{\partial \vartheta} d\vartheta\,,$$

and all terms with more one factor of ω_i – in particular, all of the ones that were not written down in (2.2) – drop out. One has:

$$\Omega = f\,(dt) + (-1)^{\alpha-1}\; df_{p_{i\alpha}}\; \omega_i (dt)_\alpha + \sum_{i<j,\alpha<\beta} (-1)^{\alpha+\beta-1} A_{i\alpha,\, j\beta}\, \omega_i \omega_j\, (dt)_{\alpha\beta} + \cdots$$ [11]).

One thus obtains:

---

[12]) For the computations of this chapter one can restrict oneself to the terms that are linear in ω_i. We have nevertheless just now written the most general form Ω that figures in the Lepage theory, by which one may convince oneself that the further terms play no role in these computations.

$$d\Omega = f_{x_i}\omega_i(dt) + (-1)^{\alpha-1} df_{p_{i\alpha}}\omega_i(dt)_\alpha +$$
$$+ \sum_{i<j,\alpha<\beta} (-1)^{\alpha+\beta-1} A_{i\alpha,j\beta}(d\omega_i \cdot \omega_j - \omega_i \cdot d\omega_j)(dt)_{\alpha\beta} +$$
$$+ \sum_{i<j,\alpha<\beta} (-1)^{\alpha+\beta-1} dA_{i\alpha,j\beta}\omega_i \cdot \omega_j(dt)_{\alpha\beta} + \dots$$

(2.4) $$= \left\{ f_{x_i} - \frac{df_{p_{i\alpha}}}{dt_\alpha} + \sum_{\alpha<\beta} A_{i\alpha,j\beta}\left(\frac{dp_{j\beta}}{dt_\alpha} - \frac{dp_{j\alpha}}{dt_\beta}\right)\right\}\frac{\partial x_i}{\partial\vartheta} d\vartheta(dt)$$

$$= \left(f_{x_i} - \frac{df_{p_{i\alpha}}}{dt_\alpha}\right)\frac{\partial x_i}{\partial\vartheta} d\vartheta(dt) \qquad [13])$$

We ultimately have:

$$J_{\bar{E}} - J_E = \operatorname{sgn}\varepsilon \int_{\mathfrak{F}} \left(f_{x_i} - \frac{df_{p_{i\alpha}}}{dt_\alpha}\right)\frac{\partial x_i}{\partial\vartheta} d\vartheta(dt)$$

$$= \int_0^\varepsilon d\vartheta \left\{ \int_G \left(f_{x_i} - \frac{df_{p_{i\alpha}}}{dt_\alpha}\right)\frac{\partial x_i}{\partial\vartheta}(dt)\right\}$$

For sufficiently small positive ε, this has the same sign as the contents of the curly brackets when $\theta = 0$, and the opposite one for negative ε. An extremum can therefore only come about when this quantity vanishes, and due to the arbitrary nature of the "variations" $\partial x_i / \partial\theta$, one concludes in a well-known way the existence of the *Euler equations:*

(2.5) $$\frac{df_{p_{i\alpha}}}{dt_\alpha} - f_{x_i} = 0.$$

The solutions of the variational problem, in the form of the integrals of these differential equations, are thus the *extremals* that we seek.

3. Transversality. The boundary of the desired surface may now be no longer assumed to be fixed; rather, it can move on a manifold H of dimension $\mu - 1 + p$ ($1 \le p \le n$). We again consider a family (2.1) of μ-surfaces whose boundaries now no longer necessarily agree, but lie on H. We let S denote the totality of all of these boundary points, and let $\mathfrak{S}$ denote the corresponding μ-surface in $\mathfrak{R}_{\mu+n+\mu n}$. On the left-hand side of (2.3), a new term appears:

---

[13]) The derivation with respect to the *parameter* t_α, i.e., along the surface E_θ, is denoted by the plain d; naturally, one has $\frac{dp_{i\alpha}}{dt_\alpha} = \frac{dp_{i\alpha}}{dt_\beta} = \frac{d^2 x_i}{dt_\alpha dt_\beta}$. One observes that in the last summation sign of (2.4) the reference to i and j is omitted; here, one sums over these two indices independently and one defines $A_{i\alpha,j\beta}$ for $j > i$ by $A_{i\alpha,j\beta} = -A_{j\alpha,i\beta}$.

$$\int_{\mathfrak{S}} \Omega \qquad (3.1)$$ [14]).

Since the surface E, in any case, must also give also a minimum for a fixed boundary the results of the previous section must remain true – *without* the new term – and we must therefore place the same demands on this term *by itself* as we did on the last integral of the previous section.

We regard the boundary of E_θ in terms of $\mu - 1$ parameters $\theta_2, \dots, \theta_\mu$ and set $\theta_1 = \theta$, such that $\theta_1, \dots, \theta_\mu$ yields a parameter representation of S ($\mathfrak{S}$, resp.):

$$t_\alpha = t_\alpha(\theta_\beta), \qquad x_i - x_i(\theta_\beta) = x_i(\theta_1, t_\alpha(\theta_\beta)), \tag{3.2}$$
$$p_{i\alpha} = \frac{\partial x_i}{\partial t_\alpha}(\theta_1, t_\beta(\theta_\rho)).$$

To abbreviate, we write:

$$\frac{\partial t_\alpha}{\partial \theta_\beta} = l_{\alpha\beta}.$$

The derivatives with respect to θ_1, which characterize the "boundary displacement," will be written by means of a δ:

$$\frac{\delta t_\alpha}{\delta \vartheta_1} = l_{\alpha 1} \qquad \text{and} \qquad \frac{\delta x_i}{\delta \vartheta_1} = \frac{\partial x_i}{\partial \theta_1} + p_{i\alpha}\, l_{\alpha 1}.$$

One convinces oneself that one also has:

$$\omega_i = \frac{\partial x_i}{\partial \vartheta_1} d\vartheta_1$$

now.

Now, we can compute (3.1). If we denote the algebraic complement of $l_{\alpha\beta}$ in its determinant l by an overbar then we can write:

$$(dt) = l\,(d\theta) = \frac{\delta t_\rho}{\delta \vartheta_1} \bar{l}_{\rho 1}(d\vartheta)$$ [15]),

$$dt_1 \dots dt_{\alpha-1}\, \omega_i\, dt_{\alpha+1} \dots dt_\mu = \frac{\partial x_i}{\partial \vartheta_1} dt_1 \cdots dt_{\alpha-1} d\vartheta_1 dt_{\alpha+1} \cdots dt_\mu = \frac{\partial x_i}{\partial \vartheta_1} \bar{l}_{\alpha 1}(d\vartheta).$$

One thus obtains:

[14]) The sign of this term, thus the orientation of $\mathfrak{S}$, is irrelevant for our purposes.

[15]) $(dt) = dt_1 \dots dt_\mu$ and $(d\theta) = d\theta_1 \dots d\theta_\mu$.

$$\int_{\mathfrak{S}} \Omega = \int_{\mathfrak{S}} \left(f \frac{\delta t_\alpha}{\delta \vartheta_1} + f_{p_{i\alpha}} \frac{\partial x_i}{\partial \vartheta_1} \right) \overline{l}_{\alpha 1} (d\vartheta)$$

(3.3)
$$= \int_{\mathfrak{S}} \left\{ f \frac{\delta t_\alpha}{\delta \vartheta_1} + f_{p_{i\alpha}} \left(\frac{\delta x_i}{\delta \vartheta_1} - p_{i\rho} \frac{\delta t_\rho}{\delta \vartheta_1} \right) \right\} \overline{l}_{\alpha 1} (d\vartheta)$$

$$= \int_{\mathfrak{S}} \left(a_{\rho\alpha} \frac{\delta t_\rho}{\delta \vartheta_1} + f_{p_{i\alpha}} \frac{\delta x_i}{\delta \vartheta_1} \right) \overline{l}_{\alpha 1} (d\vartheta),$$

with the notation that Carathéodory introduced:

$$a_{\alpha\beta} = \delta_{\alpha\beta} f - p_{i\alpha}\; f_{p_{i\beta}} .$$

We first integrate over $\theta_2, \ldots, \theta_\mu$, and must state from the outset the demand that the result must vanish when $\theta_1 = 0$. One must then have, when one denotes the boundary of the extremal E by R:

(3.4a)
$$\int_R \left(a_{\rho\alpha} \frac{\delta t_\rho}{\delta \vartheta_1} + f_{p_{i\alpha}} \frac{\delta x_i}{\delta \vartheta_1} \right) \overline{l}_{\alpha 1} d\vartheta_2 \cdots d\vartheta_\mu = 0$$

or:

(3.4b)
$$\sum_\alpha (-1)^{\alpha-1} \int_R \left(a_{\rho\alpha} \frac{\delta t_\rho}{\delta \vartheta_1} + f_{p_{i\alpha}} \frac{\delta x_i}{\delta \vartheta_1} \right) dt_1 \cdots dt_{\alpha-1} dt_{\alpha+1} \cdots dt_\mu = 0;$$

the parameters no longer appear in (3.4b).

From (3.4), it follows that there is a "transversality relation" between the surface H and the extremal E. It reads like this: any vector with the components:

(3.5)
$$(dt_\alpha, dx_i)$$

that is tangential to H at a point of R ("displacement vector") must satisfy the equation:

(3.6)
$$(a_{\rho\alpha} \delta t_\rho + f_{p_{i\alpha}} \delta x_i)\, \overline{l}_{\alpha 1} = 0 .$$

Otherwise, one could define a family E_θ such that (3.4a) is not valid.

In order to better formulate this condition geometrically, we call a direction (3.5) *transversal* to E when it satisfies the μ equations:

$$a_{\rho\alpha}\, \delta t_\alpha + f_{p_{i\alpha}}\, \delta x_1 = 0 .$$

We solve this for δt_α and write:

(3.7)
$$\delta t_\alpha + P_{i\alpha}\, \delta x_i = 0,$$

with the notation that was likewise introduced by Carathéodory (in connection with the Legendre transformation):

(3.8)
$$P_{i\alpha} = \frac{\overline{a}_{\alpha\beta}}{a} f_{p_{i\beta}} .$$

The symmetry of the notation becomes clear when one takes into consideration that the direction that is *perpendicular* to the extremal E is distinguished by:

$$\delta t_\alpha + p_{i\alpha}\, \delta x_i = 0.$$

There are n linearly independent transversal directions to any surface element $p_{i\alpha}$, e.g.:

(3.9)
$$(P_{i\alpha},\ \delta_{ij}) \qquad (j = 1, \dots, n).$$

In general, we assume that they, together with the μ directions:

$$(\delta_{\alpha\beta},\ p_{i\beta}) \qquad (\beta = 1, \dots, \mu)$$

that span the surface element, define $\mu + n$ linearly independent vectors.

Condition (3.6) now reads like this: *the* ($\mu - 1 + p$)*-dimensional surface element of H*, which naturally includes the $\mu - 1$ directions:

(3.10)
$$\left(\frac{\partial t_\alpha}{\partial \vartheta_\gamma}, \frac{\partial x_i}{\partial \vartheta_\gamma}\right) \qquad (\gamma = 2, \dots, \mu)$$

that span the boundary element of E, *must*, at the same time, *be included in the* ($\mu - 1 + n$)*-dimensional element that is spanned by them and the n transversal directions* (3.9).

In fact, relation (3.6) is necessary and sufficient for the vector (3.5) to be linearly independent of the $\mu - 1 + n$ vectors (3.10) and (3.9). One first has, for $\gamma = 2, \dots, \mu$:

$$\left(a_{\rho\alpha} \frac{\partial t_\rho}{\partial \vartheta_\gamma} + f_{p_{i\alpha}} \frac{\partial x_i}{\partial \vartheta_\gamma}\right) \overline{l}_{\alpha 1} = (a_{\rho\alpha} + f_{p_{i\alpha}} p_{i\rho}) \frac{\partial t_\rho}{\partial \vartheta_\gamma} \overline{l}_{\alpha 1} = f \frac{\partial t_\alpha}{\partial \vartheta_\gamma} \overline{l}_{\alpha 1} = 0,$$

and then, for $j = 1, \dots, n$:

$$(-a_{\rho\alpha} P_{j\rho} + f_{p_{i\alpha}} \delta_{ij}) \overline{l}_{\alpha 1} = (-f_{p_{j\alpha}} + f_{p_{j\alpha}}) \overline{l}_{\alpha 1} = 0.$$

Together with (3.6), this is a linear relation between the *columns* of the determinant that is constructed out of all of the vectors (3.5), (3.10), and (3.9) as *rows,* from which the assertion follows, due to the linear independence of the vectors (3.10) and (3.9).

4. Hadamard's necessary Legendre condition. The condition, whose necessity for the minimum shall be proved in this section, reads:

(4.1)
$$f_{p_{i\alpha} p_{j\beta}} \rho_i \rho_j \lambda_\alpha \lambda_\beta \geq 0$$

for all $\rho_1, \ldots, \rho_n, \lambda_1, \ldots, \rho_\mu$. The biquadratic form in (4.1) is nothing but the quadratic form:

$$f_{p_{i\alpha}p_{j\beta}} u_{i\alpha} u_{j\beta}$$

that is defined for such values of the variables $u_{i\alpha}$ whose rectangular matrix has rank 1.

I assume that (4.1) is not satisfied for a certain surface element of the surface E, in whose neighborhood its tangential plane is continuous; there are therefore certain numbers ρ_i and λ_α such that $\rho_i\,\rho_i = 1$, $\lambda_\alpha\,\lambda_\alpha = 1$, and:

$$f_{p_{i\alpha}p_{j\beta}} \rho_i \rho_j \lambda_\alpha \lambda_\beta = -k < 0 \,. \tag{4.2}$$

Under these assumptions, I will construct a comparison surface $\bar{E}$ such that the integral takes on a smaller value than on E. This surface $\bar{E}$ will possess kinks, which we have indeed ruled out for all of the surfaces considered. If $p_{i\alpha}$, $\bar{p}_{i\alpha}$ are its surface elements on both sides of one such kink then these two surface elements have $\mu - 1$ linearly independent directions in common; from this, it follows that the matrix $(\bar{p}_{i\alpha} - p_{i\alpha})$ has rank one, and conversely.

For the proof, only the differences $\bar{x}_i - x_i$, $\bar{p}_{i\alpha} - p_{i\alpha}$ play a role; in this section, we would like to employ the notational simplification that the surface E be determined by the equations $x_i = 0$. In addition, let the coordinate system be chosen such that point of E that was considered above has the coordinates $t_\alpha = 0$, and that the surface element $p_{i\alpha} - \rho_i\,\lambda_\alpha$, which (for the moment, is assumed to be at the origin) has the aforementioned $\mu - 1$ directions in common with the t-plane $x_i = 0$, intersects it along the $(\mu - 1)$-plane $t_1 = 0$. Thus, one has (with the Kronecker δ-symbol) $\lambda_\alpha = \delta_{1\alpha}$, hence, $p_{i\alpha} = \delta_{1\alpha}\,\rho_i$. Instead of (4.2), one now has simply:

$$f_{p_{i1}p_{j1}}(0,0,0)\,\rho_i\,\rho_i = -k < 0. \tag{4.3}$$

I consider the region in the t-plane:

$$Q: \begin{array}{l} -\tau \le t_1 \le \tau, \\ -h \le t_\gamma \le h \quad (\gamma = 2, \cdots, \mu). \end{array} \tag{4.4}$$

Q decomposes into 2μ sub-regions, existing on the points of the connecting line from the null point to the points of any one of the 2μ boundary surfaces of Q, which are characterized by the demand that in (4.4) the equal sign is always valid at one location. These 2μ sub-regions are not all congruent, but all they possess the same volume:

$$\frac{1}{\mu}\,\tau(2h)^{\alpha-1}. \tag{4.5}$$

I likewise construct a family E_θ that includes the surface $\bar{E}$ with the desired properties for $\theta = \varepsilon > 0$. Let $x_i = \theta\,\tau\,\rho_i$ for $t_\alpha = 0$ and $x_i = 0$ on the boundary of Q and outside of it, whereas x_i shall be linear on the line considered above. Thus, $x_i(\theta, t_\alpha)$ is everywhere continuous and $|x_i| \leq \theta\tau$. The $p_{i\alpha}$ are constant in each of the 2μ sub-regions. We denote the two regions that belong to $t_1 = \mp\varepsilon$ by Q', and the two regions that belong to $t_\gamma = \mp h$ by Q''_γ. One then has:

$$(4.6)\qquad \begin{aligned} &\text{in } Q': \quad p_{i\alpha} = \pm\delta_{i\alpha}\vartheta\rho_i \qquad \text{and} \quad x_i = \vartheta\rho_i \quad (\tau \pm t_1), \\ &\text{in } Q''_\gamma: \quad p_{i\alpha} = \pm\delta_{\gamma\alpha}\frac{\tau}{h}\vartheta\rho_i \quad \text{and} \quad x_i = \vartheta\frac{\tau}{h}\rho_i \quad (h \pm t_\gamma). \end{aligned}$$

The various signs belong to the two halves of each region. *I assert that ε, τ, and h can be chosen such that $J_{\bar{E}} - J_E$ becomes negative.*

Since $\varepsilon > 0$, one has:

$$J_{\bar{E}} - J_E = \int_{\mathfrak{F}} d\omega,$$

where $\mathfrak{F}$ is defined by:

$$0 \leq \theta \leq \varepsilon, \qquad (t_\alpha) \text{ in } Q,$$

when we choose θ, $t_1, \ldots, t_\mu$ to be the parameters, as in § 2. Let $\mathfrak{F}'$, $\mathfrak{F}''_\gamma$ denote the regions of $\mathfrak{F}$ that correspond to Q', Q''_γ. One then has:

$$\int_{\mathfrak{F}} d\omega = \int_{\mathfrak{F}} (f_{x_i} dx_i + f_{p_{i\alpha}} dp_{i\alpha})(dt)$$

$$= \int_{\mathfrak{F}'} \rho_i \left\{ f_{x_i}(\tau \pm t_1) \pm f_{p_{i1}} \right\} d\vartheta(dt) + \sum_{\gamma=2}^{n} \int_{\mathfrak{F}''_\gamma} \frac{\tau}{h} \rho_i \left\{ f_{x_i}(h \pm t_\gamma) \pm f_{p_{i\gamma}} \right\} d\vartheta(dt) \qquad [11]),$$

where the upper (lower, resp.) sign is to be taken in the two halves of each sub-region.

I choose $\varepsilon > 0$, $t_0 > 0$, and $h_0 > 0$ so small that for:

$$(4.7)\qquad 0 < \tau < t_0, \quad 0 < h < h_0, \quad |t_1| \leq \tau, \qquad |t_\gamma| \leq h, \qquad |x_i| \leq \varepsilon\tau, \qquad |p_{i\alpha}| \leq \varepsilon,$$

the contributions from all of the aforementioned derivatives of f lie beneath a fixed limit, and one has:

$$f_{p_{i1}p_{j1}} \rho_i \rho_j < -\frac{k}{2},$$

moreover.

By means of (4.5), the totality of the first summands of each integral will then be estimated by:

$$M_1\, \varepsilon\, \tau^2\, h^{\mu-1},$$

and for the second summands one finds the following estimate on $\mathfrak{F}''$:

$$M_2\, \varepsilon\, \tau^2\, h^{\mu-2}.$$

$M_1 > 0$ and $M_2 > 0$ do not depend on τ and h.

In the second summand of $\mathfrak{F}'$, we set:

$$f_{p_{i1}}(t_\alpha, \vartheta\rho_j(\tau \pm t_1), \pm\delta_{1\alpha}\vartheta\rho_i)$$
$$= \left\{ f_{p_{i1}}(t_\alpha, \vartheta\rho_j(\tau \pm t_1), \pm\delta_{1\alpha}\vartheta\rho_j) - f_{p_{i1}}(t_\alpha, \vartheta\rho_j(\tau \pm t_1), 0) \right\} +$$
$$+ \left\{ f_{p_{i1}}(t_\alpha, \vartheta\rho_j(\tau \pm t_1), 0) - f_{p_{i1}}(0,0,0) \right\} + f_{p_{i1}}(0,0,0) .$$

Due to the opposing signs in the two parts of $\mathfrak{F}'$, the last summand contributes zero to the integral. The second curly bracket converges with τ and h to zero uniformly in θ. The associated integral can be estimated by:

$$M_3\,(h,\,\tau)\;\varepsilon\;\tau\;h^{\mu-1}$$

with $\lim\limits_{\tau,h\to 0} M_3 = 0$. The first bracket is finally equal to $\pm\vartheta\rho_j\,\overline{f}_{p_{i1}p_{j1}}$, where the circumflex refers to certain intermediate values for the arguments outside of the region (4.7), and thus delivers a negative value for the integral sum, a value that lies above a limit:

$$M_4\,(h,\,\tau)\;\varepsilon^2\;\tau\;h^{\mu-1}$$

where $M_4 > 0$ again does not depend upon τ and h.

One immediately deduces that it is possible to choose τ and h in such a way that this last term dominates all of the other ones; with this, the proof of our assertion is achieved [16]).

Second chapter

The sufficient conditions

5. In order to arrive at the sufficient conditions one must pass through the theory of *geodesic fields.* We speak of a *field* when the $p_{i\alpha}$ are given as functions of (t_β, x_j); one is thus dealing with a $(\mu + n)$-surface $\mathfrak{R}_{\mu+n+\mu n}$. Therefore, the integrability of the associated equations $\omega_i = 0$ will not be assured, since, in general, one is therefore not dealing with an n-parameter family of m-surfaces in $\mathfrak{R}_{\mu+n}$ that satisfy the differential equations:

[16]) The "Weierstrass" necessary condition that is associated with the necessary condition of Hadamard, namely, that the De Donder-Weyl $\mathcal{E}$-function (cf., infra) must be non-negative for all $\overline{p}_{i\alpha}$ such that the matrix $\overline{p}_{i\alpha} - p_{i\alpha}$ has rank one, was recently proved in an elegant way by *L. M. Graves*, The Weierstrass condition for multiple integral variation problems. Duke Math. Journ. **5** (1939), pp. 656-660.

$$\frac{\partial x_i}{\partial t_\alpha} = p_{i\alpha}$$

(resp., the associated μ-surfaces that lie on the aforementioned $(\mu + n)$-surface in $\mathfrak{R}_{\mu+n+\mu n}$). As in "*St.*", we denote by square brackets the function or form in $\mathfrak{R}_{\mu+n}$ that results from the replacement of the functions $p_{i\alpha}(t_\beta, x_j)$. Moreover, with Lepage, we call a field *geodesic* when $[\Omega]$ is a complete differential; i.e., $d[\Omega] = 0$. This notion therefore depends essentially upon choice of arbitrary functions $A_{i\alpha, j\beta}$, etc., in (2.2): A field that is geodesic with respect to *one* form Ω is not generally geodesic for another form. A μ-surface E is said to be *embedded* in a field when it is an integral of the associated equations $\omega_i = 0$; thus, these equations need to be integrable only along this surface.

If E is embedded in a geodesic field and $\bar{E}$ is a second surface with the same boundary that moves in the field then from Stokes's theorem, one has:

$$J_E = \int_E [\Omega] = \int_{\bar{E}} [\Omega]$$

and therefore:

(5.1) $$J_{\bar{E}} - J_E = \int_{\bar{E}} \{ f(dt) - [\Omega] \} = \int_{\bar{E}} \mathcal{E}(dt)\,.$$

In this, we have (obviously, one always needs to write $(\bar{p}_{i\alpha} - p_{i\alpha})dt_\alpha$ instead of ω_i for Ω):

(5.2) $$\mathcal{E}(t_\alpha, x_i, p_{i\alpha}, \bar{p}_{i\alpha}) = \bar{f} - f - (\bar{p}_{i\alpha} - p_{i\alpha}) f_{p_{i\alpha}} - \\ -\tfrac{1}{2} A_{i\alpha, j\beta} (\bar{p}_{i\alpha} - p_{i\alpha})(\bar{p}_{j\beta} - p_{j\beta}) - \ldots$$ [17]).

It is clear that:

(5.3) $\mathcal{E} > 0$ for the $(t_\alpha, x_i, p_{i\alpha})$ of the field and all $\bar{p}_{i\alpha}$

represents a sufficient condition for a (strong) minimum in the event that one has embedded E in a geodesic field. For a weak minimum ($\bar{p}_{i\alpha} - p_{i\alpha}$ sufficiently small), it is sufficient that the $\mathcal{E}$-function is positive for $\bar{p}_{i\alpha}$ close to $p_{i\alpha}$. For this, the quadratic terms in the development of $\mathcal{E}$ in powers of $\bar{p}_{i\alpha} - p_{i\alpha}$ (in the absence of absolute and linear terms) defines a positive definite form. This "sufficient condition of Hadamard" reads:

(5.4) $$(f_{p_{i\alpha} p_{j\beta}} - A_{i\alpha, j\beta})\, u_{i\alpha} u_{j\beta} > 0.$$

The meaning of the fact that there is an arbitrariness in the choice of $A_{i\alpha, j\beta}$ will be discussed below [18]).

---

[17]) In the last term of (5.2) and (5.4) one must sum over all four indices independently, and therefore one must assume that $A_{i\alpha, j\beta}$ is skew-symmetric, not only in the i and j, but also in the α and β, hence, for $\alpha > \beta$ they are defined by $A_{i\alpha, j\beta} = -A_{i\beta, j\alpha}$. One observes that due to this skew symmetry the additional terms in (5.4) are null as long as the matrix of the $u_{i\alpha}$ has rank one. Cf. (4.1)!

Up till now, we have said nothing about the Euler equations in this section. Since a system of sufficient conditions implies the satisfaction of all that is necessary, one now concludes: A surface that is embedded in a geodesic field and satisfies (5.4) is an extremal. From this, it follows, as computation shows: Any surface that is embedded in a geodesic field is an extremal. From the equations that follow from $d[\Omega] = 0$ [19]), in order to see this, one needs only to write down those things that the vanishing of the coefficients of $\omega_i(dt)$ demand. Thus, one observes that for any function φ of the $\mathfrak{R}_{\mu+n+\mu n}$ in the field, one has:

$$d[\varphi] = \frac{\partial[\varphi]}{\partial x_j} dx_j + \frac{\partial[\varphi]}{\partial t_\beta} dt_\beta$$
$$= \frac{\partial[\varphi]}{\partial x_j} d\omega_j + \left\{ \frac{\partial[\varphi]}{\partial t_\beta} + \frac{\partial[\varphi]}{\partial x_j} p_{j\beta} \right\} dt_\beta = \frac{\partial[\varphi]}{\partial x_j} \omega_j + \frac{d\varphi}{dt_\beta} dt_\beta ,$$

where $d\varphi / dt_\beta$ means, in general, only an abbreviation for the curly brackets, which refer to differentiation on an embedded surface. By the use of this formula, a similar computation to (2.4) gives the coefficients that we spoke:

$$f_{x_i} - \frac{df_{p_{i\alpha}}}{dt_\alpha} + \sum_{\alpha<\beta} A_{i\alpha,j\beta} \left(\frac{dp_{j\beta}}{dt_\alpha} - \frac{dp_{j\alpha}}{dt_\beta} \right).$$

From their vanishing, and due to integrability, it thus follows that one in fact has on an embedded surface:

$$f_{x_i} - \frac{df_{p_{i\alpha}}}{dt_\alpha} = 0.$$

Finally, we thus have the following theorem for the sufficient conditions that an extremal give a minimum for a fixed boundary:

1. *The Legendre condition* (5.4) *for the weak minimum and the Weierstrass condition* (5.3) *for a strong minimum that are associated with any* Ω.

---

[18]) At this point, let us mention the work of *R. Debever* (Les champs de Mayer dans le calcul de variations des intégrales multiples. Bull. Acad. Roy. Belg. v. **23** (1937), pp. 809-815), in which he showed that in order to construct an n-parameter family of extremals for an arbitrary field (i.e., a piece of $\mathfrak{R}_{\mu+n}$ that covers it simply), the $A_{i\alpha,j\beta}$ (only these; the higher terms can be null) can be chosen in such a manner that the field is geodesic with respect to *this* Ω. If (5.4) is satisfied with these $A_{i\alpha,j\beta}$ then one has a weak minimum. Thus, one does not need to construct a geodesic field, but simply to provide such a family of extremals and to compute the associated $A_{i\alpha,j\beta}$, which can come about by quadratures. However, all of this is valid only for a fixed boundary – cf., the following section.

[19]) In Lepage [6]), one finds them written out in the case $\mu = 2$.

2. *The possibility of embedding in a geodesic field that is associated with this* Ω [20]).

6. In the general theory of Lepage – and therefore the theory of De Donder-Weyl that one obtains from it when one sets all arbitrary functions to zero – that was developed in the previous section, the notion of transversality does not appear. However, we know that transversality is essential for moving boundaries. In fact, the sufficient conditions of § 5 are valid only for a fixed boundary; we shall not discuss the possibility of displacing the boundary for a complete differential in μ dimensions, in general.

In order to obtain sufficient conditions for a moving boundary, one must investigate how one is to obtain something like the "complete figure" of a simple integral. For this, a generalization of the Kneser transversality theorem must be true. (Cf., "*St.*" § 8) We ask whether this can be managed by a particular choice of Ω (i.e., of the arbitrary functions).

The geodesic field that is associated with this Ω must possess the following properties (for the time being, we speak only of such fields that are generally integrable – hence, ones that exist in an n-parameter family of extremals). There is a pointwise map from the individual extremals to each other such that associated regions always yield the same value for the integral J_E – or, what amounts to the same thing, $\int [\Omega]$. From that fact, the latter integral also possesses the same value for an arbitrary μ-surface piece F that lies in the field as for a piece of any of the field extremals that one obtains when one "projects" the points of F by means of this map of extremals. (For the sake of simplicity, we consider only those surface pieces that intersect each "projection ray" only once.)

The formulas that this requirement will imply demand that $f \neq 0$. From now on, we assume that $f > 0$ (at least, for the surface element of the field in question).

We introduce the parameters $\lambda_1, \ldots, \lambda_\mu$ on any of the field extremals. As a result of the map, the parameter representation carries over to the other extremals in such a way that associated points correspond to the same parameter values. One can, moreover, choose the parameter in such a manner that on the starting extremal – and thus on all other extremals and any surface piece in the field – the integral that we spoke of becomes equal to the volume of the surface in the "λ-plane":

$$\int [\Omega] = \int d\lambda_1 \ldots d\lambda_\mu . \tag{6.1}$$

For any point of the field, hence, as functions of $\mathfrak{R}_{\mu+n}$ (which we will assume possess continuous derivatives), the λ_α are defined by:

$$\lambda_\alpha = S_\alpha(t_\beta, x_j). \tag{6.2}$$

If one introduces them, along with the x_i , as Gaussian coordinates in the field and computes $[\Omega]$ in terms of these parameters then the form $[\Omega] - d\lambda_1 \ldots d\lambda_\mu$ gives the value zero when it is integrated over any surface piece; thus, one has:

$$[\Omega] = dS_1 \ldots dS_\mu . \tag{6.3}$$

[20]) The possibility of embedding in a geodesic field for a sufficiently small piece of an extremal has only been proved for the special cases that we shall mention later [3]) [5]). To determine whether a given piece of an extremal is "sufficiently small" is a matter for the theory of the second variation.

This means nothing more than the fact that $[\Omega]$ *is of class* μ (§ 1). Conversely, when $[\Omega]$ is of class μ, one can determine functions $S_1, \ldots, S_\mu$ such that (6.3) is valid, and the geodesic field satisfies all of our requirements.

Since $[\Omega]$ is a complete differential, from the conclusion of § 1, one also has that the *rank* of $[\Omega]$ is equal to μ, and this rank is not greater than that of Ω. Thus, if one can determine the arbitrary functions in Ω in such a manner that Ω has rank μ (hence, the product of μ Pfaff forms) then any geodesic field that is associated with this Ω will satisfy all of our demands and give sufficient conditions for problems with moving boundaries.

This is, in fact, possible. However, one arrives at the desired form Ω faster than by using (6.3) when one takes into account the following consequence:

We assume that in the field the $\mathcal{E}$-function is positive for the values of its arguments that are in question, resp., that the Legendre condition is satisfied. One then knows that *the n-surface* $S_\alpha = \lambda_\alpha$ *cuts the field extremals transversally.* In fact, we take a field extremal E and an arbitrary bounding domain R on it that is a closed $(\mu - 1)$-surface. The surface $S_\alpha = \lambda_\alpha$, which goes through the points of R defines a $(\mu - 1 + n)$-“tube”. Since $f > 0$, the surface piece considered will solve the minimum problem for any boundary conditions of the following form: R shall be moving on a manifold H that extends outside of the tube. H then contacts the tube along R from the outside and satisfies the necessary transversality condition of § 3, and with that, due to the arbitrariness of the normal direction of R on E, our assertion is proved.

This leads us to ultimately expect the following of the geodesic field (we do this independently of whether the $p_{i\alpha}$ belong to a family of extremals or not): We can give a family of surfaces (6.2) such that (6.3) is valid and the surfaces (6.2) cut the surface element $p_{i\alpha}$ transversally.

From (3.9), the last requirement can be expressed by the formula:

$$\frac{\partial S_\alpha}{\partial x_i} = P_{i\alpha}\frac{\partial S_\alpha}{\partial t_\beta}, \tag{6.4}$$

with the $P_{i\alpha}$ that were described in (3.8). Thus, one obtains, for $[\Omega]$:

$$\begin{aligned}
[\Omega] = dS_1 \ldots dS_\mu &= \prod_\alpha\left(\frac{\partial S_\alpha}{\partial t_\beta}dt_\beta + \frac{\partial S_\alpha}{\partial x_i}dx_i\right) \\
&= \prod_\alpha \frac{\partial S_\alpha}{\partial t_\beta}(dt_\beta + P_{i\beta}dx_i) = \frac{\partial(S_1,\cdots,S_\mu)}{\partial(t_1,\cdots,t_\mu)}\prod_\alpha(dt_\alpha + P_{i\alpha}dx_i) \\
&= \frac{\partial(S_1,\cdots,S_\mu)}{\partial(t_1,\cdots,t_\mu)}\prod_\alpha\left(dt_\alpha + \frac{\bar{a}_{\alpha\beta}}{a} f_{p_{i\beta}}dx_i\right) \\
&= \frac{\partial(S_1,\cdots,S_\mu)}{\partial(t_1,\cdots,t_\mu)}\prod_\alpha \frac{\bar{a}_{\alpha\beta}}{a}\left(a_{\rho\beta}dt_\rho + f_{p_{i\beta}}dx_i\right)
\end{aligned}$$

$$= \frac{\partial(S_1,\cdots,S_\mu)}{\partial(t_1,\cdots,t_\mu)} \frac{1}{a} \prod_\alpha \left(f\, dt_\alpha + f_{p_{i\alpha}} \omega_i \right).$$

(We actually must set square brackets around all expressions in f and its derivatives.)

Equating the coefficients of $dt_1 \dots dt_\mu$ here and in (2.2) shows that we must have [21]):

(6.5)
$$\frac{\partial(S_1,\cdots,S_\mu)}{\partial(t_1,\cdots,t_\mu)} = \frac{[a]}{[f^{\mu-1}]}.$$

We heuristically write down the form:

(6.6)
$$\Omega = \frac{1}{f^{\mu-1}} (f\, dt_1 + f_{p_{i1}} \omega_i) \cdots (f\, dt_\alpha + f_{p_{i\alpha}} \omega_i).$$

One also has the correct second term of (2.2), which is of rank μ, and thus we have found the form that we need.

The geodesic fields that belong to this Ω are the ones that Carathéodory introduced [2]). Since we have introduced transversality independently of this, we also define them as follows: A field is called geodesic when it completes the surface element that is transversal to it into a family of n-surfaces (6.2), such that (6.4) and (6.5) are true.

Such a geodesic field defines a "complete figure" in the Carathéodory sense when it is part of a family of extremals. As for the case of a fixed boundary, in general problems one needs, however, no complete figure whatsoever in order to obtain sufficient conditions for a minimum. Furthermore, the equations $\omega_i = 0$ obviously need to be integrable along the extremal under scrutiny that one has embedded in the field.

We write down the values of the $A_{i\alpha,\, j\beta}$ that belong to the form (6.6) that we found, which indeed appear in the Legendre condition (5.4) alone:

(6.7)
$$A_{i\alpha,\, j\beta} = \frac{1}{f} \left(f_{p_{i\alpha}} f_{p_{j\beta}} - f_{p_{i\beta}} f_{p_{j\alpha}} \right).$$

One also easily computes the associated $\mathcal{E}$-function. As before, one must compute only the coefficients of dt in $\overline{f}\, dt - \Omega$, where one has replaced ω_i with $(\overline{p}_{i\alpha} - p_{i\alpha})\, dt_\alpha$, this time, in the expression (6.6) for Ω. One finds:

---

[21]) In connection with his Legendre transformation, Carathéodory [2]) set:

$$\frac{f^{\mu-1}}{a} = F(t\alpha, x_i, P_{i\alpha});$$

one then has:

$$F \cdot \frac{\partial(S_1,\cdots,S_\mu)}{\partial(t_1,\cdots,t_\mu)} = 1,$$

together with (6.4), as the Hamilton-Jacobi equation.

(6.8)
$$\mathcal{E}(t_\alpha, x_i, p_{i\alpha}, \bar{p}_{i\alpha}) = \bar{f} - \frac{1}{f^{\mu-1}}\left|\delta_{\alpha\beta} f + (\bar{p}_{i\alpha} - p_{i\alpha}) f_{p_{i\beta}}\right|.$$

A theorem of sufficient conditions for a problem with moving boundary is therefore the following one: The extremal piece shall be sufficiently small [20]) that a geodesic field that belongs to the form (6.6) can be embedded; it shall satisfy the Legendre condition:

(6.9)
$$\left| f_{p_{i\alpha}p_{j\beta}} - \frac{1}{f}(f_{p_{i\alpha}p_{j\beta}} - f_{p_{i\beta}p_{j\alpha}})\right| u_{i\alpha}u_{j\beta} > 0,$$

resp. (for a strong minimum), so the $\mathcal{E}$-function shall be positive. Furthermore, the transversality condition for the boundary shall be satisfied, along with a "second order transversality condition" that one can, since $f > 0$, formulate as follows: It shall be possible to choose the geodesic field in such a way that the manifold on which the boundary moves contacts the "tube" of geodesic transversals that goes through the boundary *from the outside.*

If the Weierstrass formula (5.1) is now valid for all comparison surfaces whose boundary lies on this selfsame tube, and which is then further added to it, is positive, since $f > 0$. For the weak minimum, it suffices for f to be assumed positive for the surface element of the field.

7. The Legendre condition and the notion of regularity. For a simple integral, one is careful to call a line element *regular* when it satisfies the Legendre condition with the > sign. Carathéodory called a curve "extremal" only when each of its points had the property: When one varies the curve in a sufficiently small interval that includes the points, one increases the value of the integral. Such a curve must obviously satisfy the Euler equations, as well as the Legendre necessary conditions (with the ≥ sign). It is sufficient that it satisfy the Euler equation and possess nothing but regular line elements; one can always embed a sufficiently small piece of such a curve in a geodesic field.

For multiple integrals things are significantly more complicated. One can now think of each surface element as being called regular when it satisfies the Hadamard condition (4.1) with the > sign. However, with this definition one only guarantees that one can also really pass an extremal in the Carathéodory sense through each regular surface element, when it would follow from (4.1) that one can determine the skew-symmetric additional term, hence, the arbitrary functions $A_{i\alpha,j\beta}$, such that (5.4) is valid [22]).

This is by no means always the case. Moreover, it is only true as long $\mu \leq 2$ or $n \leq 2$; for $\mu > 2$, $n > 2$, it is, on the contrary, false [23]).

---

[22]) Assuming the possibility of embedding in a geodesic field, which has only been proved for $A_{i\alpha,j\beta} = 0$ [5]) and the quantities (6.7) [3]).

[23]) *F. J. Terpstra,* Die Darstellung biquadratischer Formen als Summen von Quadraten mit Anwendung auf die Variationsrechnung. Math. Annalen **116** (1939), pp. 166-180. Cf., also *P. Finsler,* Über eine Klasse algebraischer Gebilde (Freigebilde). Comm. math. Helv. **9** (1937), pp. 172-187. Über das Vorkommen definiter und semidefiniter Formen in Scharen quadratischer Formen. Comm. math. Helv. **9** (1937), pp. 187-191. The investigations of Finsler were, in any case, originally proposed by Carathéodory for the aforementioned purpose.

The aforementioned requirement would be satisfied if one called a surface element regular only when there are quantities $A_{i\alpha,\,j\beta}$ such that (5.4) is valid. This definition is, however, quite complicated. It already opens a gap between "necessary" and "sufficient," and ultimately we may, as § 6 and the example of § 8 will show, not even expect that the regular extremals in this sense are also solutions of the problem for general boundary conditions.

Here, one draws one's attention to a shortcoming of the Carathéodory definition of extremal, which, of course, nowhere makes an appearance for simply integrals: It is, to a certain extent, only associated with fixed-boundary problems [24]). It thus seems divorced from multiple integrals to make a similar definition of distance; I have myself therefore always avoided the otherwise conventional terminology of calling any solution of the Euler equation an extremal.

Furthermore, when one does not wish to introduce any notion of regularity whatsoever it seems to be most expedient, in the sense of § 6, to define: A surface element is called regular when it satisfies the Legendre condition (6.9). This comparatively simple definition gives sufficiency at least in each case [25]). One must, however, be clear on one thing: many solutions to many problems – certainly for moving boundary problems – are not expected to be regular extremals, in this sense.

8. The following example shall show that there actually is a perfectly simple case in which the Hadamard sufficient condition *does not* suffice because the boundary is moving.

Let $\mu = 2$ and $n = 2$. We call the independent variables s and t, the dependent ones x and y, and for the sake of greater clarity, we let the Greek indices vary through the symbols s and t, while the Latin ones run through x and y. The basic function of the variational problem is:

$$f = p_{xs}^2 + 2\,p_{xt}^2 + 2\,p_{ys}^2 + p_{yt}^2 \,.$$

Here, all 2-planes are extremals. We consider the plane:

$$(8.1) \qquad E: \qquad x = -\,t,\, y = 2s,$$

and the field that consists of all the planes that are parallel to this plane. This field is given by the formula:

$$p_{xs} = p_{yt} = 0, \qquad p_{xt} = -\,1, \qquad p_{ys} = 2.$$

---

[24]) In this context, Tonelli's notion of "estremante" (L. Tonelli, Fondamenti di calcolo delle variazioni, 2 Band, Bologna 1923) is far-reaching, since he, to a certain extent, was dealing with the boundary conditions of his era. It would, however, be very difficult for the problem that is treated here to present general theorems for the extremants that relate to the Legendre condition.

[25]) In a certain sense, one may indeed assert that (6.9), with the ≥ sign, is *necessary*. From [3]), the embedding in a geodesic field is also possible when this quadratic form is indefinite; only the fact that its determinant is non-vanishing is required. If one introduces an extremal for a surface element in this case then the "tube" bounds geodesic transversals that go through the boundary of a sufficiently small neighborhood of the element, and also a surface piece that neighbors this extremal piece and gives a small value to the integral. Cf., the example of § 8, where this consequence is correct "in the large."

The various possible theories with a fixed boundary are distinguished from each other by *only one* arbitrary coefficient in the form Ω:

$$A_{xs,\,yt} = -A_{xt,\,ys} = A_{yt,\,xs} = -A_{ys,\,xt}\,.$$

If one assumes that this function, like f, depends only upon the $p_{i\alpha}$ then [Ω] has constant coefficients in the field considered above, and this field is therefore geodesic relative to all of these Ω. However, we watch how it behaves for various Ω with respect to the Legendre condition and the $\mathcal{E}$-function.

One has:

$$\tfrac{1}{2} f_{p_{i\alpha} p_{j\beta}} u_{i\alpha} u_{j\beta} = u_{xs}^2 + 2u_{xt}^2 + 2u_{ys}^2 + u_{yt}^2\,.$$

Hence, not only is the condition (4.1) satisfied with the > sign, but also Hadamard's sufficient Legendre condition: In the family of quadratic forms there is one that is positive definite, namely, the De Donder-Weyl one with $A_{xs,\,yt} = 0$. We compute the associated $\mathcal{E}$-function for the field considered as:

$$\mathcal{E} = \bar{p}_{xs}^2 + 2(\bar{p}_{xt} + 1)^2 + 2(\bar{p}_{ys} - 2)^2 + \bar{p}_{yt}^2\,.$$

It is positive, not only in the neighborhood of $p_{i\alpha}$, but also for all $\bar{p}_{i\alpha} \neq p_{i\alpha}$. *Thus, our extremal gives a minimum for a fixed boundary.*

In order to examine the behavior for a moving boundary, we first compute the formula for transversality. One obtains for the $p_{i\alpha}$ of our field:

$$a_{ss} = -6, \qquad a_{st} = a_{ts} = 0, \quad a_{tt} = 6,\; a = -36;$$
$$P_{xs} = P_{yt} = 0, \qquad P_{xt} = -\tfrac{2}{3}, \qquad P_{ys} = -\tfrac{4}{3}\,.$$

Whereas the surface element of the field is spanned by the two vectors $(\delta_{\alpha\beta},\, p_{i\beta})$, hence, by:

(8.2a) $\qquad (1, 0, 0, 2) \quad$ and $\quad (0, 1, -1, 0),$

the transversal to it is determined by $(-P_{j\alpha},\, \delta_{ij})$, hence, by:

(8.2b) $\qquad (0, \tfrac{2}{3}, 1, 0) \quad$ and $\quad (\tfrac{4}{3}, 0, 0, 1).$

All four vectors are linearly independent.

It is very easy to write down two functions $S_1 = S$, $S_2 = T$ that satisfy the relations (6.4) and (6.5); e.g.:

$$S = 4x - 6t, \qquad T = \tfrac{4}{3}\,t - \tfrac{3}{5}\,s.$$

Now, between them we have a "complete figure"; it exists in the family of extremal planes:

$$x = x_0 - t, \qquad y = y_0 + 2s$$

and the family of planes that is transversal to it:

$$4x - 6t = \lambda, \qquad \tfrac{4}{3}t - \tfrac{3}{5}s = \mu\,.$$

One immediately computes that for the form (6.6) in our field, one has, in fact:

$$[\Omega] = -\tfrac{18}{5}\,ds\,dt + \tfrac{12}{5}\,ds\,dx - \tfrac{24}{5}\,dt\,dy + \tfrac{16}{5}\,dx\,dy = dS\,dT\,.$$

However, if one looks into the formula of the Carathéodory theory further then one sees that the sufficient conditions of § 6 for a minimum are not satisfied for a moving boundary; indeed, one can easily give boundary conditions such that the extremal (8.1) satisfies all of the transversality conditions and nevertheless is not a solution at all. This surface is not, in fact, regular in the sense of § 7.

In fact, from (6.7), one takes:

$$A_{xs,\,yt} = \frac{4}{f}(p_{xs}\,p_{yt} - p_{xt}\,p_{ys}),$$

hence, in our field, where $f = 10$, one has:

$$A_{xs,\,yt} = \tfrac{16}{5}\,.$$

The quadratic form reads:

$$u_{xs}^2 + 2u_{xt}^2 + 2u_{ys}^2 + u_{yt}^2 - \tfrac{16}{5}(u_{xs}u_{yt} - u_{xt}u_{ys})\,.$$

For $u_{xs} = u_{yt} = u$, $u_{xt} = u_{ys} = 0$ it is equal to:

$$-\tfrac{6}{5}u^2 < 0,$$

and for $u_{i\alpha} = \rho\,\lambda_\alpha$ it is naturally positive, hence, indefinite; its determinant is $\neq 0$. Appropriately, the $\mathcal{E}$-function:

$$\mathcal{E} = \bar{p}_{xs}^2 + 2(\bar{p}_{xt} + 1)^2 + 2(\bar{p}_{ys} - 2)^2 + \bar{p}_{yt} - \tfrac{16}{5}\{\bar{p}_{xs}\bar{p}_{yt} - (\bar{p}_{xt} + 1)(\bar{p}_{ys} - 2)\}$$

is negative for the values that neighbor on $p_{i\alpha}$:

$$\bar{p}_{xs} = \bar{p}_{yt} = \varepsilon, \quad \bar{p}_{xt} = -1, \qquad \bar{p}_{ys} = 2,$$

namely:

$$\mathcal{E} = -\tfrac{6}{5}\varepsilon^2.$$

In order to simplify the expressions, we consider the affine transformation that takes the four vectors (8.2) to the four unit vectors of a $(\sigma, \tau, \xi, \eta)$-space. In this space, we consider coordinate plane σ, τ (which corresponds to the plane (8.1)) in which there is the circle $\sigma^2 + \tau^2 = 1$, and in space, it is the "cylinder" $\sigma^2 + \tau^2 = 1$. If we take as our boundary condition that the boundary curve shall lie on the cylinder that corresponds to it in the original space then the extremal (8.1) determines no minimum. If one takes the plane:

$$\bar{E} : x = \varepsilon s - t, \quad y = 2s + \varepsilon t$$

for a comparison surface then the difference $J_{\bar{E}} - J_E$ is an integral with the integrand $-\frac{6}{5}\varepsilon^2$, hence, negative.

We had not really expected an actual minimum at all, because we assumed that the boundary was moving on the tube of geodesic transversals. However, suppose we consider, instead of the cylinder, a hyperboloid:

$$\sigma^2 + \tau^2 - a(\xi^2 + \eta^2) = 1 \qquad (a > 0)$$

that contacts it from the outside along the circle, and we allow the boundary curve to move on the corresponding hyperboloid in the original space! The difference $J_{\bar{E}} - J_E$ will now be larger, and indeed adds to the former integral the integrand $10 + 2\varepsilon^2$ over an annular integration region whose volume, when developed in powers of ε, begins with a term of the form const $\cdot\, a \cdot \varepsilon^2$. One clearly needs to choose a to be sufficiently small in order to obtain a boundary condition on which the extremal (8.1) represents no solution of the problem, although all of the transversal conditions of § 6 are satisfied.

(Received on 1 May 1940.)

On the geodesic fields of multiple integrals

By Th. LEPAGE (*)

1. The perusal of two important memoirs of C. Caratheodory and H. Weyl ([1]) that establish the sufficient conditions for a weak or strong local extremum for a multiple integral whose value depends upon several functions has led me to observe ([2]) that on deeper analysis the difference between the results that were obtained by these authors stems from a simple algebraic fact: When n is greater than 1, any alternating form of degree $\mu > 1$ in $\mu + n$ indeterminates is, in general, of rank greater than μ (and similarly greater than $\mu + 1$).

The method that was followed by the authors cited is founded upon the determination of conditions that permit us to express the variation of the extremal integral by an integral that is extended over a portion of the surface being varied. It is thus the classical method of Weierstrass fields, but the geodesic fields of Caratheodory and Weyl exhibit profound differences. They are not, in general, generated by families of extremals, and the existence of transversal manifolds, which is preserved for Caratheodory fields, does not subsist for the Weyl fields. Besides these fields, one must mention the extremal fields that were considered long ago by Th. De Donder ([3]).

The method that I followed shows that the algebra of alternating differential forms and its integral aspect – that of facilitating multiple integrals – whose importance one recognizes in other domains of analysis, the theory of partial differential equations, and notably topology, may likewise be of service to the calculus of variations. All of this calculus is, in reality, nothing but a chapter in the analysis of alternating forms ([4]). In particular, upon recognizing the relationship between the notions of integrable form and geodesic field, I have been led to extend the definition of the field for multiple integrals. The existence of distinct Caratheodory and Weyl fields is obvious when one observes that the sum of two integrable forms is an integrable form. Moreover, in each case, it is found to be established by the presence of nondegenerate forms in a certain sheaf of ordinary quadratic forms. One may then introduce a system of canonical variables that permit, as in the cases that were studied before, the reduction of the problem of constructing a field that encompasses a portion of a given extremal to the integration of a first order partial differential equations that does not contain any unknown variables.

When $n > 1$, one knows that a family of extremals does not constitute a field for a simple integral. It is necessary, moreover, that a certain supplementary condition be satisfied (the nullity of the Lagrange brackets). On the contrary, for $\mu > 1$, $n > 1$ any family of extremal multiplicities is a geodesic field, but the converse is not true.

(*) Presented by L. Godeaux.

([1]) C. CARATHEODORY, *Variationsrechnung bei mehrfachen Integralen* (*Acta Szeged.* [1929]); H. WEYL, *Geodesic Fields* (*Ann. Math.*, [1935], pp. 607-629.).

([2]) Th. LEPAGE, *Sur les champs géodesiques du Calcul des Variations* (*Bull. Acad. Roy. Belg.* **22** [1936], pp. 716-729)

([3]) Th. DE DONDER, *Sur le théoreme d'independance d'Hilbert* (*C.R.*., Paris [1913], t. 156, pp. 609-611 and pp. 868-870.)

([4]) Th. DE DONDER, *Théorie des invariants integraux*, Paris, 1927; *Théorie invariantive du Calcul des Variations*, Paris, 1935.

Throughout this study, one is concerned with local problems, in the sense that one operates in a certain neighborhood of the initial values (contact point or element) in which certain conditions are found to be satisfied. Among these conditions, we mention the following ones: All of the functions envisioned are holomorphic, analytic, or, at the very least, continuously differentiable in the neighborhood of the values considered. These conditions are the ones that one habitually adopts in the theory of alternating differential forms. The problem of knowing to what degree the rules of the calculus of differential forms persist under conditions that are less restrictive than these is yet to be resolved. However, several results that have been obtained along this path justify the consideration of more general geodesic fields. This point of view is closely related to the work of Haar on certain systems of first order partial derivatives that generalize the Cauchy-Riemann equations.

Among all of the possible fields, the Caratheodory fields occupy a privileged place, which essentially stems from the fact that they are found to be defined by integrable forms of minimum class. For several reasons, they are presented as the most natural generalization of the fields that are studied in the ordinary case ($\mu = 1$, $n \geq 1$ and $\mu > 1$, $n = 1$). The existence of these fields, when incorporated in an extremal field, which was originally established by Börner ([5]), has been recently established by E. Hölder ([6]) by an elegant method that is founded upon the theory of contact transformations. Likewise, this result may be deduced from certain properties of integrable forms.

2. First of all, we recall several properties of integrable forms and indicate the notational conventions that shall adopt.

The Greek indices α, β, γ vary from 1 to μ and the Latin ones i, j, k vary from 1 to n. The function $\mathcal{L}(t_\alpha, x_i, x_{i\alpha})$ denotes a function of the $\mu + n + n\mu$ variables t_α, x_i, $x_{i\alpha}$, which are holomorphic in a certain neighborhood of a point $p_0(t_\alpha^0, x_i^0)$ in a space of $n + \mu$ dimensions, and for any system of finite values that are attributed to the variables $x_{i\alpha}$. e_0 denotes the set consisting of a point p_0 and a system $x_{i\alpha}^0$ of values for $x_{i\alpha}$. The symbols ω_i will denote n linear forms in the dx_i, dt_α, namely:

$$\omega_i = dx_i - x_{i\alpha}\, dt_\alpha, \tag{2.1}$$

in the left-hand of which, one adopts the usual convention that relates to the summation over repeated indices. The symbols Ω, Π denote alternating forms in the differentials dt_α, dx_i. Their coefficients will be holomorphic in a neighborhood of the element e_0.

These forms are the elements of a hypercomplex system – or *algebra* – that is constructed over a linear – or *vector* – space of linear forms in the dt_α, dx_i whose coefficients are holomorphic in e_0. In this algebra, addition and multiplication satisfy the usual rules of associativity and distributivity; addition is commutative, but multiplication is alternating. If ξ and η denote linear forms then, by virtue of distributivity, their product is obtained from following the rule for an ordinary product, but agreeing that:

([5]) H. BÖRNER, *Math. Ann.* **112** (1936).
([6]) E. HÖLDER, *Jahr. der deutsch. Math. Ver.* (1939).

$$dt_\alpha\, dx_i + dx_i\, dt_\alpha = dx_i\, dx_j + dx_j\, dx_i = dt_\alpha\, dt_\beta + dt_\beta\, dt_\alpha = 0,$$

from which, it results that:

$$\xi\eta + \eta\xi = 0.$$

Any element of this algebra is a linear combination with holomorphic coefficients of the products:

(2.2) $$dt_{\alpha_1} \cdots dt_{\alpha_k} dx_{i_1} \cdots dx_{i_p} .$$

Two elements are equal, up to sign, if they are obtained from each other by a permutation of the factors dt_α, dx_i . Any monomial is null whenever it contains two identical factors, from which it results that its degree is $k + p$, i.e., the number of its factors dt_α, dx_i is at most equal to $\mu + n$. The algebra thus possesses a *basis* that is composed of a unity element and all of the monomials:

(2.3) $$\begin{cases} dt_{\alpha_1} \cdots dt_{\alpha_k} dx_{i_1} \cdots dx_{i_p}, & 1 \le p+k \le \mu+n, \\ \alpha_1 < \alpha_2 < \cdots < \alpha_k, & i_1 < i_2 < \cdots < i_p, \\ \alpha = 1, 2, \ldots, \mu, & i = 1, 2, \ldots, n. \end{cases}$$

By definition, a form Ω of degree p is a sum of monomials with holomorphic coefficients that are also of degree p. If Ω and Π denote two forms of degree p and q, respectively, then we have the following rule for their product:

(2.4) $$\Omega\Pi = (-1)^{pq}\, \Pi\Omega.$$

All of the monomials (2.3) for the basis (dt_α, dx_i) are linearly independent, which is an immediate consequence of the product law and the fact that the monomial $dt_1 \ldots dt_\mu$ $dx_1 \ldots dx_n$ of degree $n + \mu$ is non-null. It results from this that a form of degree p, $1 \le p \le n$ cannot be null unless all of the coefficients of the distinct monomials are null.

We say that a form is *normal* when all of its terms are distinct monomials and the coefficient of an arbitrary monomial is antisymmetric with respect to the indices that appear in the expression for the monomial. Obviously, this mode of representation is always possible, and is unique. In what follows, we shall denote a form of degree μ by Ω. Since it will be assumed to be normal, it is written in the form:

(2.5) $$A_{12\ldots\mu}\, dt_1 \ldots dt_\mu + A_{12\ldots\alpha-1,\, i,\, \alpha+1,\ldots,\mu}\, dt_1 \ldots dt_{\alpha-1}\, dx_i\, dt_{\alpha+1} \ldots dt_\mu + \ldots$$

$$A_{12\ldots\alpha_1-1, i_1, \alpha_1+1, \ldots, \alpha_2-1, i_2, \alpha_2+1, \ldots} dt_1 \cdots dt_{\alpha_1-1} dx_{i_1} dt_{\alpha_1+1} \cdots dt_{\alpha_1-2} dx_{i_2} dt_{\alpha_1+2} \cdots + \ldots,$$

in which the A coefficients are antisymmetric in the indices, summation is over all of the combinations of the $n + \mu$ indices taken μ at a time, and $\alpha = 1, 2, \ldots, \mu$, $i = 1, 2, \ldots, n$. Therefore, any substitution of the symmetric group of degree $n + \mu$ leaves each term unchanged, and as a result, the form Ω.

Two normal forms are equal when the coefficients of their corresponding terms are equal. To simplify, the right-hand side of (2.5) may be written:

(2.6) $A\, dt_1 \dots dt_\mu + A_{i\alpha}(\alpha - 1,\, dx_i,\, \alpha + 1) + \dots + A_{\alpha_1 i_1, \alpha_2 i_2 \dots}(\alpha_1 - 1, dx_{i_1}, \alpha_1 + 1, \cdots) + \dots$

Order Ω in terms of distinct monomials of degree $\mu - 1$. Their coefficients are linear forms, the set of which goes by the name of the *associated system* for the form Ω. The number of linearly independent forms in the associated system is called the *rank* of Ω. It is the minimum number of linear forms with which it is possible to express the form; it may not be less than μ nor equal to $\mu + 1$. If it is equal to μ then the form is called *simple*, and in this case Ω is, up to a factor, the product of the μ linearly independent forms of the associated system. If $A \neq 0$ then the associated system is:

(2.7) $$\pi_\alpha = A\, dt_\alpha + A_{\alpha i}\, dx_i\,,$$

and one will have:

(2.8) $$\Omega = \frac{1}{A^{\mu-1}} \pi_1 \pi_2 \cdots \pi_\mu\,.$$

In order for a form to be simple, it is necessary and sufficient that the coefficients A of the normal expression verify the quadratic relations $g\binom{n+\mu}{\mu} = 0$ that one obtains by annulling the product $\Omega \cdot \pi$, in which the factor π denotes an arbitrary form in the associated system. Upon considering the A to be the homogeneous coordinates of a linear space of dimension $\binom{n+\mu}{\mu} - 1$, the equations $g\binom{n+\mu}{\mu} = 0$ define a $\mu \cdot n$-dimensional homogeneous rational variety; hence, it is devoid of singular points. It may not be the complete intersection of $\binom{n+\mu}{\mu} - 1 - n\mu$ hypersurfaces, except when $\mu = 2$, $n = 2$, when the variety corresponds to the lines in a 3-dimensional space (the Plücker-Klein hyperquadric). The variety $g\binom{n+\mu}{\mu} = 0$ admits a transitive continuous group of projective transformations into itself that corresponds to the group of transformations of the basis (dt_α, dx_i).

3. If the A are continuously differentiable functions of t_α and x_i then the differential of the form Ω is the form of degree $\mu + 1$:

(3.1) $$d\Omega = dA\, dt_1 \dots dt_\mu + dA_{\alpha i}(\alpha - 1,\, dx_i\,,\, \alpha + 1) + \dots$$

that one deduces from Ω by differentiating all of the A coefficients and then developing the products $dA\, dt_1 \dots dt_\mu$, etc. If it is normal for the monomials of degree $\mu + 1$ of the basis (2.3) then $d\Omega$ may be written:

$$d\Omega = \Omega_i\, dt_1 \dots dt_\mu\, dx_i + \Omega_{i\alpha}\, dt_1 \dots dt_{\alpha-1} \dots dt_\mu\, dx_i\, dx_j + \dots, \tag{3.2}$$

in which the Ω_i, $\Omega_{i\alpha}$, … denote homogeneous linear expressions in the partial derivatives of the A with respect to the t_α and x_i.

A form Ω is called *integrable* when its differential is null; therefore, when all of the coefficients Ω_i, $\Omega_{i\alpha}$ are null.

In the theory of multiple integrals the integrable forms of degree $\mu > 1$ enjoy properties that are analogous to those of total differentials in the theory of curvilinear integrals. These expressions occur notably in the generalization of the classical formulae of Green and Stokes for ordinary spaces. Let $V_{\mu+1}$ be a μ+1-dimensional manifold in the space of t_α, x_i ; i.e., a sum of continuously differentiable images of Euclidian simplexes; the integral of $d\Omega$ over $V_{\mu+1}$ then possesses a well-defined value. If we denote the frontier of $V_{\mu+1}$ by $f(V_{\mu+1})$, which is therefore the algebraic sum of the frontiers of the simplexes of $V_{\mu+1}$, then the formula that generalizes that of Stokes is:

$$\int_{f(V_{\mu+1})} \Omega = \pm \int_{V_{\mu+1}} d\Omega\,,$$

in which the sign is fixed by the convention that one makes for the orientation of the frontier.

If Ω is integrable then one will have:

$$\int_{V_\mu} \Omega = 0$$

for any closed manifold V_μ that is the frontier of a $V_{\mu+1}$ that is completely contained in a domain around the point (t_α^0, x_i^0). Under the same conditions, one establishes the existence of a *primitive form* for Ω; i.e., a form Π of degree $\mu - 1$ with holomorphic coefficients in the domain of the point (t_α^0, x_i^0) such that:

$$d\Pi = \Omega.$$

Furthermore, this primitive form is not completely determined. Indeed, under the conditions that we are imposing (viz., there is a neighborhood of the point that is homeomorphic to a Euclidian space) there is an identity between the integrable forms and the *exact differentials*; i.e., the forms of degree μ that are differentials of forms of degree $\mu - 1$. Therefore, the two relations:

$$d\Pi = \Omega, \qquad d\Omega = 0$$

are entirely equivalent. Upon normalizing the left-hand sides we thus obtain two systems of relations between the coefficients, the second of which expresses the compatibility conditions for the first system, in which the coefficients of the form Π are regarded as unknown functions.

In order to construct an integrable form Ω of degree μ it will therefore suffice to take a form Π of degree $\mu - 1$ with holomorphic coefficients and set $d\Pi = \Omega$. We give two simple examples that will be useful in the sequel. Let $S_1, S_2, \ldots, S_\mu$ be μ functions of x_i, t_α, and let:

$$\Pi_1 = S_1\, dS_2 \ldots dS_\mu, \qquad \Pi_2 = S_\alpha\, dt_1 \ldots dt_{\alpha-1}\, dt_{\alpha+1} \ldots dt_\mu$$

be forms of degree $\mu - 1$ that give the following two integrable forms:

$$\Omega_1 = dS_1\, dS_2 \ldots dS_\mu, \qquad \Omega_2 = \frac{\partial S_\alpha}{\partial t_\alpha} dt_1 \cdots dt_\mu + \frac{\partial S_\alpha}{\partial x_i}(\alpha - 1, dx_i, \alpha + 1),$$

respectively.

The rank of an integrable form depends upon the class. The associated system is a completely integrable Pfaff system whose integration amounts to looking for an expression that contains the minimum number of variables; this number is equal to the class, moreover.

4. One establishes without difficulty that all of these results are independent of the system of coordinates that was adopted to frame the neighborhood of the point p_0. This is a property that is especially important in the study of the integrable forms that are attached to a manifold, in which one may introduce local systems of coordinates ([7]).

For our purposes, it will be useful to study the effect of a change of basis. For the basis (2.3) that is defined by the dt_α, dx_i we substitute the equivalent basis that is generated by the $\mu + n$ linear forms:

$$dt_\alpha, \qquad \omega_i = dx_i - x_{i\alpha}\, dt_\alpha, \tag{4.1}$$

in which the $x_{i\alpha}$ denote $n\mu$ new variables. If we denote the forms that are obtained from (2.6) when it is normalized according to the bases (2.3) and (4.1) by $\Omega(dx)$ and $\Omega(\omega)$, respectively, then we have:

$$\Omega(dx) = \Omega(\omega) = D\, dt_1 \ldots dt_\mu + D_{i\alpha}(\alpha - 1, \omega_i, \alpha + 1) + \ldots \tag{4.2}$$

in which:

$$D_{i\alpha} = \frac{\partial D}{\partial x_{i\alpha}},$$

and:

([7]) G. DE RHAM, *Thése*, Paris (1931), and *Abh. Math. Sem. Hans. Univ.* (1938).

(4.3) $$D = A + A_{i\alpha} \cdot x_{i\alpha} + A_{i\alpha j\beta} \begin{bmatrix} x_{i\alpha} & x_{i\beta} \\ x_{j\alpha} & x_{j\beta} \end{bmatrix} + \dots$$

D is a function whose coefficients A are linear in the minors, and all have the same order as the matrix $(x_{i\alpha})$. Obviously, if $\mu = 1$ or $n = 1$ then (4.2) reduces to two primary groups of terms. In this case, Ω has rank μ, and the function D is of first degree in the $x_{i\alpha}$. In a general manner, if Ω has rank μ then we have, upon normalizing the expression (2.8) and supposing that that the element e_0 is such that $D(e_0) \neq 0$:

(4.4) $$\Omega = D^{1-\mu}\, \xi_1\, \xi_2 \dots \xi_\mu ,$$

upon setting:

(4.5) $$\begin{cases} \xi_\alpha = D\, dt_\alpha - D_{i\alpha}\omega_i = (\delta_{\alpha\beta} D - D \cdot x_{i\beta}) dt_\beta + D_{i\alpha} \cdot dx_i , \\ \delta_{\alpha\beta} = 0 \text{ for } \alpha \neq \beta \text{ and } \delta_{\alpha\alpha} = 1. \end{cases}$$

We further set:

(4.6) $a_{\alpha\beta} = \delta_{\alpha\beta}\, D - D_{i\alpha}\, x_{i\beta}$ and a = determinant of $a_{\alpha\beta}$.

Upon comparing (2.8) with (4.4) one then obtains:

$$a(e_0) - A\, D^{\mu-1}(e_0) \neq 0.$$

We further observe that no matter what the rank of Ω, and for any value that is attributed to $x_{i\alpha}$, one has the two congruences:

(4.7) $$\Omega \equiv D\, dt_1 \dots dt_\mu , \qquad d\Omega \equiv 0 \qquad (\text{modulo } \omega_i).$$

On the other hand, it is easy to obtain the general solution to this system of congruences. It is given by the family of forms $\{\Omega\}$ that take the form:

(4.8) $$\Omega_1 + \lambda_{i\alpha j\beta}(\alpha - 1,\, \omega_i ,\, \alpha + 1;\, \beta - 1,\, \omega_j ,\, \beta + 1) + \lambda_{i_1\alpha_1 i_2\alpha_2\cdots}(\alpha_1 - 1, \omega_i, \alpha_1 + 1, \cdots) + \dots$$

in which we have set:

(4.9) $$\Omega_1 = D\, dt_1 \dots dt_\mu + D_{i\alpha}(\alpha - 1,\, \omega_i ,\, \alpha + 1),$$

and the λ denote arbitrary quantities that are holomorphic in a neighborhood of the element e_0 .

Consider a form of the family $\{\Omega\}$, and normalize it according to the bases (2.3) and (4.1), respectively. Between the systems of coefficients of the two expressions there exist very simple relations that stem from the fact that the unimodular substitution $(dt_\alpha , dx_i) \to (dt_\alpha, \omega_i)$ transforms the one into the other homographically. In the case where the form considered has minimum rank. Hence, when the coefficients of any $x_{i\alpha}$ are the homogeneous coordinates of a point of the rational variety $g\begin{pmatrix} n+\mu \\ \mu \end{pmatrix} = 0$ one obtains a remarkable result:

First of all, observe that no matter what the rank of the form (2.6) the family $\{\Omega\}$ contains a unique form of rank μ. We suppose that the element e_0 satisfies the following two conditions:

$$(4.10) \qquad D(e_0) \neq 0, \qquad\qquad a(e_0) \neq 0.$$

The second condition implies that in a neighborhood of e_0 the associated system to any form of the family $\{\Omega\}$ is composed of at least μ linearly independent forms:

$$(4.11) \qquad \zeta_\alpha = D\, dt_\alpha + D_{i\alpha}\, \omega_i = a_{\alpha\beta}\, dt_\beta + D_{i\alpha}\, dx_i\, .$$

On the other hand, since $D(e_0) \neq 0$, the form:

$$(4.12) \qquad D^{1-\mu} \cdot \prod_{\alpha=1}^{\mu} (D\, dt_\alpha + D_{i\alpha}\omega_i)$$

belongs to $\{\Omega\}$, and it has rank μ; it is the unique form of this rank. Indeed, any form that has rank μ and belongs to $\{\Omega\}$ is, up to a factor, the product of μ forms (4.11), and this factor is, as one easily sees, $D^{1-\mu}$.

5. The condition $a(e_0) \neq 0$ permits us to express the form (4.12), when normalized according to the basis (dt_α, dx_i), in a very simple form. This has been done above for the form (2.6) under the hypothesis that the latter has minimum rank precisely, with the condition that $A \neq 0$. Consider, in a general fashion, a form of rank μ:

$$(5.1) \qquad f^{1-\mu} \cdot \prod_{\alpha=1}^{\mu} (f\, dt_\alpha + \pi_{i\alpha}\omega_i)\, , \qquad \text{with} \qquad \omega_i = dx_i - p_{i\alpha}\, dt_\alpha\, ,$$

in which $f \neq 0$ and the $p_{i\alpha}$ denote quantities that make the determinant:

$$(5.2) \qquad a = |\, \delta_{\alpha\beta} f - \pi_{i\alpha}\, p_{i\beta}\, |$$

different from zero. Normalizing according to the basis (dt_α, dx_i), it is obviously a form of rank μ, which may be written as:

$$(5.3) \qquad \frac{1}{F} \prod_{\alpha=1}^{\mu} (dt_\alpha + P_{i\alpha} dx_i)\, ,$$

with:

$$(5.4) \qquad F = f^{\mu-1} a, \qquad\qquad P_{i\alpha} = \frac{1}{a} \pi_{i\beta} \bar{a}_{\alpha\beta}\, ,$$

in which $\bar{a}_{\alpha\beta}$ denotes the algebraic complement of $a_{\alpha\beta}$ in the determinant a. Next, introduce the quantities:

(5.5) $$\varphi, \Phi, \Pi_{i\alpha}$$

by means of the defining relations:

(5.6) $$f + \varphi = p_{i\alpha} \cdot \pi_{i\alpha}, \qquad \Pi_{i\sigma} = \frac{f^{\mu-2}}{a} p_{i\beta} \cdot a_{\alpha\beta}, \qquad F + \Phi = P_{i\alpha} \cdot \Pi_{i\alpha}.$$

These two systems of quantities have been considered before by Caratheodory ([8]), who showed that one passes from one to the other by a rational involutive contact transformation, namely:

(5.7) $$\frac{f}{F} = \frac{\varphi}{\Phi} = \frac{F^{\mu-2}}{A}, \qquad p_{i\alpha} = \frac{1}{A}\Pi_{i\beta}\bar{A}_{\alpha\beta}, \qquad \pi_{i\alpha} = \frac{F^{\mu-1}}{A} P_{i\beta} A_{\alpha\beta},$$

in which A denotes the determinant of $\delta_{\alpha\beta} F - P_{i\alpha} \cdot \Pi_{i\alpha}$. Moreover, one verifies the relation:

(5.8) $$F(df + \pi_{i\beta}\, dp_{i\beta}) + f(dF + \Pi_{i\beta}\, dP_{i\beta}) = 0 .$$

6. Let $\mathcal{L}(t_\alpha, x_i, x_{i\alpha})$ be a holomorphic function of the t_α, x_i, $x_{i\alpha}$ (sec. 2), and consider the family $\{\Omega\}$ of forms in the dt_α, dx_i such that:

(6.1) $$\Omega \equiv \mathcal{L}\, dt_1 \dots dt_\mu, \qquad d\Omega \equiv 0 \qquad (\text{modulo } \omega_i).$$

A simple calculation gives the general solution:

(6.2) $$\begin{cases} \Omega = \mathcal{L}\, dt_1 \cdots dt_\mu + \mathcal{L}_{i\alpha}(\alpha - 1, \omega_i, \alpha + 1) + \tau, \\ \mathcal{L}_{i\alpha} = \dfrac{\partial \mathcal{L}}{\partial x_{i\alpha}}, \end{cases}$$

in which τ denotes the most general form of degree μ such that each monomial term in dt_α, dx_i contains at least two factors ω_i. One thus has:

(6.3) $$\tau = \lambda_{ij,\alpha\beta}(\alpha-1, \omega_i, \alpha+1; \beta-1, \omega_j, \beta+1) + \lambda_{ijk,\alpha\beta\gamma}(\alpha-1, \omega_i, \alpha+1; \beta-1, \omega_j, \beta+1) + \dots,$$

in which the λ denote arbitrarily-chosen holomorphic functions.

If a form Ω in the family (6.2) is integrable then the condition $d\Omega = 0$ entails that the coefficients of $\Omega(dx)$, when normalized according to the basis (dt_α, dx_i), will all be independent of $x_{i\alpha}$. Therefore, in this case $\mathcal{L}$ is a function D that is linear in the minors

([8]) C. CARATHEODORY, *loc. cit.*, pp. (?) to (?). [page numbers missing in original].

of $(x_{i\alpha})$, and the coefficients A satisfy integrability conditions, moreover. For the forms, Ω_1 and Ω_2 of sec. 3 the corresponding functions D are:

$$D_1 = \frac{d(S_1 \cdots S_\mu)}{d(t_1 \cdots t_\mu)}, \qquad D_2 = \frac{dS_1}{dt_1} + \cdots + \frac{dS_\mu}{dt_\mu}, \tag{6.4}$$

respectively, with:

$$\frac{dS_\alpha}{dt_\alpha} = S_{\alpha\beta} + S_{\alpha i}\, x_{i\beta}\,, \qquad S_{\alpha\beta} = \frac{\partial S_\alpha}{\partial t_\beta}\,.$$

The general expression for a function D that corresponds to an integrable form of degree μ is obtained immediately by virtue of the identity $d\Pi = \Omega$. Any function D of this space will be called an *exact derivative*. This expression is justified when – considering the $x_{i\alpha}$ to be the derivatives $\partial x_i / \partial t_\alpha$ of the x_i, which are interpreted as functions of the t_α – one observes that $D(t_\alpha, x_i, \partial x_i / \partial t_\alpha)$ is a rational function that is completely composed of expressions such as $\partial S_\alpha / \partial t_\alpha$.

7. Now, suppose that the function $\mathcal{L}$ is not an exact derivative. Thus, the family $\{\Omega\}$ of the preceding section does not contain any integrable form. We suppose, in addition, that $\mu > 1$ and $n > 1$. Hence, the family $\{\Omega\}$ effectively contains indeterminate coefficients λ.

It may happen that when the $x_{i\alpha}$ in a certain form Ω are replaced by functions:

$$x_{i\alpha} = \psi_{i\alpha}(t_\alpha, x_i), \tag{7.1}$$

which are holomorphic in a neighborhood of a point $p_0(t_\alpha^0, x_i^0)$, the form becomes integrable. In this case, we say that the functional system (ψ) that is given by (7.1) defines a *geodesic field for the function* $\mathcal{L}$ in a certain neighborhood of the point p_0.

A geodesic field is thus found to be defined by the data of a matrix (ψ) and the coefficients λ which are holomorphic at p_0, these elements being such that the value of Ω, which we denote by $\Omega(\psi, \lambda)$, gives rise to the identity:

$$d\Omega(\psi, \lambda) = 0. \tag{7.2}$$

8. Since any integrable form is an exact differential (sec. 3), it is possible to write the differential equations for geodesic fields in two completely equivalent forms: The first one expresses the fact that the $\psi_{i\alpha}$ and the λ are functions of the t_α and the x_i, such that $\Omega(\psi, \lambda)$ possesses a null differential, and the second one expresses the fact that this same form is the differential of a certain holomorphic form of degree $\mu - 1$. On the one hand, on normalizing the expressions $d\Omega$ and $d\Pi - \Omega$, first according to the basis (dt_α, dx_i) and then according to the basis (dt_α, ω_i), one obtains two equivalent differential systems for defining geodesic fields.

Practically, it seems that the equations $d\Omega(\psi, \lambda) = 0$ are advantageous when it is a question of verifying that a system of functions ψ, λ define a field. This is what we shall do below in order to show that any family of extremals for the "variational problem $\mathcal{L}$" is a geodesic field. The equations $d\Pi = \Omega$ will be useful in the problem of determining the sufficient conditions for the existence of a field. When written in the basis (dt_α, dx_i), the equations take on a simple form. In that case, they reduce to only one (first order) partial differential equation in the coefficients of the primitive form Π. The results that one obtains in this way are a natural generalization of the theory of the integration of the canonical equations by the method of Hamilton and Jacobi.

9. The differential of the form (6.2), in which the $x_{i\alpha}$ and the λ are regarded as holomorphic functions of the t_α, x_i in a neighborhood of a point p_0, will be normalized according the basis (dt_α, ω_i). We denote the coefficients by Ω_i, $\Omega_{ij\alpha}$, $\Omega_{ijk\alpha\beta}$, etc. Ω_i is the coefficient of the monomial $(1, 2, \ldots, \mu, \omega_i)$, $\Omega_{ij\alpha}$ is the coefficient of the monomial $(1, 2, \ldots, \alpha - 1, \alpha + 1, \ldots, \mu, \omega_i \omega_j)$, and so on.

To fix these ideas, we suppose that $\mu = 2$ and $n > 1$. In this case, the form (6.3) reduces to its first term. All of what follows persists without modification in the general case, so it will suffice to limit the expression for τ to its first term.

We thus have the following expressions for $\Omega(\psi, \lambda)$ and $d\Omega(\psi, \lambda)$:

$$(9.1)\qquad \begin{cases} \Omega = \mathcal{L}dt_1 dt_2 + (\mathcal{L}_{i2}dt_1 - \mathcal{L}_{i1}dt_2)\omega_i + \lambda_{ij}\omega_i\omega_j, \\ d\Omega = \Omega_i(1,2,\omega_i) + \Omega_{\alpha ij}(\alpha, \omega_i, \omega_j) + \Omega_{ijk}(\omega_i, \omega_j, \omega_k), \end{cases}$$

in which we have set:

$$(9.2)\qquad \begin{cases} \Omega_i = \mathcal{L}_i - \dfrac{d}{dt_\alpha}\mathcal{L}_{i\alpha} + \lambda_{ij}\left(\dfrac{\partial \psi_{j1}}{\partial t_2} - \dfrac{\partial \psi_{j2}}{\partial t_1}\right), \\ \Omega_{\alpha ij} = \dfrac{\partial \lambda_{ij}}{\partial t_\alpha} + \psi_{k\alpha}\dfrac{\partial \lambda_{ij}}{\partial x_k} + \lambda_{kj}\dfrac{\partial \psi_{k\alpha}}{\partial x_i} + \lambda_{ik}\dfrac{\partial \psi_{k\alpha}}{\partial x_j} + \dfrac{\partial}{\partial x_j}\mathcal{L}_{i\alpha} - \dfrac{\partial}{\partial x_i}\mathcal{L}_{j\alpha}, \\ \Omega_{ijk} = \dfrac{\partial \lambda_{ij}}{\partial x_k} + \dfrac{\partial \lambda_{jk}}{\partial x_i} + \dfrac{\partial \lambda_{ki}}{\partial x_j}. \end{cases}$$

We will make the following hypotheses about the functions $\psi_{i\alpha}(t_\alpha, x_i)$: The functions ψ are holomorphic in a neighborhood of a point p_0, the system:

$$(9.3)\qquad \omega_i = dx_i - \psi_{i\alpha}\, dt_\alpha = 0$$

is completely integrable, and the expressions:

$$(9.4)\qquad \mathcal{L}_i - \frac{d}{dt_\alpha}\mathcal{L}_{i\alpha},$$

in which the $x_{i\alpha}$ are replaced with the $\psi_{i\alpha}$, are all null. This hypothesis amounts to assuming that for the system $(\psi_{i\alpha})$ all of the coefficients Ω_i are null for any arbitrary functions λ_{ij}. Therefore, the form:

(9.5) $$\Omega_{\alpha ij}(\alpha,\ \omega_i\omega_j) + \Omega_{ijk}(\omega_i\omega_j\omega_k)$$

is integrable for any arbitrary λ_{ij}.

We propose to look for a system of functions λ_{ij} that are holomorphic at p_0 and are such that:

(9.6) $$\Omega_{\alpha ij} = \Omega_{ijk} = 0.$$

For such a system of solutions, the matrix $(\psi_{i\alpha})$ will therefore define a geodesic field for the function $\mathcal{L}$.

First of all, observe that the equations $\Omega_{\alpha ij} = 0$ are linear in the λ_{ij} and their first order derivatives. If the system (9.6) is homogeneous, i.e., if all of the quantities:

(9.7) $$\frac{\partial}{\partial x_j}\mathcal{L}_{i\alpha} - \frac{\partial}{\partial x_i}\mathcal{L}_{j\alpha}$$

are null, then one can take $\lambda_{ij} = 0$, so the equations $\Omega_{ijk} = 0$ will obviously be verified, and the matrix (ψ) will define a field that makes the form:

(9.8) $$\mathcal{L}\, dt_1\, dt_2 + (\mathcal{L}_{i2}\, dt_1 - \mathcal{L}_{i1}\, dt_2)\omega_i$$

integrable.

We thus suppose that the system $\Omega_{\alpha ij} = 0$ is not homogeneous. The equations of this system are found to resolve into the derivatives of the λ with respect to the variables t_α. The right-hand sides are linear in the λ and their derivatives with respect to the x_i. This system is completely integrable, or in involution, in the sense of Riquier. This is a consequence of the following fact: For any λ_{ij} the form:

(9.9) $$\Pi = \Omega_{\alpha ij}(\alpha,\ \omega_i\omega_j) + \Omega_{ijk}(\omega_i\omega_j\omega_k)$$

is an exact differential. Indeed, upon developing the identity $d\Pi = 0$ one gets:

(9.10) $$d\Pi = \Pi_{12ij}(12,\ \omega_i\omega_j) + \Pi_{\alpha ijk}(\alpha\ \omega_i\omega_j\omega_k) + \Pi_{ijkl}(\omega_i\omega_j\omega_k\omega_l) = 0.$$

Hence:

$$\Pi_{12ij} = \ldots = \Pi_{ijkl} = 0.$$

In particular:

(9.11) $$\Pi_{12ij} = \frac{\partial\Omega_{2ij}}{\partial t_1} - \frac{\partial\Omega_{1ij}}{\partial t_2} - \Omega_{1jk}\frac{\partial\psi_{k2}}{\partial x_i} + \Omega_{2jk}\frac{\partial\psi_{k1}}{\partial x_i} + \Omega_{2ik}\frac{\partial_k\psi_1}{\partial x_i} - \Omega_{1ik} = 0,$$

in which the λ_{ij} denote arbitrarily chosen functions of the t_α and x_i . One remarks that the $\partial\Omega_{2ij}/\partial t_1 - \partial\Omega_{1ij}/\partial t_2$ are homogeneous linear combinations of the $\Omega_{\alpha ij}$.

Thanks to the identities (9.11), we shall establish the following proposition:

Let $\lambda_{ij}(x_i)$ be a skew-symmetric matrix of $n(n - 1)/2$ holomorphic functions in a neighborhood of the system (x_i^0). The equations $\Omega_{\alpha ij} = 0$ possess a unique system of solutions:

$$\lambda_{ij} = \Lambda_{ij}(t_\alpha, x_i), \tag{9.12}$$

which is holomorphic in a certain domain surrounding the point $p_0(t_\alpha^0, x_i^0)$, such that:

$$\Lambda_{ij}(t_\alpha^0, x_i) = \lambda_{ij}(x_i). \tag{9.13}$$

PROOF: - Arrange the equations $\Omega_{\alpha ij} = 0$ along the following lines:

$$\Omega_{1ij} = 0, \qquad \Omega_{2ij} = 0, \qquad i, j = 1, 2, \ldots, n. \tag{9.14}$$

Let $\lambda_{ij}^1(t_1, x_i)$ be the system of solutions to the equations $\Omega_{\alpha ij} = 0$, in which we have replaced t_2 with t_2^0, solutions that are subject to the constraint that they reduce to the functions $\lambda_{ij}(x_i)$ for $t_2 = t_2^0$. This system of solutions exists and is unique, by virtue of a classical theorem of Cauchy-Kowalewsky.

Now consider the system $\Omega_{2ij} = 0$, and let:

$$\lambda_{ij}^2(t_1, t_2, x_i) \equiv \Lambda_{ij}(t_\alpha, x_i)$$

denote the solution such that:

$$\lambda_{ij}^2(t_1, t_2^0, x_i) = \lambda_{ij}^1(t_1, x_i).$$

The system $\Lambda_{ij}(t_\alpha, x_i)$ satisfies the conditions that were imposed and verifies the equations $\Omega_{2ij} = 0$. I say that it likewise verifies the equations $\Omega_{1ij} = 0$. To show this, let $[\Omega_{\alpha ij}]$ denote the functions of x_i, t_α that are obtained by replacing the λ_{ij} in $\Omega_{\alpha ij}$ with $\Lambda_{ij}(t_\alpha, x_i)$. We must therefore establish that all of these quantities are null.

First of all, the $[\Omega_{2ij}]$ are identically null, so it thus remains for us to show that the same is true for the $[\Omega_{1ij}]$. In order to do this, we remark that these quantities are null for $t_2 = t_2^0$. Now use the identities (9.11), which reduce to:

$$\frac{\partial[\Omega_{1ij}]}{\partial t_2} = \text{homogeneous linear combinations of the } [\Omega_{1ij}]. \tag{9.15}$$

Therefore, the $[\Omega_{1ij}]$ are solutions of a normal system of homogeneous linear equations. Since these solutions are null for $t_2 = t_2^0$ they are null identically.

In conclusion, the system $\Lambda_{ij}(t_\alpha, x_i)$ is precisely a solution and is the unique solution of the equations $\Omega_{\alpha ij} = 0$ that satisfies the conditions that were imposed, moreover.

It now remains for us to establish that it is always possible to choose the functions $\lambda_{ij}(x)$ in such a fashion that the corresponding solutions Λ_{ij} likewise verify the second group of equations (9.6).

Now, for $\lambda_{ij} = \Lambda_{ij}$ the form $d\Pi$ reduces to the integrable form:

$$\left(\frac{\partial\Lambda_{ij}}{\partial x_k}+\frac{\partial\Lambda_{jk}}{\partial x_i}+\frac{\partial\Lambda_{ki}}{\partial x_j}\right)\omega_i\omega_j\omega_k\,. \tag{9.16}$$

On the manifold $t_\alpha = t_\alpha^0$, it becomes:

$$\left(\frac{\partial l_{ij}}{\partial x_k}+\frac{\partial l_{jk}}{\partial x_i}+\frac{\partial l_{ki}}{\partial x_j}\right)dx_i dx_j dx_k \tag{9.17}$$

and remains integrable. From this fact, choose the $l_{ij}(x)$ in such a manner that one has:

$$\frac{\partial l_{ij}}{\partial x_k}+\frac{\partial l_{jk}}{\partial x_i}+\frac{\partial l_{ki}}{\partial x_j} = 0\,, \tag{9.18}$$

which one may do in the following manner: One is given n holomorphic functions $l_1(x)$, $l_2(x)$, …, $l_n(x)$ of the x_i , and one takes $l_{ij} = \partial l_i /\partial x_j - \partial l_j /\partial x_i$.

I say that the corresponding functions Λ_{ij} verify the equations $\Omega_{ijk} = 0$.

Indeed, the expression (9.16) is integrable, so upon differentiating and normalizing it one has homogeneous linear identities in the Ω_{ijk} and the $\partial\Omega_{ijk} / \partial t_\alpha$. These identities thus express the fact that since the Ω_{ijk} are null for $t_1 = t_1^0$, $t_2 = t_2^0$ they are identically null.

In summation: When the matrix $[\psi_{i\alpha}(x_i, t_\alpha)]$ satisfies the conditions that were indicated in (9.3) and (9.4), *in order to construct an integrable form it suffices to determine the functions Λ_{ij}, which are solutions to the system $\Omega_{ijk} = 0$, $\alpha = 1, 2, \ldots, \mu$, $i = 1, 2, \ldots, n$. For $t_\alpha = t_\alpha^0$, they reduce to the functions $l_{ij} = \partial l_i /\partial x_j - \partial l_j /\partial x_i$, in which the $l_i(x)$ denote arbitrarily chosen functions that are holomorphic in the neighborhood of* (x_i^0).

The classical Mayer transformation:

$$t_1 = t_1^0 + \tau_1, \qquad\qquad t_\alpha = t_\alpha^0 + \tau_1\tau_\alpha\,,$$

which replaces the variables t_α with τ_α, permits us to write the equations in a simpler form:

$$\frac{\partial\lambda_{ij}}{\partial\tau_1} = \varphi_{ij1} + \varphi_{ij\alpha}\,\tau_\alpha\,.$$

It thus suffices to determine the solution $\Lambda_{ij}(x_i, t_\alpha)$ of this system that reduces to $\partial l_i / \partial x_j - \partial l_j / \partial x_i$ for $\tau_1 = 0$.

Therefore, the integration of the system (9.3) is not indispensable for the construction of a geodesic field. If this system is integrable – i.e., if it possesses a family of integral surfaces that uniformly satisfy the system on a certain $n+\mu$-dimensional domain surrounding the point (t_α^0, x_i^0) – then R. Debever has observed ([9]) that the construction of a field requires only quadratures.

———————

([9]) R. Debever, *Bull. Acad. Roy. Belg.* (*Class des Sciences*) (1937), pp. 809-815.

Stationary fields, geodesic fields, and integrable forms

TH. LEPAGE (*)

(*First communication*)

1. – If $L(t_\alpha, x_i, p_{i\alpha})$ denotes a definite regular [I, 2] (1) function of the element $e(t, x, p)$ then we can associate an alternating form Ω of degree μ with it [I, 6], which is a solution to the congruences:

(1.1) $$\Omega \equiv L\, dt_1\, dt_2 \dots dt_\mu, \qquad d\Omega \equiv 0 \qquad (\text{mod } \omega_i), \qquad \omega_i = dx_i - p_{i\alpha}\, dt_\alpha .$$

This form is well-defined for $\mu = 1$ or for $n = 1$. In any other case, it depends upon the undetermined coefficients λ. To simplify the notation, we denote the form Ω, when it is limited to the first two and the first three groups of terms, by Ω_1 and Ω_2, respectively:

(1.2) $$\begin{cases} \Omega_1 = L\, dt_1 \cdots dt_\mu + L_{i\alpha}(\alpha - 1, \omega_i, \alpha + 1), & L_{i\alpha} = \dfrac{\partial L}{\partial p_{i\alpha}}, \\ \Omega_2 = \Omega_1 + \lambda_{i\alpha, j\beta}(\alpha - 1, \omega_i, \alpha + 1; \beta - 1, \omega_j, \beta + 1). \end{cases}$$

The form Ω_2 is, moreover, identical with Ω when μ or n does not exceed two. The latter group of Ω_2 is comprised of $\binom{n}{2}\binom{\mu}{\mu-2}$ distinct monomials, so we assume that the indeterminates λ give rise to the following relations:

(1.3) $$\lambda_{i\alpha, j\beta} = \lambda_{j\beta, i\alpha} = -\lambda_{j\alpha, i\beta} = -\lambda_{i\beta, j\alpha} .$$

If, in a neighborhood of the element e, one has:

(1.4) $$L(e) \cdot a(e) \neq 0, \qquad a = |\,\delta_{\alpha\beta} L - L_{i\alpha}\, p_{i\beta}\,|,$$

then the other μ Pfaff equations:

(1.5) $$L\, dt_\alpha + L_{i\alpha}\, \omega_i = (\delta_{\alpha\beta} L - L_{i\alpha}\, p_{i\beta})\, dt_\beta + L_{i\alpha}\, dx_i = a_{\alpha\beta}\, dt_\beta + L_{i\alpha}\, dx_i = 0,$$
$$\alpha = 1, 2, \dots, \mu, \qquad i = 1, 2, \dots, n.$$

(*) Presented by L. GODEAUX.

(1) This work makes reference to the Note entitled "*Sur les champs géodésiques des integrales multiple*," that was published in this Bulletin (1941), pages 27-46. I will preserve the same notations, and the references are denoted by the Letter "I" followed by the section number. Nevertheless, here I shall call a field such as the ones that were defined in [I, 7] *stationary*, the qualifier "geodesic" being reserved for the fields that render the minimum-rank form Ω^* of the family [I, 6.2] integrable.

are linearly independent, they belong to the associated system [I, 2, page 32] of the form Ω, and:

(1.6) $$\Omega^* = L^{1-\mu}\prod_{\alpha=1}^{\mu}(a_{\alpha\beta}\,dt_\beta + L_{i\alpha}dx_i) = \frac{1}{F}\prod_{\alpha=1}^{\mu}(dt_\alpha + P_{i\alpha}dx_i)\,,$$

where:

(1.7) $$F = \frac{L^{\mu-1}}{a}\,, \qquad P_{i\alpha} = \frac{\bar{a}_{\alpha\beta}}{a}L_{i\beta}\,, \qquad [\text{I, 5}],$$

is the unique form of minimum rank μ in the family (1.1). Obviously, for $\mu = 1$ or $n = 1$, one will have $\Omega^* = \Omega$. In any other case, Ω^* may be deduced from Ω upon determining the λ by means of:

$$\lambda_{i\alpha,\, j\beta} = \frac{1}{L}(L_{i\alpha}L_{j\beta} - L_{i\beta}L_{j\alpha}), \quad \ldots$$

2. – The invariant character of the differentiation of the form [I, 3 and 4] confers a significance to the congruences (1.1) ([2]) that is independent of any transformation $(x, t, p) \to (\bar{x}, \bar{t}, \bar{p})$ that preserve the term:

(2.1) $$L\,dt_1 \ldots dt_\mu = \bar{L}\,d\bar{t}_1 \cdots d\bar{t}_\mu$$

and transforms the Pfaff system ω_i into the system:

(2.2) $$\bar{\omega}_i = d\bar{x}_i - \bar{p}_{i\alpha}d\bar{t}_\alpha\,, \qquad i = 1, 2, \ldots, n.$$

This property appears again when one considers Ω to be a surface integral element that is extended over a μ-manifold E_μ in the space of (t, x) [I, 3].

Two cases are distinguished: In the first case, the function L is linear in the minors of all orders of the matrix $(p_{i\alpha})$, and we denote them by $D(t_\alpha, x_i, p_{i\alpha})$ [I, 4].

The results are well known, and amount to an extension of the classical Gauss-Green-Stokes formula ([3]). The corresponding forms Ω are characterized by the following properties: A certain form of the family (1.1) has all of the coefficients independent of the $p_{i\alpha}$; in particular, when $\mu = 1$ or $n = 1$, it is the unique form Ω_1, normalized according to the basis (dt_α, dx_i) [I, 2 and 4]. The set of functions D corresponds to the set of forms of degree μ in the dt_α and dx_i whose coefficients are functions of only x and t.

In the second case, the coefficients of Ω, when normalized according to (dt, dx), do depend upon the $p_{i\alpha}$. Therefore, when considered to be the surface integral element, the form Ω will depend upon not only the contact elements $e(t, x, \partial x_i/\partial t_\alpha)$ to the portion of the surface E_μ, along which the integration is performed, but also upon the quantities $p_{i\alpha}$

([2]) As well as to conditions (1.4) and to the form of minimum rank (1.6).

([3]) E. GOURSAT, *J. de Liouville*, t. I (1915), and *Leçons sur le problème de Pfaff*, especially chapter III: *Formes symboliques de différentielles*, Paris, 1922.

that define a μ-dimensional planar element – or *μ-vector* – that is attached to the point (t_α, x_i) of E_μ .

In a general manner, we let $E_\mu, \bar{E}_\mu$ denote the portions of the surface in the space of (t, x) that are defined by the systems:

$$x_i = x_i(t_1, \ldots, t_\mu), \qquad \bar{x}_i = \bar{x}_i(t_1, \cdots, t_\mu), \qquad i = 1, 2, \ldots, n, \tag{2.3}$$

which have frontiers $f(E_\mu)$, $f(\bar{E}_\mu)$, and assume that the $x_i(t_\alpha)$, $\bar{x}_i(t_\alpha)$ are continuously differentiable when the point (t_α) describes two cylindrical regions G, $\bar{G}$. One supposes that the points $(x, t), (\bar{x}, t)$ belong to a certain open connected region R of the space of (t, x) for which all of the coefficients of the forms envisioned will be continuously differentiable.

One arrives at the notion of a field upon considering the $p_{i\alpha}$ to be functions that are continuously differentiable on R. We agree to say that the field $p_{i\alpha}(t, x)$ *envelops* E_μ when, taking into account the primary equations (2.3), one has:

$$p_{i\alpha}(x_i(t_\beta), t_\beta) = \frac{\partial x_i}{\partial t_\alpha} \tag{2.4}$$

for any point of G.

With these conditions, the surface integral ([4]):

$$I(E_\mu) = \int_{E_\mu} [\Omega] \tag{2.5}$$

is, up to sign – this sign being fixed by the choice of orientation for $f(E_\mu)$ – the value of the μ-fold integral:

$$I(E_\mu) = \int_{E_\mu} L\left(t_\alpha, x_i, \frac{\partial x_i}{\partial t_\alpha}\right) dt_1 \cdots dt_\mu \ . \tag{2.6}$$

The portion of the surface $\bar{E}_\mu$ being varied will not, in general, be enveloped by the field $(p_{i\alpha})$. The formula (2.6) is therefore invalid for $\bar{E}_\mu$, but instead one will have:

$$I(\bar{E}_\mu) = \int_{\bar{E}_\mu} [\Omega_1] = \int_G L\left(t, \bar{x}, \frac{\partial \bar{x}}{\partial t}\right) dt + \int_{\bar{E}_\mu} E_1\left(t, x, p, \frac{\partial \bar{x}}{\partial t}\right) dt_1 \cdots dt_\mu , \tag{2.7}$$

in which:

$$E_1(t, x, p, \bar{p}) = L(t, x, \bar{p}) - L(t, x, p) - L_{i\alpha}(\bar{p}_{i\alpha} - p_{i\alpha}) . \tag{2.8}$$

One will obtain an analogous formula upon replacing $[\Omega_1]$ with $[\Omega]$, which is an arbitrary form of the family Ω, when referred to the field $(p_{i\alpha})$, and the λ denote arbitrary

([4]) We denote the result of the substitution $p_{i\alpha} = p_{i\alpha}(t, x)$ by brackets [ ].

chosen functions of the t and x. Nevertheless, in this case, the function E_1 may be replaced with the expression:

$$E = E_1 - \lambda_{i\alpha,\, j\beta}(x,\, t) \cdot \begin{bmatrix} \bar{p}_{i\alpha} - p_{i\alpha} & p_{i\beta} - \bar{p}_{i\beta} \\ \bar{p}_{j\alpha} - p_{j\alpha} & p_{j\beta} - \bar{p}_{j\beta} \end{bmatrix} \ldots \tag{2.9}$$

One observes that if L is linear, hence, of type D, then the functions E, E_1 are either identically null or they go to zero for all values of $\bar{p}$, p for which the matrix:

$$(\bar{p}_{i\alpha} - p_{i\alpha}) \tag{2.10}$$

has rank one ([5]).

Taking (2.5), (2.6), and (2.7) into account, we obtain the expression for the variation of $I(\bar{E}_\mu) - I(E_\mu)$:

$$\Delta I = I(\bar{E}_\mu) - I(E_\mu) = \int_{\bar{E}_\mu} [\Omega] - \int_{E_\mu} [\Omega] + \int_{\bar{E}_\mu} E\, dt_1 \cdots dt_\mu\,. \tag{2.11}$$

3. – *Stationary and geodesic fields.* If a certain form $[\Omega]$ is integrable when referred to the field $(p_{i\alpha})$ and corresponding to a choice of λ then formula (2.11) gives:

$$\Delta I = \int_{\bar{E}_\mu} E\left(t, x, p, \frac{\partial x}{\partial t}\right) dt_1 \cdots dt_\mu \tag{3.1}$$

upon supposing that E_μ and $\bar{E}_\mu$ possess the same frontier ([6]). In this case, we say that the field enveloping E_μ is *stationary*.

It is obvious that a stationary field $\mathcal{E}$ does not render any form of the family Ω integrable no matter what the λ, when considered as functions of t and x. There will be reason to consider, in particular, the fields that render either the form Ω_1, the form Ω, or the form of minimum rank Ω^* integrable; we denote them by the symbols $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}^*$, respectively. The fields $\mathcal{E}^*$ will be called *geodesic*. They are nothing but the fields that were introduced by C. Caratheodory in order to extend to multiple integrals his theory of equidistant geodesics ([7]), in which he made systematic use of various questions that touch upon the Calculus of Variations for simple integrals.

([5]) One observes that the condition $d\Omega \equiv 0 \pmod{\omega}$ entails that the functions E have the property of being of second order in the $\bar{p}_{i\alpha} - p_{i\alpha}$.

([6]) Since the frontier is variable this formula persists when the integrable form $[\Omega]$ possesses characteristics. (*cf. infra,* sec. 9).

([7]) C. CARATHEODORY, *Über die diskontinuerlichen Lösungen in der Variationsrechnung.* Dissertation, *Göttingen,* 1904. *Über das allgemeine Problem in der Variationsrechnung.* Gött. Nachr. 1905. *Die Methode der geodätische Äquidistanten und das Problem von Lagrange,* Acta Math. (47) (1926). *Variationsrechnung und partielle Differential gleichungen erster Ordnung,* Leipzig, 1937.

4. – In order to calculate the principal part of ΔI when $\bar{E}_\mu$ ranges over a family that depends upon τ and reduces to E_μ when $\tau = 0$, one may follow the path that was indicated by E. Goursat ([8]), which brings out the role of the two tensorial operators d and e (the interior product, or contraction of an alternating tensor with a vector). If one considers the coefficients of the form Ω to be the components of an alternating covariant tensor, and if:

(4.1) $$\Delta(T^\alpha, X^i, \alpha = 1, 2, \ldots, \mu; i = 1, 2, \ldots, n)$$

denotes a contravariant vector, then the two alternating forms:

(4.2) $$\begin{cases} e_\Delta(\Omega) = T^\alpha \dfrac{\partial\Omega}{\partial(dt_\alpha)} + X^i \dfrac{\partial\Omega}{\partial(dx_i)}, \\ e_\Delta(d\Omega) = T^\alpha \dfrac{\partial(d\Omega)}{\partial(dt_\alpha)} + X^i \dfrac{\partial(d\Omega)}{\partial(dx_i)}, \end{cases}$$

in which $\partial\Omega/\partial(dt_\alpha), \ldots, \partial(d\Omega)/\partial(dx_i)$ denote the coefficients of dt_α and dx_i in Ω and $d\Omega$, which have degree $\mu - 1$ and μ, respectively. The same is true for the alternating form of degree μ:

(4.3) $$\Delta\Omega = d(e_\Delta\Omega) + e_\Delta(d\Omega).$$

We interpret Δ as being a deformation vector for E_μ. We let $E_\mu(\tau) = \bar{E}_\mu$ denote the portion of the surface that is described by the points that are situated along the family of trajectories of the differential system:

(4.4) $$\frac{dt_\alpha}{T^\alpha} = \frac{dx_i}{X^i} = d\tau,$$

and correspond to the points of E_μ ($\tau = 0$) for each $\tau = \text{const.}$

From Stokes's formula:

(4.5) $$\int_{E_\mu} \Delta\Omega = \int_{f(E_\mu)} e_\Delta\Omega + \int_{E_\mu} e_\Delta(d\Omega)\,.$$

This is the expression for the *first variation* of $I(E_\mu)$ ([9]). One observes that only the first two groups of terms in Ω, which is normalized according to the basis (dt, ω), appear in the right-hand side of (4.5). Therefore, the first variation δI has the expression:

Geometrische Optik. Erg. d. Math. IV, Berlin, 1937. Geodesic fields for multiple integrals are introduced in: *Über die Variationsrechnung bei mehrfachen Integralen.* Acta Szeged **4** (1928), pp. 193-216.

([8]) E. GOURSAT. *Sur certaines systèmes d'equations aux différentielles totales et sur une généralisation du problème de Pfaff.* Ann. Fac. Sc. Toulouse, 1915. *Leçons sur le problème de Pfaff.* Paris, 1922.

([9]) P. V. PAQUET, *Les formes différentielles Ω_n dans le Calcul de Variations.* Bull. Acad. Roy. Belg. 1941, pp. 65-84.

$$\delta I = \int_{f(E_\mu)} [e_\Delta \Omega_1] + \int_{E_\mu} e_\Delta d\Omega_1 , \tag{4.6}$$

in which the quantities:

$$\begin{cases} [e_\Delta \Omega_1] = (L \cdot T^\alpha + L_{i\alpha} \cdot \omega_i(\Delta))\, dt_1 \cdots dt_{\alpha-1} dt_{\alpha+1} \cdots dt_\mu , \\ e_\Delta [d\Omega_1] = \left(\dfrac{d}{dt_\alpha} L_{i\alpha} - \dfrac{\partial L}{\partial x_i} \right) \omega_i(\Delta) dt_1 \cdots dt_\mu , \\ \omega_i(\Delta) = X^i - \dfrac{\partial x_i}{\partial t_\alpha} \cdot T^\alpha , \end{cases} \tag{4.7}$$

are calculated along E_μ .

One may deduce that any surface E_μ that is enveloped by a stationary or geodesic field is a solution of the following system (Euler-Lagrange equations) ([10]):

$$\frac{d}{dt_\alpha} L_{i\alpha} - \frac{\partial L}{\partial x_i} = 0. \tag{4.8}$$

If the first variation is null then if the frontier varies with τ – i.e., under the deformation Δ – then one will necessarily have:

$$[e_\Delta \Omega_1] = (LT^\alpha + L_{i\alpha}\, \omega_i(\Delta))\, dt_1 \ldots dt_{\alpha-1}\, dt_{\alpha+1} \ldots dt_\mu = 0. \tag{4.9}$$

This supplementary condition expresses the notion that the vector Δ belongs to a planar element defined by the contact element $f(E_\mu)$, and by the multivector whose components verify the first μ equations (1.5) of the associated system for the form Ω. For $\mu = 1$ it is the *transversality condition*. We agree to say that any vector that belongs to the multivector that is defined by (1.5) is *transversal* to the element $(p_{i\alpha})$ that is attached to the same point. If conditions (1.4) are satisfied at each point of E_μ then all of the vectors that are transversal to the contact element at a point of E_μ belong to an n-vector that is defined by the system of quantities $P_{i\alpha}$ (1.7) at this point.

([10]) One may further verify this follows: In a field $\mathcal{E}$ for an integrable Ω let D be the corresponding exact derivative function [I, 6]. One has:

$$[\Omega] = D\, dt_1 \ldots dt_\mu + D_{i\alpha}(\alpha - 1,\ \omega_i ,\ \alpha + 1) + D_{i\alpha j\beta}(\omega_i ,\ \omega_j) + \ldots$$

such than : $\quad L = D, \quad L_{i\alpha} = D_{i\alpha},$

but: $\quad \dfrac{d}{dt_\alpha} D_{i\alpha} - \dfrac{\partial D}{\partial x_i} = 0,$

which gives formula (4.8).

5. – *Remark.* By the prior construction of a field $\mathcal{E}$ that envelops E_μ, the preceding method reduces the integral $I(E_\mu)$ to a surface integral whose element $[\Omega]$ is an integrable form. In order to apply Stokes's theorem in the transformation of the first variation and obtain the expressions of the preceding sections it is indispensable that we assume that E_μ is of class C^2 ([11]), and that the field S is of class at least C^2. The same is true as far as the calculation of the function E is concerned. For this calculation one only appeals to the property of the integrable form Ω that it gives rise to the equality:

$$\int_{\bar{E}} \Omega = \int_{E} \Omega \tag{5.1}$$

when E_μ, $\bar{E}_\mu$ possess the same frontier. Let $p_{i\alpha}(t, x)$ be a system of C^0 functions on R such that one has (2.4), in addition, and suppose, moreover, that these values of $p_{i\alpha}$, when introduced into the coefficients of a certain form Ω give:

$$\int_{V_\mu} [\Omega] = \int_{\bar{V}_\mu} [\Omega] \tag{5.2}$$

for any portions of the surfaces V and $\bar{V}$ that have the same frontier. We then say that $[\Omega]$ is a C^0 *integrable form.* Formula (3.1) persists and the principal term on the right-hand side of (2.11) is null for $\tau = 0$, but this time one may no longer transform the expression by the use of Stokes's theorem, since $d[\Omega]$ no longer has any meaning, as the coefficients of $[\Omega]$ are no longer differentiable. However, one may, upon generalizing a lemma of A. Haar ([12]), deduce that E_μ is a solution of a first order partial differential equation that reduces to the system (4.8) when E_μ is C^2 ([13]).

6. – *The discriminants of a form* Ω. *The problem* $\bar{L} = L + D$. Denote the Jacobians with respect to $p_{i\beta}$ of the systems of coefficients of the group of terms in Ω, which are normalized according to (dt_α, ω_i) and (dt_α, dx_i), respectively, by $\Delta(L)$, $\mathcal{D}(\Omega)$, with:

$$\Delta(L) = \left|\frac{\partial^2 L}{\partial p_{i\alpha}\partial p_{j\beta}}\right|, \qquad \mathcal{D}(\Omega) = \left|\frac{\partial \mathcal{P}_{i\alpha}}{\partial p_{j\beta}}\right|, \tag{6.1}$$

$$\begin{cases} \Omega = L\, dt_1 \cdots dt_\mu + L_{i\alpha}(\omega_i) + \lambda_{i\alpha,j\beta}(\omega_i\omega_j) + \cdots \\ \quad = \mathcal{L}\, dt_1 \cdots dt_\mu + \mathcal{P}_{i\alpha}(dx_i) + \Lambda_{i\alpha,j\beta}(dx_i dx_j) + \cdots \end{cases} \tag{6.2}$$

([11]) A function whose derivatives of order k are all continuous in a domain is said to be of class C^k there. It is C^0 if it is only assumed to be continuous. E_μ is C^k if the $x_i(t)$ *are* C^k.

([12]) A. HAAR, *Zur Variationsrechnung.* Abh. Math. Sem. Hamburg Univ., **8** (1930).
TIBOR RADÓ, *Bemerkung über die Differentialgleichungen zweidimensionalen Variationsprobleme.* Acta Litt. Sc. Szeged **3** (1925).
TIBOR RADÓ, *On the Problem of Plateau*, Ergeb. der Math. (2) (1933.

([13]) P. GILLIS, *Sur les formes différentielles et la formule de Stokes*, Mémoires in-8^0, Acad. Roy. Belg., 1942.

One has:

$$(6.3) \qquad \Delta(L) = \mathcal{D}(\Omega_1), \qquad \mathcal{D}(\Omega) = \mathcal{D}(\Omega_2).$$

$\mathcal{D}(\Omega)$ is the discriminant of the ordinary quadratic form the $n\mu$ indeterminates $U_{i\alpha}$:

$$(6.4) \qquad L_{i\alpha,\,j\beta} \cdot U_{i\alpha}\, U_{j\beta} - \lambda_{i\alpha,\,j\beta}\,(U_{i\alpha}\, U_{j\beta} - U_{j\alpha}\, U_{i\beta}).$$

Let $D(t, x, p)$ be an exact derivative [I, 3], i.e., a linear function that corresponds to an integrable form $d\pi$, where π denotes a form of degree $\mu - 1$ whose coefficients are functions of only the x_i, t_α. Consider the function:

$$(6.5) \qquad \bar{L} = L + D.$$

For $\mu = 1$ or $n = 1$ one will have:

$$(6.6) \qquad \Delta(\bar{L}) = \Delta(L) = \mathcal{D}(\Omega),$$

but these expressions will differ, in general, for the other values of μ and n. We let $\bar{\Omega}$ denote the form:

$$(6.7) \qquad \bar{\Omega} = \Omega + d\pi.$$

On the other hand, we have that:

$$(6.8) \qquad \mathcal{D}(\Omega) = \mathcal{D}(\bar{\Omega}).$$

It is obvious that since the sum of two integrable forms is an integrable form any stationary field that envelops E_μ for the problem L does the same for the problem $\bar{L}$, as well, and for any arbitrary function D. *This is no longer the case for geodesic fields.* Indeed, the addition of a form $d\pi$ will alter the rank of an integrable form ([14]) and its associated system, in general. If one refers the expression of the first variation in the case where the frontier is variable then one confirms that the boundary terms $[e_\Delta\Omega]$ (sec. 4, formula (4.7)) are found to be modified, in general, upon passing from L to $\bar{L}$. The directions Δ that are transversal to the element $(p_{i\alpha})$ vary from one problem to the other. Thus, a field that is geodesic for L will no longer be geodesic for the problem $\bar{L}$; it will behave like an ordinary field.

7. – *The reduction of an integrable form* Ω. *Characteristics.* To simplify the notation, we set:

$$(7.1) \qquad x_{n+1} = t_1, \;\ldots,\quad x_{n+\mu} = t_\mu, \qquad n + \mu = N.$$

In any field $\mathcal{E}$ there exists at least one integrable form Ω, which we represent by:

([14]) Except in the cases $\mu = 1$ or $n = 1$, when the sum of the two forms of rank μ has rank μ, although the remark persists nonetheless because the associated systems still differ.

(7.2) $$\Omega = A_{i_1 \cdots i_\mu}\, dx_{i_1} \cdots dx_{i_\mu}\,, \qquad i = 1, 2, \ldots, N,$$

in which the A are alternating in the μ and i indices.

The integrability condition:

(7.3) $$d\Omega = \mathcal{A}_{i_1 \cdots i_\mu i_{\mu+1}}\, dx_{i_1} \cdots dx_{i_{\mu+1}} = 0$$

is equivalent to the system:

(7.4) $$\mathcal{A}_{i_1 \cdots i_{\mu+1}} = \frac{\partial A_{i_1 \cdots i_\mu}}{\partial x_{i_{\mu+1}}} \pm \frac{\partial A_{i_2 \cdots i_\mu i_{\mu+1}}}{\partial x_{i_1}} + \cdots + \frac{\partial A_{i_{\mu+1} i_1 \cdots i_{\mu-1}}}{\partial x_{i_\mu}} = 0,$$

where the minus sign relates to only the case of odd μ.

With this notation, the associated system to Ω is the Pfaff system:

(7.5) $$A_{i_1 \cdots i_{\mu-1}}\, dx_i = 0, \qquad i_1, \ldots, i_{\mu-1} = 1, 2, \ldots, N.$$

The properties of integrable forms may be deduced from the following proposition ([15]): *Any integrable form whose coefficients are regular in a neighborhood of a point* (X) *is transformable into a form with constant coefficients in the domain of this point.*

It then results that two integrable forms that are defined at the same point and possess all of the properties that are required for algebraic equivalence there will be transformed from one into the other. From this, we immediately deduce the following useful result:

First of all, *the system* (7.5) *is completely integrable*. Indeed, it is equivalent to a system for which all of the coefficients are constants. The number ρ of linearly independent equations of this system is the rank of the form Ω, and it is, moreover, the minimum number of variables that it takes to express the form. The $N-\rho$-dimensional integral multiplicities are the *characteristics* of the form Ω; the system (7.5) sometimes takes the name of the *characteristic system* of the form. The covariant character of the associated system shows that its integration is related to the search for a reduced expression, since the variables that figure in such an expression – viz., the *reduced or characteristic variables* – necessarily constitute a complete system of first integrals of the system (7.8).

Observe that for any field $\mathcal{E}$ that envelops E_μ the associated system to the corresponding integrable form comprises the first μ equations of (1.5), which define the transversal n-vector. Therefore, all of the characteristic multiplicities of a field are tangent to the transversal n-vector along E_μ. In particular, *for a geodesic field* $\mathcal{E}^*$ *the characteristic system may be identified with the system of* μ *equations* (1.5). It is thus completely integrable when it is referred to the functions ($p_{i\alpha}$) or ($P_{i\alpha}$) that define $\mathcal{E}^*$. The n-dimensional characteristics uniformly cover a domain R that surrounds any E_μ. They are transversal to the element ($p_{i\alpha}$) of the field at each point (x, t). Nevertheless, this

([15]) CL. M. CRAMLET, *Note on the integrability conditions of implicit differential equations; Differential invariant theory of alternating tensors*. Bull. Amer. Math. Soc. (1938).

property does not completely characterize a geodesic field because the system (1.5) might not be completely integrable for a certain field E for which the corresponding form (1.1) is not necessarily of minimum rank, and the preceding conclusions persist in this case. In order to completely characterize a field S^*, in addition to the integrability condition for the associated system (1.5) it is necessary to introduce the following condition: *the function* F^{-1} (sec. 1) *is a multiplier of the system.*

8. – Another consequence of the proposition that was stated at the beginning of the preceding section concerns the type of reduced expression that a form will take for various values that are attributed to μ and n. These expressions will be useful to us in the study of the differential systems of the fields.

The proposition that was invoked boils down to the problem of finding the distinct algebraic types of the alternating forms when the dx and dt are regarded as the indeterminates, in the algebraic sense ([16]).

We have three cases to distinguish, namely:

(8.1) (*a*) $\mu = 1$ or $n = 1$, (b) $\mu = 2$ or $n = 2$, (c) μ and $n > 2$.

In the first cases, (a) and (b), the knowledge of the rank ρ suffices to fix the type o the reduced form. This is no longer true for the forms (c), except when $\rho = \mu$ (for a form Ω^* of minimum rank) ([17]).

For (a), the form is a monomial since the rank is equal to the degree. For (b), if $\mu = 2$, $n \geq 2$ then the rank is necessarily even:

$$\rho = 2\gamma, \tag{8.2}$$

indeed, it the rank of the alternating matrix (A_{i_1,i_2}). By a non-singular substitution of the indeterminates of the form it may be reduced to a *canonical expression:*

$$\xi_1\,\xi_2 + \xi_3\,\xi_4 + \ldots + \xi_{2\gamma-1}\,\xi_{2\gamma}, \tag{8.3}$$

([16]) The method that is followed here is obviously subordinate to the hypotheses that were made to establish the proposition of sec. 7. Cramlet has established it by basing it on the general existence theorems for differential systems in involution – hence, on the Cauchy-Kowalevska theorem – when the given data are analytic and regular. However, the proposition persists under less restrictive hypotheses, at least in the case where the knowledge of the rank determines the types of reductions; it suffices to suppose that the forms are of class C^2. It might be interesting to establish the proposition in full generality with hypotheses of this nature.

([17]) It is not without interest to observe that this classification appears again in the study of the elements (e) in relation to the character of the ordinary quadratic form (6.4), For (a), an element is called *regular, singular,* or *irregular* (Cf., C. CARATHEODORY, *Variationsrechnung*) according to whether (6.4) is definite, semi-definite, or indefinite. For (b) and (c), the elements (e) are classified according to the character of the form (6.4), *since the* $U_{i\alpha}$ *are related by the condition that the matrix* ($U_{i\alpha}$) *have rank one.* For (b), if the form (6.4) is definite with the indicated conditions then there will exist at least one system of values for the $\lambda_{i\alpha,\,j\beta}$ such that (6.4) is definite. This is no longer true for (c).

in which the ξ denote 2γ linearly independent forms of the associated system. The group of automorphisms of this form depends upon $2\gamma(2\gamma + 1)$ parameters and presents obvious analogies with the orthogonal group ([18]).

For $n = 2$, $\mu > 2$ the rank is μ or $\mu + 2$. It cannot be $\mu + 1$ since the rank of a form of degree μ in $\mu + 1$ indeterminates is necessarily equal to μ. The (reduced) canonical expression may be obtained by passing to the *complementary form* of degree $n = 2$.

In a general manner, the complementary form of:

(8.4) $$A_{i_1\cdots i_\mu}\xi_1\cdots\xi_{i_\mu}, \qquad i = 1, 2, \ldots, N,$$

which has degree μ in the x indeterminates, is the form of degree $N - \mu = n$:

(8.5) $$A_{j_1\cdots j_n}\zeta_{j_1}\cdots\zeta_{j_n}$$

in the N indeterminates ζ that is deduced from (8.4) by setting:

(8.6) $$A_{j_1\cdots j_n} = \pm A_{i_1\cdots i_\mu}$$

and taking the + or – sign according to whether the permutation:

(8.7) $$i_1 \ldots i_\mu, \qquad j_1 \ldots j_\mu$$

is even or odd relative to the principal permutation $12 \ldots N$.

One sees without difficulty that if one performs a substitution S ($\xi \to \bar{\xi}$) on the ξ, and, at the same time, the adjoint substitution S' ($\zeta \to \bar{\zeta}$) on the ζ *that preserves the bilinear form:*

(8.8) $$\xi_1\zeta_1 + \xi_2\zeta_2 + \ldots \xi_N\zeta_N$$

then the complementary form will thus be reproduced, only multiplied by the determinant of the substitution S.

Having done this in order to obtain a reduced form, it will suffice to determine a reduced form for the complementary form. In the case that we are considering, the latter is quadratic, and we may suppose that it amounts to the canonical form (8.3). As a result, *the canonical form for* $n = 2$, $\mu > 2$ *is:*

(8.9) $$\zeta_3\zeta_4\ldots\zeta_N + \zeta_1\zeta_2\zeta_5\zeta_6\ldots\zeta_N + \zeta_1\zeta_2\ldots\zeta_{2\gamma-2}\zeta_{2\gamma+1}\ldots\zeta_N.$$

The rank of this form is equal to $N - 2 = \mu$ when (8.9) is reduced to only one monomial; otherwise it is equal to $\mu + 2$.

Contrary to what happened in the preceding case, one may not say anything, *a priori*, about the rank of an alternating form for $\mu > 2$ and $n > 2$. Observe further that for $\mu = 1$

([18]) HERMANN WEYL, *The Classical Groups. Their Invariants and Representations*. Chap. VI. *The symplectic group*, Princeton 1939.

or $n = 1$, and for $\mu = 2$, $n > 1$, the algebraic equivalence of the two forms is assured whenever the rank is the same, which always makes sense for $\mu > 1$ and $n = 1$, moreover. This is no longer always true for $n = 1$, $\mu > 1$ whenever μ exceeds three. For example, for $\mu = 4$, $N = 6$, one has two distinct types of rank 6:

(8.10) $$(\xi_1\xi_2 + \xi_3\xi_4)\xi_5\xi_6\,, \qquad (\xi_1\xi_2 + \xi_3\xi_4 + \xi_5\xi_6)^2.$$

In order to assure the equivalence it will therefore be necessary for the two forms to possess the same number of linear divisors. The first expression in (8.10) has the two linear divisors ξ_5 and ξ_6; the second one has none.

The preceding results translate immediately for integrable forms. For example, any integrable quadratic form amounts to the canonical form in the characteristic variables S_1, …, $S_{2\gamma}$:

(8.11)′ $$dS_1\, dS_2 + \ldots + dS_{2\gamma-1}\, dS_{2\gamma}.$$

For $\mu = \rho$, in particular for all of the forms (a), one will have the expression:

(8.11)″ $$dS_1 \ldots dS_\mu\,.$$

For $\mu = 3$, $n = 2$, one will have the following two expressions:

(8.11)‴ $$dS_1\, dS_2\, dS_3\,, \qquad dS_1(dS_2\, dS_3 + dS_4\, dS_5);$$

For $\mu = 4$, $n = 2$, one will have the three distinct expressions:

(8.11)$^{\mathrm{IV}}$ $$dS_1\, dS_2\, dS_3\, dS_4\,,\ \ (dS_1 dS_2 + dS_3\, dS_4)\, dS_5\, dS_6\,,\ \ (dS_1 dS_2 + dS_3\ dS_4 + dS_5\ dS_6)^2,$$

in which the S_α denote, in each case, the distinct first integrals of the associated system for the integrable form Ω.

9. – *Characteristics of the forms* $[\Omega]$ *and transversality*. Let $\mathcal{E}$ be a stationary field that envelops E_μ for which the form $[\Omega]$ is integrable. If the rank ρ of this form is less than $n + \mu$ – hence, if the variables x_i, t_α are not characteristic – then there will exist $(n + \mu - \rho)$-dimensional characteristic multiplicities such that there will be one and only one of them at each point of a domain. We suppose, moreover, that the domain R (sec. 2) is uniformly covered by this ρ-fold family of multiplicities.

A linear element (dt_α, dx_i) – or if one prefers, a vector (T_α, X_i) – that issues from a point (t, x) will be called *characteristic* if it verifies the associated system for $[\Omega]$. Thus, there exist $n + \mu - \rho$ linearly independent characteristic elements at each point of R, the set of which constitutes the characteristic $(n + \mu - \rho)$-vector $\Gamma_{n+\mu-\rho}$ of the form $[\Omega]$. Any characteristic element verifies the first μ equations (1.5) of the associated system, from which it results that it is a transversal element to the field $(p_{i\alpha})$. In particular, at every point of E_μ where conditions (1.4) are assumed to be satisfied $\Gamma_{n+\mu-\rho}$ belongs to the n-vector that is transversal to the contact element $(p_{i\alpha})$ of E_μ. In this case, the μ-fold family of n-dimensional characteristic surfaces:

$$S_\alpha(t, x) = S_\alpha(t_0, x_0) \tag{9.1}$$

that issue from the points (t_0, x_0) of E_μ provide an image of the field $\mathcal{E}^*$. For $\mu = 1$, this image is completed by the fact that the trajectories of the ordinary differential system:

$$\omega_i = dx_i - p(t_i, x_i)\, dt = 0, \qquad i = 1, 2, \ldots, n \tag{9.2}$$

are extremals. The field is generated by an n-fold family of extremals that is intersected transversally by a countably infinite family of n-dimensional characteristic surfaces (waves). In any other case, one may no longer say that a field $\mathcal{E}$ or $\mathcal{E}^*$ is generated by a family of extremals because, in general, the Pfaff system:

$$\omega_i = dx_i - p_{i\alpha}(t_i, x_i)\, dt_\alpha = 0, \quad i = 1, 2, \ldots, n, \ \alpha = 1, 2, \ldots, \mu \tag{9.3}$$

will not be completely integrable. However, contrary to what happens when $\mu > 1$, $n > 1$, any n-fold family of extremals E_μ is a stationary field [I, 9]. A field of extremals that renders integrable a form Ω with characteristics – which is always true when $\mu > 1$, $n = 1$, $\mu > 1$, n odd, $\mu = \rho$ – is connected at more than one point to the figure (family of waves transversal to an extremal) that realizes any stationary field when $\mu = 1$. Notably, we show this for the situation that relates to the extension of formula (3.1) to the case in which the surface $\bar{E}_\mu$ being varied has a frontier that is distinct from $f(E_\mu)$.

10. – Let $\Delta(T^\alpha, X^i)$ be a characteristic vector. The $n + \mu$ functions T^α, X^i are continuously differentiable in R so they verify the ρ linearly independent equations:

$$A_{i_1\cdots i_{\mu-1}i} X^i = 0, \qquad i = 1, 2, \ldots, N, \qquad N = n + \mu, \tag{10.1}$$

where, as in sec. 7, we have set:

$$\begin{cases} x_{n+1} = t_i, & \cdots \quad x_{n+\mu} = t_\mu, \\ X^{n+1} = T^1, & \cdots \quad X^{n+\mu} = T^\mu. \end{cases} \tag{10.2}$$

We obviously suppose that the field $\mathcal{E}$ is endowed with characteristics, i.e., $\rho < n + \mu$. R is traversed by a family of trajectories (or characteristic curves) of the differential system:

$$\frac{dx_i}{X^i} = d\tau, \qquad i = 1, 2, \ldots, N. \tag{10.3}$$

A characteristic curve γ that issues from the point (x_i^0) is obviously completely contained in the characteristic multiplicity $\Gamma_{n+\mu-\rho}$ that issues from that point. Follow along the characteristics (γ) that issue from the points of E_μ, and consider an arbitrary section $\bar{E}_\mu$ of this family. *The value of the integral* $I(E_\mu)$ *is equal to the surface integral:*

$$I(E_\mu) = \int_{\bar{E}_\mu} [\Omega], \tag{10.4}$$

where, as above, $[\Omega]$ *denotes the integrable form of field* $\mathcal{E}$ *that envelops* E_μ.

Now, one has (sec. 3):

$$\Delta I = I(\bar{E}_\mu) - I(E_\mu) = \int_{\bar{E}_\mu} [\Omega] - \int_{E_\mu} [\Omega] + \int_{\bar{E}_\mu} E\, dt \cdots dt_\mu, \tag{10.5}$$

so, by (10.4):

$$\Delta I = \int_{\bar{E}_\mu} E\left(t, x, p_{i\alpha}, \frac{\partial \bar{x}_i}{\partial t_\alpha}\right) dt \cdots dt_\mu. \tag{10.6}$$

Therefore, *if* Ω *is the integrable form of the field* $\mathcal{E}$ *that envelops* E_μ *then formula* (3.1) *persists when the frontiers* $f(\bar{E}_\mu)$ *are found to all be situated on the manifold that is generated by a family of characteristic curves that issue from the points of the frontier* $f(E_\mu)$.

Formula (10.4) is a consequence of the fact that $[\Omega]$ may be expressed with the aid of the $n+\mu+\rho$ characteristic variables and their differentials alone, taking into account the invariance of the differentiation of the form and the change of variables rule for surface integrals ([19]).

Remark. If one interprets Δ as the symbol of an infinitesimal transformation:

$$\Delta f = x^i \frac{\partial f}{\partial x_i} \tag{10.7}$$

then the tensorial relation (sec. 4.3) gives:

$$\Delta\Omega = 0. \tag{10.8}$$

The form $[\Omega]$ is called (cf., E. Cartan ([20])) *invariant* for the differential system (10.3).

Relation (10.8) *further persists when* $e_\Delta\Omega$ *is an integrable form without being null.* In this case, the vector Δ is no longer characteristic, *although* (10.6) *persists nonetheless*

([19]) [Ambiguous reference to footnotes in original.]

([20]) Following the terminology of E. GOURSAT (cf., *Problème de Pfaff*, chap. V, sec. 60-62), $\int [\Omega]$ is an *integral invariant* that is *"attached to the trajectories"* of the system (10.3), and a *complete integral invariant* in the sense of E. CARTAN (*Leçons sur les invariants intégraux*). When $[\Omega] = d\pi$, $\int \pi$ is a *relative* invariant attached to the trajectories.

under the condition that we take a section of the family of trajectories that issue from $\bar{E}_\mu$ *for* τ = const. *to be the surface that is varied.* One may further say that the integral:

$$\int_{E_\mu} [\Omega] \tag{10.9}$$

is an integral invariant for the infinitesimal transformation Δf (or a Poincaré integral invariant for (10.3)). Such transformations always exist for an integrable form, but they are no longer determined by purely algebraic operations, as they are for characteristic transformations. The set of all of them contains a system of infinitesimal transformations that is isomorphic to the $n+\mu$-dimensional translation group. The knowledge of these transformations may simplify the problem of the integration of the associated system to [Ω] since the forms $e_\Delta\Omega$, $e_{\Delta_1}(e_{\Delta_2}\Omega)$, etc., are all invariant.

On the De Donder-Weyl fields and their construction by the method of characteristics

by LÉON VAN HOVE (*)

1. Consider the μ-uple integral from the calculus of variations:

$$I = \int_D L(t_\alpha, x_i, p_{i\alpha}) dt_1 \cdots dt_\mu \qquad \alpha = 1, \ldots, \mu, \quad i = 1, \ldots, n, \tag{1.1}$$

where the $x_i(t_\alpha)$ are unknown functions and the $p_{i\alpha}$ are their derivatives $\partial x_i / \partial t_\alpha$. One knows the utility of the notion of *field* in the study of the extrema of (1.1).

For multiple integrals with several unknown function (n and $\mu \geq 2$), this notion was introduced by Th. De Donder in 1913 ([1]), and was reprised in the same form by Hermann Weyl in 1934 ([2]). Meanwhile, in 1929 Carathéodory ([3]) defined geodesic fields that were distinct from the preceding ones that have a remarkable geometric significance. In 1936, Th. Lepage showed the existence of an extended family of fields; those of De Donder-Weyl and Carathéodory are only special cases ([4]). Recalling the terminology adopted by Th. Lepage in his last note, we call all of the fields of the family *stationary*. We use the notation adopted by Carathéodory and Lepage.

Stationary fields are introduced as follows for the problem (1.1). One sets:

$$\omega_i = dx_i - p_{i\alpha}\, dt_\alpha \qquad p_{i\alpha} = L_{p_{i\alpha}}, \tag{1.2}$$

and one considers the alternating differential forms Ω of degree μ that satisfy the congruences:

$$\Omega \equiv L\, dt_1 \ldots dt_\mu \pmod{\omega_1, \ldots, \omega_n}, \qquad d\Omega \equiv 0 \pmod{\omega_1, \ldots, \omega_n}.$$

It is given by the formula (one always sums over repeated indices):

(*) Presented by Th. DE DONDER.

([1]) DE DONDER, Th. – *Sur le théorème d'indépendance de HIlbert.* C.R. Acad. Sci. Paris, t. 156, 1913, pp. 609-611 and 868-870.

– *Théorie invariantive du Calcul des Variations*, 1st ed., Paris 1930, new edition, 1935.

GEHENIAU, J. – *Sur la generalization de Th. De Donder du théorème d'indépendance de HIlbert*, C.R. Acad. Sci. Paris, session on 6 July 1936. pp. 32-34.

([2]) WEYL, H. – *Observations on Hilbert's Independence Theorem and Born's Quantization of FIeld Equations*, PHYSICAL REVIEW, vol. **46**, 2nd ser., 1934, pp. 505-508.

– *Geodesic Fields*, ANN. OF MATH., v. **36**, 1935, pp. 607-629.

([3]) CARATHÉODORY, C. – *Über die Variationsrechnung bei mehrfachen Integralen*, ACTA SZEGED, **4** (1929), pp. 193-216.

([4]) LEPAGE, Th. – *Sur les champs géodésiques du Calcul de Variations*, Bull. de l'Acad. R. de Belg., CL. DES SCI., XXII, 1936, pp. 710-729, 1036-1046.

– *Sur les champs géodésique des integrals multiples*, ibid., XXVII, 1941, pp. 27-46.

– *Champs stationaires, champs géodésiques et formes integrales*, ibid., XXVIII, 1942, pp. 73-92 and 247-265.

$$\Omega = L\, dt_1 \dots dt_\mu + p_{i\alpha}\, dt_1 \dots dt_{\alpha-1}\, \omega_i\, dt_{\alpha+1} \dots dt_\mu + A_{i\alpha j\beta}\, dt_1 \dots dt_{\alpha-1}\, \omega_i\, dt_{\alpha+1} \dots dt_{\beta-1}\, \omega_j\, dt_{\beta+1} \dots dt_\mu + \dots \tag{1.3}$$

The $A_{i\alpha j\beta}(t_\gamma, x_k, p_{k\gamma})$ and the coefficients of the unwritten terms (containing more than two factors of ω_i) are arbitrary.

Let $p_{i\alpha}(t_\gamma, x_k)$ be a field. It is stationary for a form Ω of the congruence (1.3) when – if $[\Omega]$ denotes what the form becomes for $p_{i\alpha} = p_{i\alpha}(t_\gamma, x_k)$ – $[\Omega]$ is integrable; i.e., when $d[\Omega] = 0$.

A stationary field envelops a manifold V_μ whose equations are $x_i = x_i(t_\alpha)$ if it satisfies:

$$p_{i\alpha}(t_\beta, x_j(t_\beta)) = \frac{\partial x_i}{\partial t_\alpha}$$

on this manifold.

One knows that V_μ is then *extremal*. If the $x_j(t_\beta)$ have second derivatives then they satisfy the *Euler-Lagrange equations:*

$$L_{x_i} - \frac{d\pi_{i\alpha}}{dt_\alpha} = 0. \tag{1.4}$$

Meanwhile, the stationary fields are not composed of a family of extremals, except when the system $[\omega_i] = 0$ is completely integrable.

Among the stationary fields, one obtains the *De Donder-Weyl fields upon annulling the arbitrary coefficients in* (1.3); they correspond to the form:

$$\Omega_0 = L\, dt_1 \dots dt_\mu + \pi_{i\alpha}\, dt_1 \dots dt_{\alpha-1}\, \omega_i\, dt_{\alpha+1} \dots dt_\mu\ . \tag{1.5}$$

As for the *Carathéodory fields*, they are given by the simple form Ω^* of the congruence (1.3).

In 1935, Weyl ([5]) constructed a field of the first type that envelops the manifold $x_i = 0$, which is assumed to be extremal. A year later, Boerner ([6]) enveloped an arbitrary extremal in a geodesic Carathéodory field; like Weyl, he used the method of Cauchy characteristics, but the calculations are long. Hölder ([7]) simplified the construction of Boerner by a change of variables. He showed that one comes down to the construction of a field for a simple integration problem. Th. Lepage recently showed ([8]) that this reduction, which is just as valid for De Donder-Weyl fields as it is for geodesic fields, is the consequence of a simple property of integrable forms.

In what follows, *we construct a De Donder-Weyl field that envelops an arbitrary extremal with the aid of the method of characteristics.* We will show in a later article that

([5]) WEYL, H. – *Geodesic Fields*, loc. cit.

([6]) BOERNER, H. – *Über die Extremalen und geodätischen Felder in der Variationsrechnung der mehrfachen Integrale*. MATH. ANN., bd. 112, 1936, pp. 187-220.

([7]) HÖLDER, E. – *Die infinitesimalen Berührungstransformationen der Variationsrechnung.* JAHRESBERICHT DER DEUTSCH, MATH. VER., bd. 49, 1939, pp. 162-178.

([8]) LEPAGE, TH. – *Champs stationnaires, champs géodésiques et formes intégrables*, loc. cit., pp. 262-265.

the change of variables that was indicated by Hölder reduces the construction of a geodesic field to the present construction. To appreciate the domain of application of the method of characteristics, we discuss the class of fields constructed. To that effect, we will have to use the following proposition:

If a manifold of class C_1 is enveloped by a field of class $C_{\rho-1}$ then it has class C_ρ (9).

Indeed, on the manifold V_μ the derivatives $\partial x_i / \partial t_\alpha = p_{i\alpha}(t_\beta, x_j(t_\beta))$ have class C_1, so V_μ has class C_2, and by induction, class C_ρ.

In what follows, we assume that L has class C_k and the extremal is enveloped by a field of class C_ρ. All of the constructions that we make will be local; we place ourselves in the neighborhood of an element $e_0(t_\alpha^0, x_i^0, p_{i\alpha}^0)$ that is tangent to V_μ.

2. De-Donder-Weyl field. Replace the ω_i in the Ω_0 by their values in (1.2); (1.5) then become:

(2.1) $$\Omega_0 = -\phi\, dt_1 \dots dt_\mu + \pi_{i\alpha}\, dt_1 \dots dt_\alpha\, dx_i\, dt_{\alpha+1} \dots dt_\mu, \qquad \phi = \pi_{i\alpha}\, p_{i\alpha} - L.$$

Assume that in a neighborhood of e_0 one has:

(2.2) $$|\, L_{p_{i\alpha}}\, p_{i\beta} \,| \neq 0.$$

One may then solve the equations:

(2.3) $$\pi_{i\alpha} = L_{p_{i\alpha}}(t_\beta, x_j, p_{j\beta})$$

with respect to the $p_{i\alpha}$. The change of variables that replaces the $p_{i\alpha}$ with the $\pi_{i\alpha}$ is of class C_{k-1}. The function ϕ (Hamilton function) is expressed by the aid of the canonical variables t_α, x_i, $p_{i\alpha}$. One immediately finds upon differentiating it that:

(2.4) $$\phi_{t_\alpha} = -L_{t_\alpha}, \qquad \phi_{x_i} = -L_{x_i}, \qquad \phi_{\pi_{i\alpha}} = p_{i\alpha}.$$

It is therefore of class C_k, like L.

This being the case, suppose that one has a De Donder-Weyl field that envelops the extremal V_μ, whose equations are $x_i = x_i(t_\alpha)$. With the aid of (2.3), it provides one with functions $\pi_{i\alpha}(t_\beta, x_j)$ that verify on V_μ:

(2.5) $$\pi_{i\alpha} = L_{p_{i\alpha}}\left(t_\beta, x_j(t_\beta), \frac{\partial x_j}{\partial t_\beta}\right).$$

It renders Ω_0 integrable:

(9) Recall that a continuous function is of class C_0 and that a function is of class C_ρ when its ρ^{th} derivatives exist and are of class C_ρ.

$$d[\Omega_0] = -\left\{\frac{\partial[\phi]}{\partial x_i} + \frac{\partial[\pi_{i\alpha}]}{\partial t_\alpha}\right\} dx_i dt_1 \cdots dt_\mu + \frac{\partial[\pi_{i\alpha}]}{\partial x_j} dx_j dt_1 \cdots dt_{\alpha-1} dx_j dt_{\alpha+1} \cdots dt_\mu = 0$$

or:

$$\frac{\partial[\phi]}{\partial x_i} + \frac{\partial[\pi_{i\alpha}]}{\partial t_\alpha} = 0, \qquad \frac{\partial[\pi_{i\alpha}]}{\partial x_j} = \frac{\partial[\pi_{j\alpha}]}{\partial x_i}.$$

There thus exist μ functions $S_\alpha(t_\beta, x_j)$ that satisfy the equations:

$$\frac{\partial S}{\partial x_i} = [\pi_{i\alpha}], \qquad \frac{\partial S}{\partial t_\alpha} + [\phi] = 0, \tag{2.6}$$

or equivalently:

$$[\Omega_0] = dS_1\, dt_2 \ldots dt_\mu + \ldots + dt_1 \ldots dt_{\alpha-1}\, dS_1\, dt_{\alpha+1} \ldots dt_\mu + \ldots + dt_1 \ldots dt_{\mu-1}\, dS_\mu\,. \tag{2.7}$$

From (2.6), it results that the S_α satisfy the *partial differential equation:*

$$\frac{\partial S_\alpha}{\partial t_\alpha} + \phi\left(t_\beta, x_j, \frac{\partial S_\beta}{\partial x_j}\right) = 0, \tag{2.8}$$

which is nothing but the *generalized Jacobi equation that was introduced by Th. De Donder* ([10]).

The problem to be solved is the following one: Find the solutions of equation (2.8) that satisfy the equalities:

$$\frac{\partial S_\alpha}{\partial x_i} = \pi_{i\beta}(t_\beta) = L_{p_{i\alpha}}\left(t_\beta, x_j(t_\beta), \frac{\partial x_j}{\partial t_\beta}\right) \tag{2.9}$$

on V_μ .

From (2.6), one then obtains the field by setting $[\pi_{i\alpha}] = \partial S_\alpha / \partial x_i$ and imposing the conditions (2.5). If the S_α are of class C_r then the field obtained is of class C_{r-1} .

3. Construction. Choose $\mu - 1$ functions $S_{\alpha'}(t_\beta, n_j)$, ($\alpha' = 2, \ldots, \mu$) that satisfy (2.9) on V_μ . We indicate how one obtains them in the following section, where we discuss the class of the functions being used. (2.8) becomes an equation in S_1 :

([10]) DE DONDER, TH. – *Réduction de la variation seconde – Généralization du théorème direct de Jacobi*, BULL. AC. R. BELG., CL. DES SC., 5th session, t. XVI, 5 April 1930, pp. 436-445.

– *Théorie invariantive du Calcul des Variations,* op. cit., 1st edition, pp. 85, eq. 444; new edition, pp. 120, eq. 711.

(3.1) $$\frac{\partial S_1}{\partial t_1} + \frac{\partial S_{\alpha'}}{\partial t_{\alpha'}} + \phi\left(t_\beta, x_j, \frac{\partial S_1}{\partial x_j}, \frac{\partial S_{\alpha'}}{\partial x_j} \right) = 0.$$

It possesses Cauchy characteristics along which the $t_{\alpha'}$ are constants and which one may refer to the parameter t_1 ; their equations are:

(3.3) $$\frac{dx_i}{dt_\alpha} = \phi_{\pi_{i1}}, \qquad \frac{d\pi_{i\alpha}}{dt_\alpha} = -\phi_{x_i}.$$

The curves $t_{\alpha'}$ = const. *in* V_μ *are the characteristics.*

Indeed, equations (1.4) become, in canonical variables:

(3.3) $$\frac{dx_i}{dt_\alpha} = \phi_{\pi_{i\alpha}}, \qquad \frac{d\pi_{i\alpha}}{dt_\alpha} = -\phi_{x_i}.$$

Like the $S_{\alpha'}$ that satisfy (2.9), along the curves $t_{\alpha'}$ = const. these equations give:

$$\frac{dx_i}{dt_1} = \phi_{\pi_{i1}}, \qquad \frac{d\pi_{i1}}{dt_1} = -\phi_{x_i} - \frac{d}{dt_{\alpha'}}\frac{\partial S_{\alpha'}}{\partial x_i}$$

$$= -\phi_{x_1} - \phi_{\pi_{j\alpha'}}\frac{\partial^2 S_{\alpha'}}{\partial x_i \partial x_j} - \frac{\partial^2 S_{\alpha'}}{\partial x_i \partial t_{\alpha'}}.$$

These are nothing but equations (3.3).

Having said this, take a function $s(t_{\alpha'}, x_j)$ such that one has:

$$\frac{\partial s}{\partial x_i} = \pi_{i1}(t_1^0, t_{\alpha'})$$

at the points of V_μ where $t_1 = t_1^0$, and integrate equations (3.2) with the initial conditions:

$$t_1 = t_1^0, \qquad t_{\alpha'} = \tau_{\alpha'}, \qquad x_i = \xi_i, \qquad \pi_{i1} = \frac{\partial s}{\partial \xi_i}(\tau_{\alpha'}, \xi_i).$$

From Cauchy's theorem, the characteristics obtained define the solution S_1 of (3.1), which reduces to s for $t_1 = t_1^0$; these characteristics consist of the curves $t_{\alpha'}$ = const. of V_μ. On V_μ, S_1 thus satisfies:

$$\frac{\partial S_1}{\partial x_i} = \pi_{i1}(t_\beta).$$

The construction is thus achieved. One may show that, after choosing the $S_{\alpha'}$ the determination of S_1 is equivalent to the construction of a field for a simple integral. It is a consequence of a property of differential forms ([11]).

4. Class of the field constructed. As we said before, we assume that L is of class C_k and V_μ is of class C_ρ, without being of class $C_{\rho+1}$. From the proposition that was established at the end of no. 1, *the field may have class at most* $C_{\rho-1}$. We seek to find the class of the field constructed above. We use the result from the theory of Cauchy characteristics: For a first-order partial differential equation of class C_q ($q \geq 2$) and initial conditions of class C_q, the solution is of class C_q. For the functions $S_{\alpha'}$ of no. 3, one may choose the functions to be of class $C_{\rho-1}$ if $k \geq \rho$.

$$S_{\alpha'}(t_\beta, x_j) = [x_i - x_i(t_\beta)]\ \pi_{i\alpha'}(t_\beta)\,.$$

Equation (3.1) is of class $C_{\rho-2}$ in its arguments t_β, x_i, $\partial S_1 / \partial t_1$, $\partial S_1 / \partial x_j$. Upon choosing s to be the $C_{\rho-1}$ function:

$$s(t_{\alpha'}, x_j) = [x_i - x_i(t_1^0, t_{\alpha'})]\ \pi_{i1}(t_1^0, t_{\alpha'})\,,$$

one obtains a function of class $C_{\rho-2}$ for S_1, at least if $\rho - 2 \geq 2$.

The field thus constructed has class $C_{\rho-3}$, *under the condition that* $k \geq \rho$.

The method is applicable only if $\rho \geq 4$.

A different choice of $S_{\alpha'}$ and s, may, in certain cases, ameliorate this result: If the $S_{\alpha'}$ are functions of class C_ρ and s is a $C_{\rho-1}$ function, with $k \geq \rho - 1$, $\rho \geq 3$, then the field obtained is of class $C_{\rho-2}$. If the $S_{\alpha'}$ are $C_{\rho+1}$ functions and s is a C_ρ function, with $k \geq \rho \geq 2$, then it has class $C_{\rho-1}$.

([11]) LEPAGE, TH. – *Champs stationnaires, champs géodésiques et formes intégrables.* loc. cit., no. 22, pp. 263.

On the Carathéodory fields and their construction by the method of characteristics

Note by LÉON VAN HOVE (*)

1. In a preceding note (1), we gave the construction of a De Donder-Weyl field that envelops an arbitrary extremal with the aid of the method of characteristics. That construction was given for a particular extremal.

In the present note, we recall, following Lepage, how the geodesic fields of Carathéodory are introduced with the aid of the alternating differential forms. We then consider a change of variables (dependent and independent), and then we show that with the aid of the algorithm of differential forms one can establish the invariance of geodesic fields and the conditions that their existence assumes without long calculations. To conclude, we show that a change of variables due to Hölder (2) reduces the construction of a Carathéodory field to the one that we already carried out for a De Donder-Weyl field.

Recall that the integral:

$$(1.1) \qquad I = \int_D L(t_\alpha, x_i, p_{i\alpha})\,dt_1 \cdots dt_\mu\,, \qquad p_{i\alpha} = \frac{\partial x_i}{\partial t_\alpha}\,, \qquad \alpha = 1, \ldots, \mu,\ \ i = 1, \ldots, n.$$

Lepage has associated the family of differential forms (3):

$$(1.2) \qquad \Omega = L\, dt_1 \ldots dt_\mu + \pi_{i\alpha}\, dt_1 \ldots dt_{\alpha-1}\, \omega_i\, dt_{\alpha+1} \ldots dt_\mu + \ldots\,,$$

which are defined by the congruences:

$$(1.3) \qquad \Omega = L\, dt_1 \ldots dt_\mu \qquad (\mathrm{mod}\ \omega_1, \ldots, \omega_h),$$

$$(1.4) \qquad d\Omega = 0 \qquad (\mathrm{mod}\ \omega_1, \ldots, \omega_h).$$

A field $p_{i\alpha}(t_\beta, x_j)$ is called *stationary* for a form Ω of the family if Ω becomes integrable when one makes $p_{i\alpha} = p_{i\alpha}(t_\beta, x_j)$. Let V_μ be a manifold $x_i = x_i(t_\alpha)$ of class C_l (l

(*) Presented by Th. DE DONDER.

(1) VAN HOVE, L. – *Sur la construction des champs de De Donder-Weyl par la méthode des caractéristiques*, ACAD. R. DE BELG., CL. DES SC., 7 August 1945, t. XXXI.

(2) HÖLDER, E., *Die infinitesimalen Berührungstransformationen der Variationsrechnung*. JAHRESBERICHT DER DEUTSCHE MATH. VER., bd. 49, 1939, pp. 162-178.

(3) One has set: $\pi_{i\alpha} = L_{p_{i\alpha}}$, $\omega_i = dx_i - p_{i\alpha}\, dt_\alpha$. The unwritten terms contain at least two factors of ω_i and have arbitrary coefficients.

Cf., LEPAGE, Th., *Sur les champs géodésiques du Calcul du Variations.* BULL. DE L'ACAD. R. DE BELG. CL. DES SC., XXII, 1936, pp. 710-720, 1036-1046.

– *Sur les champs géodésiques des intégrales multiples*, ibid., XXVII, 1941, pp. 27-46.

– *Champs stationnaires, champs géodésiques et formes intégrables*, ibid., XXVIII, 1942, pp. 73-92, 247-265.

≥ 2) and let $p_{i\alpha}(t_\beta, x_j)$ be a field – stationary or not – of class at least C_1 that envelops V_μ ([4]). We assume that $L(t_\alpha, x_i, p_{i\alpha})$ is of class C_k ($k \geq 2$). Let $[\Omega]$ denote what a form Ω of the family (1.2) becomes when one makes $p_{i\alpha} = p_{i\alpha}(t_\beta, x_j)$. An immediate calculation gives:

$$(1.5) \qquad d[\Omega] \equiv \left(L_{x_i} - \frac{d\pi_{i\alpha}}{dt_\alpha} \right) \omega_i \, dt_1 \ldots dt_\mu \qquad (\text{mod } \omega_1, \omega_2, \omega_3, \ldots, \omega_{h-1}, \omega_h)$$

at the points of V_μ.

The parentheses contain the left-hand sides of the Euler-Lagrange equations. Therefore, if V_μ is enveloped by a stationary field then it is extremal. (1.5) then becomes:

$$(1.6) \qquad d[\Omega] \equiv 0 \qquad (\text{mod } \omega_1, \omega_2, \omega_3, \ldots, \omega_{h-1}, \omega_h)$$

for a field that is or is not stationary.

2. Geodesic field of Carathéodory. Suppose that $L \neq 0$ for the contact element e_0 whose coordinates are $(t_\alpha^0, x_i^0, p_{i\alpha}^0)$, and agree to remain in a neighborhood of that element. The associated system of any form (1.2) then consists of forms $L\, dt_\alpha + p_{i\alpha}\, \omega_i$ ($\alpha = 1, \ldots, \mu$); as a consequence, the family contains one and only one simple form, which is written:

$$(2.1) \qquad \Omega^* = \frac{1}{L^{\mu-1}} \prod_1^\mu (L\, dt_\alpha + \pi_{i\alpha}\omega_i)$$
$$= \frac{1}{L^{\mu-1}} \prod_1^\mu \left\{ (L\, \delta_{\alpha\beta} - \pi_{i\alpha} p_{i\beta}) dt_\beta + \pi_{i\alpha} dx_i \right\}.$$

Lepage has confirmed that the Carathéodory fields are the ones that make this form integrable ([5]).

To simplify the expression (2.1), with Carathéodory we set:

$$(2.2) \qquad a_{\beta\alpha} = L\, \delta_{\alpha\beta} - \pi_{i\alpha}\, p_{i\beta},$$

and make the hypothesis that for the element e_0:

$$(2.3) \qquad a = |\, a_{\beta\alpha} \,| \neq 0.$$

We may then pass from the variables $p_{i\alpha}$, L to the variables $P_{i\alpha}$, F (canonical variables of Carathéodory) by the transformation:

$$(2.4) \qquad \pi_{i\alpha} = P_{i\beta}\, a_{\beta\alpha}, \qquad F = \frac{L^{\mu-1}}{a}.$$

([4]) I.e., one has: $\partial x_i / \partial t_\alpha = p_{i\alpha}[t_\beta, x_j(t_\beta)]$ on V_μ.

([5]) CARATHÉODORY, C., *Über die Variationsrechnung bei mehrfachen Integralen.* ACTA SZEGED, 4 (1929), pp. 193-216.

LEPAGE, Th. – Cf., pp. 2, ([3]).

The form Ω^* becomes:

$$\Omega^* = \frac{1}{F}\prod_1^{\mu}(dt_\alpha + P_{i\alpha}dx_i)\,. \tag{2.5}$$

In order for the Carathéodory transformation to be invertible we shall make the hypothesis that the Jacobian $D = |\,\partial P_{i\alpha} / \partial p_{j\beta}\,|$ must different from zero. This will permit us to express F as a function of t_α, x_i, $P_{i\alpha}$. In the determinant (2.3), let $A_{\alpha\beta}$ denote the algebraic minor of $a_{\alpha\beta}$ divided by a. One obtains by differentiating the first relation in (2.4):

$$\frac{\partial P_{i\alpha}}{\partial p_{j\beta}} = A_{\alpha\lambda}(\delta_{i1} + P_{i\rho}\,p_{l\rho})\;L_{p_{l\lambda}}\,p_{i\beta} + P_{i\beta}\,P_{j\alpha} - P_{i\lambda}\,A_{\alpha\lambda}\,\pi_{j\beta}\,.$$

However:

$$P_{i\beta}\,P_{j\alpha} - P_{i\lambda}\,A_{\alpha\lambda}\,\pi_{j\beta} = -\frac{1}{L}A_{\alpha\lambda}\,(\delta_{i1} + P_{i\rho}\,p_{i\rho})\,(\pi_{l\lambda}\,\pi_{j\beta} - \pi_{i\beta}\,\pi_{j\alpha}).$$

Thus ([6]):

$$\begin{aligned} D &= |\,A_{\alpha\beta}(\delta_{ij} + P_{i\rho}\,p_{j\rho})\,| \cdot |\,L_{p_{i\alpha},p_{j\beta}} - \frac{1}{L}(\pi_{i\alpha}\,\pi_{j\beta} - \pi_{i\beta}\,\pi_{j\alpha})\,| \\ &= |\,A_{\alpha\beta}\,|^n\,|\,\delta_{ij} + P_{i\rho}\,p_{j\rho}\,|^{\mu} \cdot |\,L_{p_{i\alpha},p_{j\beta}} - \frac{1}{L}(\pi_{i\alpha}\,\pi_{j\beta} - \pi_{i\beta}\,\pi_{j\alpha})\,|\,. \end{aligned} \tag{2.6}$$

Now, one has the identities:

$$\begin{aligned} &|\,\delta_{ij} + P_{i\rho}\,p_{j\rho}\,|\,|\,\delta_{ij}\,L - p_{i\alpha}\,\pi_{j\alpha}\,| = L^n, \\ &|\,\delta_{\alpha\beta} + P_{i\alpha}\,p_{i\beta}\,|\,|\,L\;\delta_{\alpha\beta} - \pi_{j\alpha}\,p_{j\alpha}\,| = L^{\mu}\,. \end{aligned}$$

Since $L \neq 0$, they entail the inequalities:

$$|\,\delta_{ij} + P_{i\rho}\,p_{j\rho}\,| \neq 0, \qquad |\,\delta_{\alpha\beta} + P_{i\alpha}\,p_{i\beta}\,| \neq 0. \tag{2.7}$$

The condition $D \neq 0$ finally reduces to:

$$|\,L_{p_{i\alpha},p_{j\beta}} - \frac{1}{L}(\pi_{i\alpha}\,\pi_{j\beta} - \pi_{i\beta}\,\pi_{j\alpha})\,| \neq 0.$$

The Carathéodory theory is therefore based on the following three hypotheses, which we assume to be satisfied at e_0:

$$L \neq 0, \qquad a \neq 0, \qquad |\,L_{p_{i\alpha},p_{j\beta}} - \frac{1}{L}(\pi_{i\alpha}\,\pi_{j\beta} - \pi_{i\beta}\,\pi_{j\alpha})\,| \neq 0. \tag{C}$$

([6]) The first two determinants of (2.6) have the order $n \cdot \mu$, where the rows carry the indices i, α and the columns, the indices j, β.

If the functions $p_{i\alpha}(t_\beta, x_j)$ define a geodesic field then they turn Ω^* into a simple integrable form $[\Omega^*]$. One knows that there then exist functions $S_\alpha(t_\beta, x_j)$, ($\alpha = 1, \ldots, \mu$), which satisfy:

(2.8) $$[\Omega^*] = dS_1 \cdot dS_2 \ldots dS_\mu ,$$

or, by identification (7):

(2.9) $$| S_{\alpha\beta} | \cdot | F | = 1,$$

(2.10) $$S_{\alpha'} = [P_{i\beta}]\, S_{\alpha\beta} .$$

These functions are solutions of a first-order partial differential equation that we will not have to utilize: indeed, we can reduce it to the equation for De Donder-Weyl fields.

Once more, recall the geometric significance of the transformation (2.4): It makes an n-dimensional element $E(t_\alpha, x_i, P_{i\alpha})$ at a point correspond to the μ-dimensional element $e(t_\alpha, x_i, p_{i\alpha})$ that is attached to the same point and defined by the equations:

$$dt_\alpha + P_{i\alpha}\, dx_i = 0.$$

One calls E the element *transverse* to e. E and e have no common direction, which is a result of (2.7).

3. Change of dependent and independent variables. Since the construction of a geodesic field is based on a change of variables, we indicate in this paragraph how a point-wise transformation T:

(T) $$t_\alpha = \tau_\alpha(\bar{t}_\beta, \bar{x}_j) , \qquad x_i = \xi_i(\bar{t}_\beta, \bar{x}_j) ,$$

is prolonged to a contact transformation T_e that acts on the μ-dimensional element $e(t_\alpha, x_i, p_{i\alpha})$ and an analogous transformation T_E that relates to the n-dimensional elements $E(t_\alpha, x_i, P_{i\alpha})$.

We assume that the transformation T, which is of class C_q ($q \geq 1$), has a non-zero Jacobian at the point $P_0(t_\alpha^0, \lambda_j^0)$. In a neighborhood of P_0 it admits an inverse T^{-1}, which we write:

(T^{-1}) $$\bar{t}_\alpha = \bar{\tau}_\alpha(t_\beta, x_j) , \qquad \bar{x}_i = \bar{\xi}_i(t_\beta, x_j) .$$

As above, we set $\omega_i = dx_i - p_{i\alpha}\, dt_\alpha$, and analogously $\Omega_\alpha = dt_\alpha + P_{i\alpha}\, dx_i$. Likewise:

$$\bar{\omega}_i = d\bar{x}_i - \bar{p}_{i\alpha} d\bar{t}_\alpha , \qquad \bar{\Omega}_\alpha = d\bar{t}_\alpha + \bar{P}_{i\alpha} d\bar{x}_i .$$

We may now establish the following proposition that relates to the elements e:

If one has:

(7) Cf., for example, GOURSAT, E., *Leçons sur le problème de Pfaff.* Paris, Hermann, 1922, Chap. III. One sets: $S_{\alpha\beta} = \partial S_\alpha / \partial t_\beta$, $S_{\alpha i} = \partial S_\alpha / \partial x_i$.

$$\left|\frac{\partial\bar{\tau}_\alpha}{\partial t_\beta}+\frac{\partial\bar{\tau}_\alpha}{\partial x_j}p_{j\beta}\right|\neq 0 \tag{3.1}$$

in a neighborhood of the element $e_0(t_\alpha^0, x_i^0, p_{i\alpha}^0)$ *then one may prolong T into one and only one contact transformation* T_e*, for which:*

$$\bar{\varpi}_i=\left(\frac{\partial\bar{\xi}_i}{\partial x_j}-\bar{p}_{i\alpha}\frac{\partial\bar{\tau}_\alpha}{\partial x_j}\right)\omega_j\,,\qquad \omega_i=\left(\frac{\partial\xi_i}{\partial\bar{x}_j}-p_{i\alpha}\frac{\partial\tau_\alpha}{\partial\bar{x}_j}\right)\bar{\varpi}_j\,. \tag{3.2}$$

One has, moreover, in the neighborhood of e_0 *:*

$$\left|\frac{\partial\bar{\xi}_i}{\partial x_j}-\bar{p}_{i\alpha}\frac{\partial\bar{\tau}_\alpha}{\partial x_j}\right|\neq 0,\qquad \left|\frac{\partial\xi_i}{\partial\bar{x}_j}-p_{i\alpha}\frac{\partial\tau_\alpha}{\partial\bar{x}_j}\right|\neq 0, \tag{3.3}$$

$$\left|\frac{\partial\bar{\tau}_\alpha}{\partial t_\beta}+\frac{\partial\bar{\tau}_\alpha}{\partial x_j}p_{j\beta}\right|\neq 0,\qquad \frac{\partial(\bar{p}_{i\alpha})}{\partial(\bar{p}_{j\beta})}\neq 0.$$

We set, to abbreviate:

$$\begin{aligned}
X_{ij}&=\frac{\partial\xi_i}{\partial\bar{x}_j}-p_{i\alpha}\frac{\partial\tau_\alpha}{\partial\bar{x}_j}, & T_{\alpha\beta}&=\frac{\partial\tau_\alpha}{\partial\bar{t}_\beta}+\frac{\partial\tau_\alpha}{\partial\bar{x}_j}\bar{p}_{j\beta},\\
X_{i\alpha}&=\frac{\partial\xi_i}{\partial\bar{t}_\alpha}+\frac{\partial\xi_i}{\partial\bar{x}_j}\bar{p}_{j\alpha}, & \bar{X}_{ij}&=\frac{\partial\bar{\xi}_i}{\partial x_j}-\bar{p}_{i\alpha}\frac{\partial\bar{\tau}_\alpha}{\partial x_j},\\
\bar{T}_{\alpha\beta}&=\frac{\partial\bar{\tau}_\alpha}{\partial t_\beta}+\frac{\partial\bar{\tau}_\alpha}{\partial x_j}p_{j\beta}, & \bar{X}_{i\alpha}&=\frac{\partial\bar{\xi}_i}{\partial t_\alpha}+\frac{\partial\bar{\xi}_i}{\partial x_j}\bar{p}_{j\alpha}.
\end{aligned} \tag{3.4}$$

One has identically:

$$d\bar{t}_\alpha=\bar{T}_{\alpha\beta}dt_\beta+\frac{\partial\bar{\tau}_\alpha}{\partial x_j}\omega_j\,, \tag{3.5}$$

$$\bar{\varpi}_i=\bar{X}_{ij}\omega_j+(\bar{X}_{i\beta}-\bar{p}_{i\alpha}\bar{T}_{\alpha\beta})\,dt_\beta\,, \tag{3.6}$$

$$\omega_i=X_{ij}\bar{\varpi}_j+(X_{i\beta}-p_{i\alpha}T_{\alpha\beta})\,d\bar{t}_\beta\,. \tag{3.7}$$

Since (3.1) is in effect, we may make the element $e(t_\alpha, x_i, p_{i\alpha})$ correspond to the element $\bar{e}(\bar{t}_\alpha, \bar{x}_i, \bar{p}_{i\alpha})$ that is given by (T) and the supplementary equations:

$$\bar{X}_{i\beta}=\bar{p}_{i\alpha}\bar{T}_{\alpha\beta}\,. \tag{T_e}$$

One then makes $\omega_i = 0$ in (3.6); one then has $\bar{\omega}_i = 0$. Substituting these values in (3.7), we easily obtain, with the aid of (3.5):

$$(T_e') \qquad X_{i\beta} = p_{i\alpha}\,T_{\alpha\beta}\,.$$

(3.6) and (3.7) thus reduce to equations (3.2). They obviously entail that $X_{ij}\bar{X}_{jk} = \delta_{jk}$, from which, the first two inequalities (3.3) result, and since (T_e) may be also written:

$$\bar{p}_{i\alpha}\frac{\partial\bar{\tau}_\alpha}{\partial t_\beta} - \frac{\partial\bar{\xi}_i}{\partial t_\beta} = \bar{X}_{ij}\,p_{j\beta}\,,$$

the transformation T_e is bijective.

On the other hand, (T_e') is equivalent to (T_e). We have already shown that (T_e) implies (T_e'); one likewise shows that (T_e') implies (T_e). Since the transformation is bijective, one will also have that $|\,T_{\alpha\beta}\,| \neq 0$, which is the third inequality in (3.3). The fourth one is a consequence of the equations:

$$\bar{X}_{ij}\cdot\delta_{\beta\gamma} = \frac{\partial\bar{p}_{i\alpha}}{\partial p_{j\gamma}}\bar{T}_{\alpha\beta}$$

that one obtains by differentiating (T_e) with respect to $p_{j\gamma}$.

Finally, one easily sees with the aid of (3.2) that T transforms a manifold V_μ with equations $x_i = x_i(t_\alpha)$ and tangent elements near e_0 into a manifold $\bar{V}_\mu$ that admits equations $\bar{x}_i = \bar{x}_i(\bar{t}_\alpha)$ of the same form, and that the elements (t_α, x_i, $\partial x_i / \partial t_\alpha$) and $(\bar{t}_\alpha, \bar{x}_i, \partial\bar{x}_i / \partial\bar{t}_\alpha)$ correspond under T_e. The transformation T_e is therefore a contact transformation in fact. It is the only contact transformation that prolongs T and relates the elements e, as (3.6) shows when one makes $\omega_i = \bar{\omega}_i = 0$.

We have an analogous proposition for the elements E. It will suffice for us to state:

If one has:

$$(3.8) \qquad \left|\frac{\partial\bar{\xi}_i}{\partial x_j} - \frac{\partial\bar{\xi}_i}{\partial t_\alpha}P_{j\alpha}\right| \neq 0$$

in a neighborhood of the element $E_0(t_\alpha^0, x_i^0, P_{i\alpha}^0)$ *then one may prolong* T *into one and only one contact transformation* T_E *for which:*

$$(3.9) \qquad \bar{\Omega}_\alpha = \left(\frac{\partial\bar{\tau}_\alpha}{\partial t_\beta} + \bar{P}_{i\alpha}\frac{\partial\bar{\xi}_i}{\partial t_\beta}\right)\Omega_\beta\,, \qquad \Omega_\alpha = \left(\frac{\partial\tau_\alpha}{\partial\bar{t}_\beta} + P_{i\alpha}\frac{\partial\xi_i}{\partial\bar{t}_\beta}\right)\bar{\Omega}_\beta\,.$$

One has, moreover, in a neighborhood of E_0 :

$$\left|\frac{\partial\bar{\tau}_\alpha}{\partial t_\beta}+\bar{P}_{i\alpha}\frac{\partial\bar{\xi}_i}{\partial t_\beta}\right|\neq 0, \qquad \left|\frac{\partial\tau_\alpha}{\partial\bar{t}_\beta}+P_{i\alpha}\frac{\partial\xi_i}{\partial\bar{t}_\beta}\right|\neq 0,$$

(3.10)

$$\left|\frac{\partial\xi_i}{\partial\bar{x}_j}-\frac{\partial\xi_i}{\partial\bar{t}_\beta}\bar{P}_{j\alpha}\right|\neq 0, \qquad \frac{\partial(\bar{P}_{i\alpha})}{\partial(P_{j\beta})}\neq 0.$$

We finally remark that if T is of class C_q then T_e and T_E are of class C_{q-1} .

4. Transformed variational problem. Consider the integral (1.1) in a neighborhood of the element $e_0(t_\alpha^0, x_i^0, p_{i\alpha}^0)$ and apply a change of variables T (no. 3) to it, where we assume only that it satisfies the condition (3.1) in a neighborhood of e_0 . As we have seen, there then exists a well-defined contact transformation T_e that prolongs T. The problem (1.1) is then transformed into another one whose integrand becomes, by means of the transformation T_e:

$$\bar{L}(\bar{t}_\alpha, \bar{x}_i, \bar{p}_{i\alpha}) = L(t_\alpha, x_i, p_{i\alpha})\left|\frac{\partial\tau_\alpha}{\partial t_\beta}+\frac{\partial\tau_\alpha}{\partial\bar{x}_j}\bar{p}_{j\beta}\right|. \tag{4.1}$$

Since Ω is a form of the family (1.2) attached to L, it satisfies the congruences (1.3) and (1.4). However, the ω_i are linear combinations of the $\bar{\omega}_i$ in (3.2). The transform $\bar{\Omega}$ of Ω by T thus satisfies the congruences:

$$\bar{\Omega} = L\,d\tau_1 \ldots d\tau_\mu = \bar{L}\,d\bar{t}_1\cdots d\bar{t}_\mu, \qquad d\bar{\Omega}\equiv 0 \pmod{\bar{\omega}_1\ldots\bar{\omega}_n}$$

and, for that reason, belongs to the analogous family that one may attach to $\bar{L}$. Thus, the latter is the transform under T of the family (1.2), and *a field that is stationary for L is transformed into a field that is stationary for $\bar{L}$* .

Since V_μ is an extremal for L, the relation (1.6) is valid at all of its points and any field that envelops it; the transformation gives:

$$d[\bar{\Omega}]\equiv 0 \pmod{\bar{\omega}_1\bar{\omega}_2,\ \bar{\omega}_1\bar{\omega}_3, \ldots \bar{\omega}_{n-1}\bar{\omega}_n}$$

The transformed manifold $\bar{V}_\mu$ is therefore extremal for $\bar{L}$.

We now assume that L satisfies the conditions (C) on which the Carathéodory theory (no. 2) is based in a neighborhood of e_0 , and that the transformation T satisfies (3.1) around e_0 and (3.8) around the element E_0 that is transverse to e_0 . We are then assured of the existence of T_e and T_E , and we may utilize the inequalities (3.3) and (3.9). Applying T to the form Ω^* (2.1), we obtain a simple form $\bar{\Omega}^*$ that, from the preceding discussion, belongs to a family attached to $\bar{L}$. Since, from (4.1), $\bar{L} \neq 0$ in the neighborhood of $\bar{e}_0$ [8] that form must be written:

$$\bar{\Omega}^* = \frac{1}{\bar{L}^{\mu-1}} \prod_1^\mu (\bar{L}\, d\bar{t}_\alpha + \bar{\pi}_{i\alpha}\bar{\omega}_i) = \frac{1}{\bar{L}^{\mu-1}} \prod_1^\mu (\bar{a}_{\beta\alpha}\, d\bar{t}_\alpha + \bar{\pi}_{i\alpha} d\bar{x}_i), \tag{4.2}$$

where we have set:

$$\bar{\pi}_{i\alpha} = \bar{L}_{\bar{p}_{i\alpha}}, \qquad \bar{a}_{\beta\alpha} = \bar{L}\,\delta_{\alpha\beta} - \bar{\pi}_{i\alpha}\bar{p}_{i\beta}.$$

However, one may also obtain $\bar{\Omega}^*$ by starting with (2.5); one then obtains, with the aid of T_E:

$$\bar{\Omega}^* = \frac{1}{\bar{F}} \prod_1^\mu (d\bar{t}_\alpha + \bar{P}_{i\alpha} d\bar{x}_i), \qquad \text{with} \qquad \frac{1}{\bar{F}} = \frac{\left| \dfrac{\partial \tau_\alpha}{\partial \bar{t}_\beta} + P_{i\alpha} \dfrac{\partial \xi_i}{\partial \bar{t}_\beta} \right|}{F}. \tag{4.3}$$

Upon identifying (4.2) and (4.3), one finds [9]:

$$\frac{1}{\bar{F}} = \frac{\bar{a}}{\bar{L}^{\mu-1}}, \qquad \bar{\pi}_{i\alpha} = \bar{P}_{i\beta}\,\bar{a}_{\beta\alpha}.$$

The first inequality gives $\bar{a} \neq 0$. The second one shows that if e and E are transversal relative to the problem L then the transformed elements $\bar{e}$ and $\bar{E}$ are also transversal relative to $\bar{L}$. One finally has:

$$\frac{\partial(\bar{P}_{i\alpha})}{\partial(p_{j\beta})} = \frac{\partial(\bar{P}_{i\alpha})}{\partial(P_{j\beta})} \cdot \frac{\partial(P_{i\alpha})}{\partial(P_{j\beta})} \cdot \frac{\partial(p_{i\alpha})}{\partial(\bar{p}_{j\beta})} \neq 0.$$

Thus, from the calculations that were made in no. 2 it results that the condition:

$$\left| \bar{L}_{\bar{p}_{i\alpha}\bar{p}_{j\beta}} - \frac{1}{\bar{L}}(\bar{\pi}_{i\alpha}\bar{\pi}_{j\beta} - \bar{\pi}_{i\beta}\bar{\pi}_{j\alpha}) \right| \neq 0$$

is likewise satisfied. $\bar{L}$ *thus satisfies the conditions* (*C*) *and a field that is geodesic for L is transformed into a field that is geodesic for* $\bar{L}$.

If C_q is of class of T, C_k, that of L, C_l, that of a manifold V_μ, and C_r, that of a field $p_{i\alpha}(t_\beta, x_j)$ then the classes of $\bar{L}$, $\bar{V}_\mu$, and the transformed field will be $C_{k'}$, $C_{l'}$, and $C_{r'}$, respectively, with:

---

[8] $\bar{e}_0$ denotes the transformed element of e_0.

[9] $\bar{a}$ denotes the determinant of $|\bar{\alpha}_{\alpha\beta}|$.

(4.4) $k' = \min(k, q-1), \qquad l' = \min(l, q), \qquad r' = \min(r, q-1).$

5. Construction of a geodesic field. In order to satisfy equation (2.8), which characterizes geodesic fields, one is given $\mu - 1$ of the functions S_α – say, $S_{\alpha'}$ ($\alpha' = 2, \ldots, \mu$) – arbitrarily and one seeks to determine the remaining function S_1 . S_1 is the solution of a first-order partial differential equation. One may solve it directly by the methods of characteristics ([10]); however, the question is simplified considerably by a change of variables that is due to Hölder (cf., reference ([2]), pp. 1), which consists of setting $\overline{t}_2 = S_2$, …, $\overline{t}_\mu = S_\mu$.

Equation (2.8) then becomes:

(5.1) $$[\overline{\Omega}^*] = d\overline{S}_1 d\overline{t}_2 \cdots d\overline{t}_\mu .$$

It may be solved upon remarking that in the left-hand side there must appear a linear form multiplied by $d\overline{t}_2 \cdots d\overline{t}_\mu$. This linear form must be integrable when one considers $\overline{t}_2$, …, $\overline{t}_\mu$ to be constant parameters. The problem is thus equivalent to the construction of a field for a simple integral ([11]).

One may also comment upon the similarity between (5.1) and the equation of De Donder-Weyl fields ([12]) and how it reduces to that construction; this is the method that we shall pursue.

Let there be an extremal V_μ with equations $x_i = x_i(t_\alpha)$. Suppose that conditions (C) are satisfied in a neighborhood of an element $e_0(t_\alpha^0, x_i^0, p_{i\alpha}^0)$ that is tangent to V_μ, and let $E[t_\alpha, x_i(t_\beta), P_{i\alpha}(t_\beta)]$ be elements transversal to the elements e that are tangent to V_μ ; in particular, E_0 be the element that is transversal to e_0 . To construct a geodesic field that envelops V_μ , we choose $\mu - 1$ functions $S_{\alpha'}(t_\beta, x_j)$, ($\alpha' = 2, \ldots, \mu$) that satisfy the inequalities:

(5.2) $$|S_{\alpha'\beta'}| \neq 0$$

(5.3) $$|S_{\alpha'\beta'} + S_{\alpha' j} \; p_{j\beta'}^0| \neq 0$$

at the point $P_0(t_\alpha^0, x_i^0)$, and satisfy equations (2.10) on V_μ , which may be written:

(5.4) $$S_{\alpha' j} = P_{i\beta}(t_\gamma) \cdot S_{\alpha'\beta'}$$

on it; later on, we shall point out a means of constructing such functions. The point-wise transformation T with the equations:

(5.5) $$\overline{t}_1 = t_1 , \qquad \overline{t}_{\alpha'} = S_{\alpha'}(t_\gamma, x_j), \qquad \overline{x}_i = x_i$$

([10]) BOERNER, H., *Über die Extremalen und geodätischen Felder in der Variationsrechnung der mehrfachen Integrale*, MATH. ANN., bd. 112, 1936, pp. 187-220.

([11]) Cf., for example, reference ([1]) on pp. 1, eq. (2.7).

([12]) LEPAGE, Th., *Champs stationnaires, champs géodésiques et formes intégrables*, BULL. DE L'ACAD. R. DE BELG., CL. DES SC., XXVIII, 1942, pp. 263.

has a non-zero Jacobian; it verifies (3.1) at e_0 and (3.8) at E_0. The results of no. 4 thus apply, and T replaces L with a problem $\bar{L}$ for which the manifold $\bar{V}_\mu$, which is the transform of V_μ, is extremal.

If we apply T_E to the functions $P_{i\alpha}(t_\beta)$ then we obtain functions $\bar{P}_{i\alpha}(\bar{t}_\beta)$, and one finds, upon transforming (5.4), the equations:

$$\bar{P}_{i\alpha'}(\bar{t}_\beta) = 0. \tag{5.6}$$

Substitute this in the relations $\bar{\pi}_{i\gamma'} = \bar{P}_{i\beta}\,\bar{a}_{\beta\gamma}$; at the point of $\bar{V}_\mu$ it becomes:

$$\bar{\pi}_{i\gamma'}(\delta_{ij} + \bar{P}_{j\alpha}\,\bar{p}_{i\alpha}) = 0, \qquad (\gamma' = 2, \ldots, \mu),$$

and from (2.7), $\bar{\pi}_{i\gamma'} = 0$. This being the case, the latter condition (C), when realized for the element $\bar{e}_0$, reduces to:

$$\left|\bar{L}_{\bar{p}_{i\alpha}\bar{p}_{j\beta}}\right| \neq 0.$$

That inequality is precisely the hypothesis that serves as the point of departure in the De Donder-Weyl theory (cf., pp. 1, reference ([1]), no. 2). Thus, envelop $\bar{V}_\mu$ with a De Donder-Weyl field, while taking the arbitrary functions $\bar{S}_2, \ldots, \bar{S}_\mu$ to be identically null (this is permissible, since $\bar{\pi}_{i\gamma'} = 0$ on $\bar{V}_\mu$). The field thus obtained satisfies ([13]):

$$[\bar{\Omega}_0] = d\bar{S}_1\, d\bar{t}_2 \cdots d\bar{t}_\mu\ ,$$

and since the form $[\bar{\Omega}_0]$ is simple, it must be equal to $[\bar{\Omega}^*]$, the only simple form in the family (no. 2). The field obtained is thus likewise geodesic. If we revert to the old variables then it remains geodesic, but generally ceases to be a De Donder-Weyl field.

6. Discussion of the class of the field constructed. Suppose that the function L is of class C_k and the extremal V_μ is of class C_l. We know (cf., pp. 1, ([1]), no. 1) that *a geodesic field that envelops V_μ has class at most C_{l-1}*.

Let C_q be the class of the transformation (5.5). As we saw in no. 4, the classes of $\bar{L}$ and $\bar{V}_\mu$ are $C_{k'}$ and $C_{l'}$, respectively, with $k' = \min(k, q-1)$, $l' = \min(l, q)$.

Recall the discussion that concludes the preceding note (cf., pp. 1, reference ([1]), no. 4) in the particular case of the De Donder-Weyl field constructed above. The class of the equation that determines $\bar{S}_1$ is k'.

([13]) Cf., reference ([1]) on page 1, nos. 2 and 3.

To evaluate the class of the initial conditions ([14]), we remark that $\bar{\pi}_{i1}$ is of class $C_{s'}$ on V_μ , with $s' = \min(k' - 1, l' - 1)$; the initial conditions are also of that class. The field constructed thus has class $C_{s'-1}$ if $s' \geq 2$. Now, return to the old coordinates. The geodesic field will have the class C_r with:

$$r = \min(q - 1, s' - 1) = \min(l - 2, q - 3)$$

if we assume that $k \geq l$.

In summation, the class of the field thus constructed depends on q – i.e., the choice of functions $S_{\alpha'}(t_\beta , x_j)$ that were in question in no. 5. To make this choice, one may use the following method, which was pointed out by Boerner ([15]). Let $\overset{1}{t}_\alpha$, $\overset{1}{x}_i = x_i$, $(\overset{1}{t}_\alpha)$ denote the coordinates of the points of V_μ and solve the equations:

$$t_\alpha = \overset{1}{t}_\alpha - P_{i\alpha}(\overset{1}{t}_\beta)(x_i - \overset{1}{x}_i)$$

with respect to the $\overset{1}{t}_\alpha$. This is possible, since the Jacobian of the system is the second determinant of (2.7). If $k \geq l$ then the $P_{i\alpha}(t_\beta)$ are of class C_{l-1} , and we obtain functions:

$$\overset{1}{t}_\alpha = \overset{1}{t}_\alpha (t_\beta , x_j),$$

which are of the same class. If the $s_{\alpha'}(\overset{1}{t}_\beta)$, $(\alpha' = 2, \ldots, \mu)$ are functions of class C_{l-1} then the functions of the same class:

$$S_{\alpha'}(t_\beta , x_j) = s_{\alpha'}[\overset{1}{t}_{\alpha'}(t_\beta , x_j)] \tag{6.1}$$

satisfy (5.4) on V_μ , as one easily confirms.

On the other hand – always on V_μ – one has:

$$(\delta_{\alpha\beta} + P_{i\alpha}\, p_{i\beta})\frac{\partial \overset{1}{t}_\beta}{\partial t_\rho} = \delta_{\alpha\beta} , \qquad \frac{\partial \overset{1}{t}_\beta}{\partial t_\rho} + \frac{\partial \overset{1}{t}_\beta}{\partial x_j}\, p_{j\rho} = \delta_{\beta\rho} ,$$

$$| S_{\alpha'\beta'}| = \left| s_{\alpha'\rho}\frac{\partial \overset{1}{t}_\beta}{\partial t_{\beta'}} \right|, \qquad \left| S_{\alpha'\beta'} + S_{\alpha' i}\, \overset{0}{p}_{i\beta'} \right|$$

$$= \left| s_{\alpha'\rho}\left(\frac{\partial \overset{1}{t}_\beta}{\partial t_{\beta'}} + \frac{\partial \overset{1}{t}_\beta}{\partial t_i}\, \overset{0}{p}_{i\beta'} \right) \right| .$$

([14]) Cf., the function $s(t_{\alpha'}, x_j)$ in reference (1) on pp. 1, no. 4.
([15]) …and which must satisfy (5.2), (5.3), and (5.4).

From the former equalities, the matrices $\left(\frac{\partial t_\beta^1}{\partial t_\rho}\right)$ and $\left(\frac{\partial t_\beta^1}{\partial t_\rho} + \frac{\partial t_\beta^1}{\partial x_j} p_{j\rho}\right)$ have the maximum rank μ. From the latter, one may choose the functions $s_{\alpha'}$ in such a way that the conditions (5.2) and (5.3) are satisfied.

The functions (6.1) thus chosen have class C_{l-1}. Therefore, $q = l - 1$ and $r = l - 4$. The condition $s' \geq 2$ becomes $l \geq 5$. The result is the following:

If $k \geq l \geq 5$ then the construction provides a geodesic field of class C_{l-4} .

The direct construction, without the change of variables, has the advantage of providing a field of class C_{l-3} if $k \geq l \geq 4$. However, it necessitates long calculations (cf., pp. 9, reference ([1])).

Calculus of variations, differential forms, and geodesics fields

By Paul DEDECKER (Brussels)

1. Introduction. – The theory of geodesic fields, which was introduced by Weierstrass in order to study the conditions that assure the minimum or maximum of a certain integral, introduces the manifold of contact points by the "canonical" method that is attributed to Hamilton and Jacobi, as well as H. Poincaré and E. Cartan. The extension of the results of Weierstrass to the case of multiple integrals is accomplished by two approaches. The first one, which was discovered by Hilbert for the case of a double integral of an arbitrary function, was extended to the general case of p-fold integral of $q = n - p$ arbitrary functions by de Donder, Hadamard, Volterra, and H. Weyl. The second one, which is due to the work of Caratheodory, has the advantage that it is possible to interpret the "independent variables" and "arbitrary functions" on the same basis. The use of the exterior differential forms of E. Cartan enabled Lepage to show that, in reality, there exists a continuous family of possible generalizations that includes both of the preceding cases as particular examples. The Lepage method was developed by Boerner, Debever, Hölder, Van Hove, Wagner, and the author.

Here, one begins to construct a "universal generalization" of the Weierstrass theory whose context is that of a fiber bundle E that has the manifold V_n^p of p-dimensional contact elements to a manifold V_n as its base and a numerical space as its fiber, which reduces to a point when $p = 1$ or $p = n - 1$, moreover. The manifold E is given a globally defined differential form ω of degree p that characterizes the celebrated expression for the integral invariant of E. Cartan: $\omega = p_i\, dq^i - H\, dt$; all of the obvious Lepage generalizations lead back to sections of the manifold E canonically. There exists a distinguished closed subset $Z \in E$, which is analogous to the set of "irregular" contact elements in the classical problem ($p = 1$). For the points of Z, one locally defines (instantaneous) canonical coordinates, a Hamiltonian function, and the exterior differential equations of Kähler-Cartan, which generalize the Hamilton-Jacobi equations, except that their indeterminate is a differential form of degree $p - 1$. There exists an indicatrix manifold I that is submerged in the manifold $\bar{V}_n^p$ of simple vectors on V_n and a figuratrix manifold F that is submerged in the manifold $\bar{U}_n^p$ of arbitrary p-covectors in V_n. If one takes the intersections, I_0 and F_0, of I and F with the fibers, $A_0 \subset \bar{V}_n^p$ and $A'_0 \subset \bar{U}_n^p$, over a point $x_0 \in V_n$ then the duality between I_0 and F_0 is further complicated since one uses an arbitrary p, including $p = 1$ and $p = n - 1$. The manifold F_0 is derived from a linear manifold L that is isomorphic to the fibers of E. It is, moreover, developable, i.e., it is swept out by the tangent hyperplanes that are fixed along each generatrix L. A generatrix L (or the corresponding hyperplane) corresponds to each point p of I_0, and a hyperplane that is tangent to I_0 at p corresponds to each point $p' \in L$. There exists canonical map λ: $E \to F$ that sends Z onto the edge of regression of F, which is a local isomorphism outside of F.

The present exposé intersects the memoir of Wagner [30] in several points. An essential difference in its intrinsic viewpoint is that the parametric Lagrangian function $\mathcal{L}$

is defined uniquely on $\bar{V}_n^p$, and not on the manifold $\bar{U}_n^p$ of arbitrary covectors. There are equally many points of intersection with the exposé of Reeb at this colloquium.

2. Definition of the variational problem. – In order to begin our considerations, we need an infinitely differentiable manifold V_n. On any open set A in V_n we define the algebra of infinitely differentiable singular cubic chains in A ([1]).

Recall that a **differentiable singular cube** (in the sequel, "differentiable" will always mean "infinitely differentiable"[2] ([3])) u if dimension p in A is the restriction to I^p (I denotes the closed interval [0, 1] of the number line $\mathbb{R}$) of a differentiable map of a neighborhood $V(I^p) \subset \mathbb{R}^p$ into A: u: $V(I^p) \to A$. The support of u is the image $u(V(I^p)) \subset A$. A cube will be called **regular** if the map that defines it is of rank p everywhere. It will be called **degenerate** if the map $(t^1, \ldots, t^p) \mapsto u(t^1, \ldots, t^p) \in A$ does not depend upon t^1. A **differentiable singular cubic chain** is a formal finite linear combination of singular cubes with real coefficients. It is said to have dimension p or to be a (regular, degenerate) p-chain if all of the cubes with non-null coefficients are of dimension p (regular, degenerate). To a cube u one associates the $2n$ $(p-1)$-cubes:

$$A_iu: (t^1, \ldots, t^{p-1}) \mapsto u(t^1, \ldots, t^{i-1}, 0, t^i, \ldots, t^{p-1})$$
$$B_iu: (t^1, \ldots, t^{p-1}) \mapsto u(t^1, \ldots, t^{i-1}, 1, t^i, \ldots, t^{p-1}),$$

which are called the **faces** of u, and the chain:

$$\partial u = \sum_{i=1}^{n} (-1)^n (A_iu - B_iu),$$

which is called the boundary of u. For $\tau \in I$, we further let u_τ denote the $(p-1)$-cube:

$$u_\tau: (t^1, \ldots, t^{p-1}) \mapsto u(\tau, t^2, \ldots, t^{p-1}),$$

and we call it the **slice** of height τ of the cube u. The chain $u_1 - u_0 - \partial u$ will be notated by λu (the **lateral** part of u), and finally, $\gamma_\tau u$ denotes the **reduced** p-cube:

$$\gamma_\tau u: (t^1, \ldots, t^{p-1}) \mapsto u(\tau t^1, t^2, \ldots, t^{p-1}).$$

The operators ∂, λ, A_i, B_i, γ_τ, and $u \mapsto u_\tau$ are then extended to chains by linearity, and one has $\partial(\partial c) = 0$ for all chains c. A **differentiable homotopy** of a differentiable p-chain

[1] For the notion of a manifold, see C. Ehresmann [15] and C. Chevalley [10], ch. III (the latter considers the analytic case, but the modifications that are needed in order to consider the infinitely differentiable case are immediate). For the notion of a differential algebra, see C. Chevalley [10], ch. V, and H. Cartan [7]. For the notion of a cubic chain, see J.P. Serre [25], ch. II.

[2]

[3] Moreover, the word "differentiable" will be often be omitted when there is no possible cause for confusion.

c is a differentiable chain $\bar{c}$ such that $A_1\bar{c} = \bar{c}_0 = c$. This homotopy is called **restricted** if $\lambda\bar{c}$ is degenerate.

For any p-form ω of V_n and any p-vector $\mathbf{X}_p$ we notate the **value** of ω on $\mathbf{X}_p$ by $<\omega, \mathbf{X}_p>$ and let Φ_ω denote the function $\Phi_\omega(\mathbf{X}_p) = <\omega, \mathbf{X}_p>$. If $\mathbf{x}_k$ is a k-vector that is tangent to V_n at q then one refers to the **interior product** of ω by $\mathbf{x}_k$, which is notated by $\omega \llcorner \mathbf{x}_k$, when one describes the $(p-k)$-form on the tangent space to V_n at q that is defined by:

$$< \omega \llcorner \mathbf{x}_k, \xi_{p-k}> = <\omega, \mathbf{x}_k \wedge \xi_{p-k}>,$$

for all $(p-k)$-vectors ξ_{p-k} that are tangent to V_n at q. Therefore, a $(p-k)$-form $\omega \llcorner \mathbf{x}_k$ on V_n corresponds to every k-vector field $\mathbf{X}_k$ on V_n ([4]).

To every (differentiable) differential p-form ω on V_n there corresponds a function Φ_ω on the manifold $\bar{V}_n^p$ of simple p-vectors on V_n; likewise, every p-cube u of V_n may be canonically prolonged to a p-cube $\tilde{u}$ in $\bar{V}_n^p$. When we compose the map $\tilde{u}$ with Φ_ω we obtain a function $I^p \to \mathbb{R}$ whose integral, in the elementary sense:

$$\int_{\mathrm{I}^p} (\Phi_\omega \tilde{u}) dt^1 \cdots dt^p$$

yields (by linearity) the definition of the integral $\int_c \omega$ of the p-chain ω over the p-chain c, which is a bilinear function of c and ω. This definition gives is Stokes's formula:

$$\int_{\partial c} \omega = \int_c d\omega$$

for any $(p+1)$-chain c and p-form ω, as well. If V_n and W_n are two differentiable manifolds and α is a differentiable map $\alpha: V_n \to W_n$ then to every p-chain c in V_n there corresponds an image p-chain αc in W_n, and to every p-form ω on W_n there corresponds an inverse image p-form $\omega\alpha$. One has the properties:

$$\alpha(\partial c) = \partial(\alpha c) \qquad\qquad d(\omega\alpha) = (d\omega)\alpha$$

$$\int_{\alpha c} \omega = \int_c \omega\alpha \, .$$

The pair (V_n, α) is an **embedded manifold** in W_n, and if $\omega\alpha = 0$ then one says that (V_n, α) is an **integral manifold** of the form ω. Each p-cube u in V_n transforms each q-form ω on V_n into a q-form ωu on $V(I^p)$, and if $\omega u = 0$ then one says that u is an **integral cube** for ω; these notions extend to chains immediately.

[4] For the notion of an interior product, see N. Bourbaki [3], Algebre, Livre II, ch. III, pp. 105, and H. Cartan [7]. (Among these authors, the value of a p-form ω on a p-vector $\mathbf{x}_p$ is notated by $<\mathbf{x}_p, \omega>$ instead of $<\omega, \mathbf{x}_p>$. As a result, the interior product that we notate on the right is notated on the left by them.) See also E. Cartan [6], pp. 83-84.

Definition. – Let ω be a differentiable p-form on the manifold V_n. One uses the term **variational problem** to describe the study of differentiable p-chains c that satisfy the following condition: for any restricted homotopy $\bar{c}$ of c the derivative of the function:

$$f(\tau) = \int_{\bar{c}\tau} \omega$$

is zero for $\tau = 0$. The chains that satisfy this condition are called the **extremal chains** of the problem. On the other hand, if one starts with an ideal $\mathcal{U}$ in $\Lambda(V_n)$ then the **variational problem** $\{V_n, \omega, \mathcal{U}\}$ consists of the study of p-chains that satisfy the condition when and only when each slice $\bar{c}_\tau$ of the homotopy annihilates the ideal $\mathcal{U}$ (i.e., it is an integral manifold for each form in the ideal $\mathcal{U}$). These chains are said to be **extremal modulo** $\mathcal{U}$. A p-manifold (W_p, α) that is embedded in V_n is called an **extremal manifold** when for every p-chain c of W_p the p-chain αc in V_n is extremal. The problem $\{V_n, \alpha\}$ is called **free** and the problem $\{V_n, \alpha, \mathcal{U}\}$ is called **bound.**

The bound problem may be formulated in the following form, which does not seem to be more general on a paracompact differentiable manifold, but is, in fact, when one replaces differentiability with analyticity. Let U_i ($i \in I$) be an open covering of V_n and for each U_i let there be given an ideal $\mathcal{U}_i \subset \Lambda(U_i)$, in such a manner that in $U_{ij} = U_i \cap U_j$ the ideals $\mathcal{U}_i$ and $\mathcal{U}_j$ generate the same ideal $\mathcal{U}_{ij} \subset \Lambda(U_{ij})$. Let there be given a p-form ω_i in each U_i such that one has $\omega_i - \omega_j \in \mathcal{U}_{ij}$ in U_{ij}. If we notate the families ω_i ($i \in I$) and U_i ($i \in$ I) by (ω) and $(\mathcal{U})$, and if c is a p-chain that makes ω_i zero in each U_i then one gives an obvious meaning to the symbol $\int_c (\omega)$. This results in the notion of the variational problem $\{V_n, (\omega), (\mathcal{U})\}$. In the sheaf language of Leray-H. Cartan ([5]), we are dealing with the sheaf-algebra $\mathcal{G}$ of germs of differentiable forms on V_n. The U_i define a subsheaf ideal $\mathcal{A}$ in $\mathcal{G}$, and the ω_i define a section of the quotient sheaf $\mathcal{G}/\mathcal{A}$. We then can define the notions of the variational problem $\{V_n, \omega, \mathcal{A}\}$ and extremals modulo a sheaf ideal $\mathcal{A}$ in $\mathcal{G}$.

3. Problems in fiber bundles. – In this case, a bound problem that one studies may be reduced to a free problem, in the following sense: One constructs a fiber bundle E over a base V_n and a p-form θ on E such that the extremals of the free problem $\{E, \theta\}$ (globally) project onto V_n in extremals of the bound problem $\{V_n, \omega, \mathcal{A}\}$. The preceding construction is classically known by the name of the **method of Lagrange multipliers**.

On a fiber bundle E with base V_n and projection π we use the subalgebra $H(E) < L(E)$ that consists of **semi-basic** differential forms, i.e., ones that are locally expressed in terms

[5] For the notion of sheaf (which was first introduced by Leray), see H. Cartan [8], [9].

of just the coordinate differentials of the base, although the coefficients may depend on the local fiber coordinates (as well as the those of the base). This algebra is not closed under the operator d. We also distinguish the **generic** p-cubes u of E, i.e., the ones for which the p-cube πu is regular, and finally, the **generic embedded** p-manifolds (W_p, α), i.e., the ones for which the map $\pi\alpha$ has rank p everywhere. By the term, **semi-restricted homotopy** of a p-chain c of E, we mean a homotopy $\bar{c}$ whose projection $\pi\bar{c}$ onto the base is restricted.

Definition. – Let ω be a semi-basic form of degree p on a fiber bundle E. The variational problem and the notions of extremal chains and extremal manifolds, as defined above, remain unscathed if one replaces if the restricted homotopies are replaced with semi-restricted homotopies.

4. Fundamental properties. – Suppose that we have a homotopy $\bar{c}$ of a p-chain c and a p-form ω on a manifold V_n. Stokes's theorem gives:

(1) $$f(t) = \int_{\bar{c}_t} \omega = \int_c \omega + \int_{\gamma_t \bar{c}} d\omega + \int_{\lambda\gamma_t \bar{c}} \omega$$

and, if the homotopy is restricted:

(2) $$f(t) = \int_{\bar{c}_t} \omega = \int_c \omega + \int_{\gamma_t \bar{c}} d\omega .$$

The above formulas also hold in a fiber bundle for a semi-basic form and a semi-restricted homotopy.

Consider the positive unit vector field X in $\mathbb{R}^{p+1}$ that is parallel to the first axis. A vector $(t^1, \ldots, t^{p+1})$ at the origin is transported by a $(p+1)$-cube $\bar{u}$ onto a vector $U(t^1, \ldots, t^{p+1})$ that is tangent to V_n, which permits us to associate a cube $U = X(\bar{u})$ in the fiber bundle $\bar{V}_n^1$ of vectors that are tangent to V_n with the cube $\bar{u}$. In a similar fashion, one associates slices of U, $U_t = X(\bar{u}_t)$, with slices $\bar{u}_t$; these notions extend by linearity to corresponding notions on chains. On the other hand, to each $(p+1)$-form $\psi \in \Lambda(V_n)$ we associate the p-form $i\psi \in H(\bar{V}_n^1)$ whose value for $\mathbf{X} \in \bar{V}_n$ is the interior product $\psi \llcorner \mathbf{X}$. If we agree to symbolically set:

(3) $$\int_{X(\bar{c}_t)} i\psi = \int_{\bar{c}_t} \psi \llcorner \mathbf{X}_t,$$

i.e., to write the left-hand side as if it were essentially an integral over a chain of V_n then one obtains:

Proposition 1.

The derivative of the function:

$$F(t) = \int_{\gamma_t \bar{c}} \psi$$

is equal to:

$$\frac{dF}{dt} = \frac{d}{dt}\int_{\gamma_t \bar{c}} \psi = \int_{\bar{c}_t} \psi \lrcorner \mathbf{X}_t.$$

This results from combining formulas (2) and (3).

Theorem 1.

In order for a p-chain c to be an extremal for the problem $\{V_n, \omega\}$ *it is necessary and sufficient that c nullifies* $(d\omega) \lrcorner \mathbf{X}$ *for all vector fields* $\mathbf{X}$ *on* V_n. *The same condition defines submanifold* (W_p, α) *as extremal. On the other hand, if* V_n *is fibered and semi-basic then this condition also determines extremals in the sense of fiber spaces.*

Proposition 2.

For any homotopy $\bar{c}_t$ *of c one has, at t* = 0:

$$\left.\frac{df}{dt}\right|_{t=0} = \frac{d}{dt}\int_{\bar{c}_t} \omega = \int_c (d\omega) \lrcorner \mathbf{X}_0 + \int_{\partial c} \omega \lrcorner \mathbf{X}_0.$$

Corollary.

If c is an extremal chain of the problem $\{V_n, \omega\}$ *then one has:*

$$\left.\frac{df}{dt}\right|_{t=0} = \int_{\partial c} \omega \lrcorner \mathbf{X}_0;$$

conversely, if each slice $\bar{c}_t$ *is extremal then one has:*

$$\int_{\bar{c}_1} \omega - \int_{\bar{c}_0} \omega = \int_{\lambda \bar{c}} \omega .$$

The set of differential forms $(d\omega) \lrcorner \mathbf{X}$ that correspond to each vector field $\mathbf{X}$ on V_n generates an ideal $A(d\omega) \subset \Lambda(V_n)$ that we call the **first associated system** of $d\omega$. Let (W_{p+1}, α) be an embedded submanifold of dimension p+1 in V_n, and suppose that we are given a foliation whose leaves W_p are of dimension p. If each of the submanifolds (W_{p+1}, α) annuls the ideal $A(d\omega)$ then the submanifold (W_{p+1}, α) is an integral manifold of $d\omega$. One may define the k^{th} **associated system** $A_k(d\omega)$ to be the ideal that is generated by the forms $(d\omega) \lrcorner \mathbf{X}_k$ that correspond to each field of k-vectors $\mathbf{X}_k$ on V_n, and one has an analogous property for a foliation with leaves W_{p+1-k} of dimension $p+1-k$. One has $A(d\omega) = A_1(d\omega)$. The former system corresponds to the classical characteristic systems that one encounters in a completely integrable Pfaffian system; the latter one defines the Cartan-Kähler system of exterior equations.

The ideal $A(d\omega)$ does not have a finite set of generators, in general. An ideal $A_U(d\omega) \subset A(U)$ is generated in each open set U of V_n. If U is the support of the local coordinates $x^1, \ldots, x^n$, then one can express $d\omega$ in terms of these variables, and $A_U(d\omega)$ will be generated by the n forms:

$$\frac{\partial(d\omega)}{\partial(dx^i)} \qquad (i = 1, 2, \ldots, n).$$

Proposition 3.

The extremal manifolds of a free problem $\{V_n, \omega\}$ *are the integral manifolds of the first associated system of* $d\omega$.

5. **Geodesic section. Excess function.** – Let ω be a semi-basic p-form on a fiber bundle E with base V_n and projection π. A section s over an open U in the base (i.e., a differentiable map s: $U \to E$ that gives the identity when composed with π) is called **geodesic** relative to the ω if the form ωs on U is closed: $d(\omega s) = (d\omega)s = 0$. A p-chain c in E is called **embedded** (or **incorporated**) in s if the image of πc is in U and $s\pi c = c$. Sometimes one says **geodesic field** instead of "geodesic section."

Suppose we are given a geodesic field and let $\bar{c}$ be a semi-restricted homotopy of a p-chain c that is incorporated in s. It is possible to express the difference:

$$\Delta_c = \int_{\bar{c}_1} \omega - \int_c \omega$$

in the form of an integral that depends only upon the maximal slice $\bar{c}_1$. Indeed, one has:

$$\int_c \omega = \int_{s\pi c} \omega = \int_{\pi c} \omega s = \int_{\pi \bar{c}_1} \omega s - \int_{\pi \bar{c}} d\omega s = \int_{\pi \bar{c}_1} \omega s = \int_{s\pi \bar{c}_1} \omega$$

from which:

$$\Delta_{\bar{c}} = \int_{\bar{c}_1} \omega - \int_{s\pi \bar{c}_1} \omega \,.$$

Let $\bar{V}_n^p$ ($\bar{E}^p$, resp.) denote the fiber bundle of simple p-vectors that are tangent to V_n (E, resp.), and let $\bar{E}$ denote the fiber bundle with base E that is the inverse image of $\bar{V}_n^p$ under the projection π: $E \to V_n$. There exists a canonical map of $\bar{E}^p$ onto E. Each p-cube $V(I^p) \to E$ canonically prolongs to a p-cube $V(I^p) \to \bar{E}^p$ that defines a p-cube σ_u: $V(I^p) \to \bar{E}^p$ when composed with the canonical map $\bar{E}^p \to \bar{E}$. Similarly, the p-cube $s\pi u$ defines a cube $\sigma_{s\pi u}$. Let F denote the fiber bundle that is the inverse image of the diagonal $\Delta \subset V_n \times V_n$ under the projection of the product $\bar{E} \times \bar{E} \times \mathbb{R}$ onto $V_n \times V_n$, which is regarded as a fibration. The pair (u, s) then corresponds to a cube $\sigma(u)$: $(t^\alpha) \mapsto (\sigma_u(t^\alpha), \sigma_{s\pi u}(t^\alpha), t^\alpha)$ in F, and, by linearity, if c is a p-chain of E then the pair (c, s) corresponds to a p-chain $s(c)$ of F. Conversely, to any semi-basic form ω on E there corresponds the function $\mathfrak{E}_\omega$ on F whose value at $(\mathbf{X}_p, \mathbf{X}'_p, t^\alpha) \in F$ is:

$$\mathfrak{E}_\omega(\mathbf{X}_p, \mathbf{X}'_p, t^\alpha) = \omega \llcorner \mathbf{X}_p - \omega \llcorner \mathbf{X}'_p .$$

Proposition 4.

With the preceding notations, one has:

$$\Delta_{\bar{c}} = \int_{\bar{c}_1} \omega - \int_c \omega = \int_{\sigma(\bar{c}_1)} \mathfrak{E}_\omega dt^1 \wedge \cdots \wedge dt^p .$$

The function $\mathfrak{E}_\omega$ is the **excess function** that is associated with the semi-basic form ω.

6. Classical problems. – We begin with a differentiable manifold V_n, to which we associate the fiber bundle V_n^p of oriented p-dimensional contact elements in V_n, and the fiber bundle $\bar{V}_n^p$ ($\bar{V}_n^{p*}$, resp.) of simple (non-null, resp.) p-vectors that are tangent to V_n. One notates the projections of V_n^p and $\bar{V}_n^p$ onto the base manifold by π and $\bar{\pi}$, and the canonical map $\bar{V}_n^p \to V_n^p$ by η. For $\bar{X}_p \in \bar{V}_n^p$, the p-element $X_p = \eta(\bar{X}_p)$ is said to be **subordinate** to $\bar{X}_p$.

To each p-chain c of V_n there canonically corresponds a p-chain $\bar{c}^*$ of $\bar{V}_n^p$ and - if it is regular – a generic p-chain $c^* = \eta(\bar{c}^*)$ of V_n^p. Similarly, to each regular oriented submanifold W_p of V_n there corresponds an oriented generic submanifold W_p^*, which is defined by its oriented contact p-element. If W_p is not oriented then the corresponding manifold W_p^*, along with π, defines a two-sheeted covering of W_p. Every submanifold of this type in V_n^p – whether it is oriented or not – will be called **multiple.** (It is necessarily generic.)

Let φ be a local coordinate system $x_\varphi^1, \cdots, x_\varphi^n$, that is defined in an open set U_φ of V_n, and let $\bar{U}_\varphi^*$ be the (open) set of p-vectors $\bar{X}_p$ whose origin is in U and whose composition with φ, namely, $X^{12\cdots p}$, is greater than zero; furthermore, let $U_\varphi^* = \eta(\bar{U}_\varphi^*)$ be the (open) set of subordinate contact p-elements. Each $X_p \in U_\varphi^*$ contains one and only one system of vectors $\xi_1, \ldots, \xi_p$ whose φ-coordinate matrix is:

$$(\xi_{\alpha\varphi}^r) = (\delta_\alpha^\beta, \xi_{\alpha\beta}^i)_{1\le\alpha\le p,\, p+1\le i\le n}^{i\le r\le n} \qquad (\delta_\beta^\alpha) = p\times p \text{ identity}.$$

The $(x^r_\varphi, \xi^i_{\alpha\varphi})$ constitute a local coordinate system φ^* in U^*_φ. It is clear that if Φ is the set of local coordinate systems on V_n then the family U^*_φ $(\varphi \in \Phi)$ covers V^p_n.

In the algebra $H(U^*_\varphi)$, let ω^i_φ denote the $n-p$ Pfaffian forms:

$$\omega^i_\varphi = dx^i_\varphi - \xi^i_{\alpha\varphi} dx^\alpha_\varphi$$

and let $\omega^{i_1\cdots i_k}_{\alpha_1\cdots\alpha_k}$ $(1 \le k \le \min(p, n-p))$ denote the p-forms that are obtained by replacing $dx^{\alpha_1}_\varphi, \ldots, dx^{\alpha_k}_\varphi$ with $\omega^{i_1}_\varphi, \ldots, \omega^{i_k}_\varphi$ in the expression $\omega_\varphi = dx^1 \wedge \ldots \wedge dx^p$. The ω^i_φ generate an ideal I_φ in $H(U^*_\varphi)$ and the $\omega^{i_1\cdots i_k}_{\alpha_1\cdots\alpha_k}$ generate an ideal J_φ. in $U^*_{\varphi\psi} = U^*_\varphi \cap U^*_\psi$; this generates the ideals $I_{\varphi\psi}$ and $J_{\varphi\psi}$, respectively.

Proposition 5.

In order for any submanifold A of dimension p in V^p_n to be multiple it is necessary and sufficient that it be generic and that its intersection with any U^*_φ be an integral manifold for I_φ or J_φ.

In particular, this proposition contains the following result, which is used in the ultimate proofs.

Proposition 6.

Any p-dimensional submanifold A in that is generic and integral for J_φ is also integral for the ideal I_φ (which is strictly larger than J_φ).

We propose to study the bound variational problem of the type $\{V^p_n, \Omega_\varphi, J_\varphi\}$, in which W_p is a semi-basic p-form on U^*_φ such that $\Omega_\varphi - \Omega_\psi \in J_{\varphi\psi}$ in $U^*_{\varphi\psi}$. This problem includes all of the classical problems: geodesics in a Riemannian space, Hamilton's principle in mechanics, minimal surfaces, etc.; these problems are usually presented as free problems in V_n.

Remark. – The data that J_φ defines is equivalent to that of a subsheaf $\mathcal{F}$ of the sheaf $\mathcal{H}$ of germs of semi-basic forms on V^p_n, and Ω_φ defines a section Ω of the quotient sheaf $\mathcal{H}/\mathcal{F}$.

To every semi-basic p-form θ in an open set $U^*_\varphi \subset V^p_n$ (i.e., to every function that associates a p-form $\theta(X)$ on the tangent space to V_n on $\pi(X)$ to every $X \in U^*_\varphi$) there corresponds the function:

$$\mathcal{L}(\theta) = \mathcal{L}: \bar{U}_\varphi^* \to \mathbb{R} \qquad (\bar{U}_\varphi^* = \eta^{-1}(\bar{U}_\varphi^*))$$
$$\mathcal{L}(\bar{X}) = \theta(\eta \bar{X}) \,\mathsf{L}\, \bar{X}\,.$$

The kernel of the map $\theta \to \mathcal{L}(\theta)$ is composed of p-forms in J_φ. As a result, to every family $\Omega = \Omega_\varphi$ ($\varphi \in \Phi$) such that $\Omega_\varphi - \Omega_\psi \in J_{\varphi\psi}$ there corresponds a function $\mathcal{L}(\Omega)$ that is well-defined on $\bar{V}_n^p$. This function is positive homogeneous of degree one, i.e., $\mathcal{L}(a\bar{X}) = a\mathcal{L}(\bar{X})$ for every real number $a \leq 0$. Conversely, to every positive homogeneous function $\mathcal{L}$ of degree one on $\bar{V}_n^p$ we may associate the function $L(\mathcal{L})_\varphi = L_\varphi$ and the form $\Omega(\mathcal{L})_\varphi \equiv \Omega_\varphi$, which is defined in U_φ^* by:

$$L_\varphi(\eta \bar{X}) = L\left(\frac{\bar{X}}{X^{12\cdots p}}\right), \qquad \Omega_\varphi = L_\varphi(X)\, dx_\varphi^1 \wedge \cdots dx_\varphi^p\,.$$

The family $\Omega(\mathcal{L}) = \Omega(\mathcal{L})_\varphi$ ($\varphi \in \Phi$) then corresponds to L, and one has $\Omega(\mathcal{L}(\Omega)) = \mathcal{L}(\Omega(\mathcal{L})) = \mathcal{L}$. In conclusion, the set of families Ω and the set of functions $\mathcal{L}$ are modules over the ring of differentiable functions on V_n^p, and the maps $\Omega \to \mathcal{L}(\Omega)$, $\mathcal{L} \to \Omega(\mathcal{L})$, are inverse isomorphisms between the modules. For any problem $\{V_n^p, \Omega_\varphi, J_\varphi\}$ the corresponding function $\mathcal{L}(\Omega)$ is called the **global Lagrangian function**, whereas the function L_φ that is defined in is called the **local Lagrangian function**. One notes that $\mathcal{L}(\Omega)$ is uniquely defined for the simple p-vectors on V_n, but not for arbitrary p-vectors ([6]).

7. The Lepage congruences and the fiber spaces E, $\mathfrak{E}$. – Consider p-forms $\bar{\Omega}$ on any open set U^* that satisfy the conditions:

(1) $$\bar{\Omega}_\varphi \equiv \Omega_\varphi \pmod{I_\varphi},$$

which may be written:

$$\bar{\Omega}_\varphi = L_\varphi dx_\varphi^1 \wedge \cdots dx_\varphi^p + \sum L_{i_1\cdots i_k}^{\alpha_1\cdots\alpha_k}\, \omega_{\alpha_1\cdots\alpha_k}^{i_1\cdots i_k}$$

or:

(2) $$\bar{\Omega}_\varphi = L_\varphi \omega_\varphi + \sum_m L_{m\varphi} \omega_\varphi^m\,.$$

[6] In most work, the function $\mathcal{L}$ is assumed to be defined on the manifold $\bar{V}_n^p$ of arbitrary p-vectors. For example, see R. Debever [11], A. Kawaguchi [21], V. Wagner [28]. Our method eliminates, *a priori*, the mystery of the "indeterminate" partial derivatives of the function $\mathcal{L}$, which relate to the coordinates in the fiber of V_n^p.

$\sum_m$ indicates a summation over the set M of symbols $m = {}^{i_1\cdots i_k}_{\alpha_1\cdots\alpha_k}$ for which $\alpha_1 \leq \ldots \leq \alpha_k \leq i_1 \leq \ldots \leq i_k$ ($1 \leq k \leq \min(p, n-p)$); the $L_{m\varphi}$ are arbitrary functions on U^*_φ.

If we represent the coordinates of a point $L_\varphi \in \mathbb{R}^m$ by $L_{m\varphi}$ then formula (2) defines a form $\bar{\Omega}_\varphi$ on $U^*_\varphi \times \mathbb{R}^m$; to any form $\Omega_{0\varphi}$ on U^*_φ that satisfies (1) there corresponds a section β_0: $U^*_\varphi \to U^*_\varphi \times \mathbb{R}^m$ such that $\Omega_{0\varphi} = \bar{\Omega}_\varphi\beta_0$. If φ and ψ belong to Φ then the local coordinate change $f^*_{\varphi\psi}$ in $U^*_{\phi\psi}$ (from φ coordinates to ψ coordinates) may be canonically prolonged to a coordinate change $f_{\varphi\psi}$ such that $\bar{\Omega}_\psi = \bar{\Omega}_\varphi f_{\varphi\psi}$. With the canonical projection $\tilde{\omega}_{1\varphi}: U^*_\varphi \times \mathbb{R}^m \to U^*_\varphi$ one has $f^*_{\varphi\psi}\tilde{\omega}_{1\psi} = \tilde{\omega}_{1\varphi} f_{\varphi\psi}$. We identify a point $P_\varphi \in U^*_\varphi \times \mathbb{R}^m$ with a point $P_\psi \in U^*_\psi \times \mathbb{R}^m$ if these points belong to $U^*_\varphi \times \mathbb{R}^m$ and the coordinate of P_ψ are transformed into the coordinates of P_φ by $f^*_{\varphi\psi}$. One verifies that these identifications are consistent and therefore define a fiber bundle E that has a base V^p_n, fiber $\mathbb{R}^m$, projection $\tilde{\omega}_1$ (which locally reduces to $\tilde{\omega}_{1\varphi}$), and the affine group of $\mathbb{R}^m$ for its structure group ([7]). The forms $\bar{\Omega}_\varphi$ induce a unique form $\bar{\Omega}$, which is globally defined on E, of degree p, and semi-basic relative to $\pi_1 = \pi \cdot \tilde{\omega}_1$.

Proposition 7.

Suppose that F is a differentiable manifold, α is a differentiable map of F into V^p_n, and Θ is a p-form that is semi-basic relative to $\pi \cdot \alpha$, and that they satisfy the following conditions:

1) *At each point of F the rank of α is equal to $p(n-p) = \dim V^p_n$.*
2) *For all $\varphi \in F$ one has $\Theta \equiv \Omega_{\varphi\alpha}(\mathrm{mod}\ I_{\varphi\alpha})$ in $\alpha^{-1}(U^*_\varphi)$.*

From these conditions there exists a unique map β: $F \to E$ such that:

$$\tilde{\omega}_1 \cdot \beta = \alpha \qquad \textit{and} \qquad \Theta = \bar{\Omega}\beta.$$

Let J'_φ be the ideal that is generated by J_φ in the algebra $\Lambda(U^*_\varphi)$. Consider the set of semi-basic p-forms $\bar{\Omega}_\varphi$ on $U^*_{\phi\psi}$ that satisfy the **Lepage congruences:**

$$\bar{\Omega}_\psi \equiv \Omega_\varphi, \qquad\qquad d\bar{\Omega}_\psi \equiv 0\ (\mathrm{mod}\ J'_\varphi),$$

which may be written:

[7] For the construction of fiber bundles from their local pieces, see C. Ehresmann [15], pp. 6 (associated fiber bundles) and N.E. Steenrod [25], pp. 14 (existence theorem).

$$\bar{\Omega}_\psi = L_\varphi\,\omega_\varphi + \frac{\partial L_\varphi}{\partial \xi^i_{\alpha\varphi}}\,\omega^i_{\alpha\varphi} + \sum_{m\in M'} L_{m\varphi}\omega^m_\varphi\,, \tag{3}$$

in which $\sum_{m\in M'}$ indicates a summation that is restricted to the subset $M' \subset M$ of symbols $m = {}^{i_1\cdots i_k}_{\alpha_1\cdots\alpha_k} \in M$ for which $2 \leq k \leq \min(p, n-p)$. If $\Lambda_{m\varphi}$ $(m \in M')$ are the coordinates of point $\Lambda_\varphi \in \mathbb{R}^{m'}$ then formula (3) defines a p-form $\tilde{\Omega}_\varphi$ of $U^*_\varphi \times \mathbb{R}^{m'}$ that is semi-basic with respect to $\pi \cdot \omega_{2\varphi}$ ($\tilde{\omega}_{2\varphi}$ denotes the canonical map $U^*_\varphi \times \mathbb{R}^m \to U^*_\varphi$). Any local coordinate change $f_{\varphi\psi}$ may be canonically prolonged to a coordinate change $f^*_{\varphi\psi}$ in $U^*_{\varphi\psi} \times \mathbb{R}^m$ such that:

$$\tilde{\Omega}_\varphi = \tilde{\Omega}_\psi f_{\varphi\psi} \quad \text{and} \quad f^*_{\varphi\psi}\tilde{\omega}_{2\psi} = \tilde{\omega}_{2\varphi} f^*_{\varphi\psi}\,.$$

By identifications that are analogous to the preceding ones one constructs a fiber bundle $\mathfrak{E}$ that has base V^p_n, fiber $\mathbb{R}^{m'}$, projection $\tilde{\omega}_2$ (which locally reduces to $\tilde{\omega}_{2\varphi}$), and the affine group of $\mathbb{R}^{m'}$ for its structure group. The forms $\tilde{\Omega}_\varphi$ induce a unique form $\tilde{\Omega}$ that is globally defined on $\mathfrak{E}$, of degree p, and semi-basic relative to $\pi_2 = \pi \cdot \tilde{\omega}_2 : \mathfrak{E} \to \mathrm{V}_n$.

Proposition 8.

Let $\mathfrak{F}$ be a differentiable manifold, let α be differentiable map of $\mathfrak{F}$ into V^p_n, and let Θ be a p-form on $\mathfrak{F}$ that satisfy the following conditions:

1) *At each point of $\mathfrak{F}$ the rank of α is $p(n-p)$.*
2) *For all $\varphi \in \Phi$ one has:* $\quad \Theta \equiv \Omega_{\varphi\alpha} \quad d\Theta \equiv 0 \pmod{J'_{\varphi\alpha}}$ *in* $\alpha^{-1}(U^*_\varphi)$.

From these conditions there exists a unique map β: $\mathfrak{F} \to \mathfrak{E}$ such that:

$$\tilde{\omega}_2 \cdot \beta = \alpha \qquad \text{and} \qquad \Omega = \tilde{\Omega}\,\beta.$$

Remark. – For $p = 1$ or $p = n - 1$, the forms $\bar{\Omega}_\varphi$ and $\tilde{\Omega}_\varphi$ may be written as:

$$\bar{\Omega}_\varphi = L_\varphi dx^1_\varphi \wedge \cdots \wedge dx^p_\varphi + L^\alpha_{i\varphi}\omega^i_{\alpha\varphi}$$

and

$$\tilde{\Omega}_\varphi = L_\varphi\,\omega_\varphi + \frac{\partial L_\varphi}{\partial \xi^i_{\alpha\varphi}}\,\omega^i_{\alpha\varphi}\,,$$

respectively. In these two cases the set M' is empty, and the manifold $\mathfrak{E}$ is identified with V_n^p. Moreover, for $p = 1$, $\tilde{\Omega}_\varphi$ is nothing but the form $p_i\, dq^i - Hdt$, which is the relative integral invariant of E. Cartan.

The manifold $\mathfrak{E}$ is mapped bijectively and canonically onto the submanifold of E that is locally defined by the equations:

$$L_{i\varphi}^{\alpha} = \frac{\partial L_\varphi}{\partial \xi_{\alpha\varphi}^{i}};$$

in the sequel, $\mathfrak{E}$ will be identified with this submanifold.

Theorem II.

1) *Every integral manifold of the free problem* $\{E, \bar{\Omega}\}$ *is an integral manifold of* J_φ, *but it may not be generic with respect to* π_1, *unless we have the (necessary) condition that it be situated in* $\mathfrak{E}$.
2) *Every extremal* $\tilde{W}_p$ *of the free problem* $\{\mathrm{E}, \bar{\Omega}\}$ *that is generic with respect to* π_1 *may be projected onto a "multiplicity" in* V_n^p *that is an extremal of the bound problem* $\{V_n^p, \Omega_\varphi, J_\varphi\}$.
3) *Every submanifold of* $\tilde{W}_p \subset \mathfrak{E}$ *that has the property that the projection of* $\tilde{W}_p$ *onto* V_n^p *is an extremal of the bound problem* $\{V_n^p, \bar{\Omega}_\varphi, J_\varphi\}$ *is an extremal of the free problem* $\{E, \bar{\Omega}\}$.

Denote the set of pairs (i, α) by N. We calculate $d\tilde{\Omega}$ in $\mathfrak{E}$ and form the $N\times N$ matrix A_φ of the coefficients of $d\xi_\alpha^i \wedge \omega_{\beta\varphi}^j$ in local coordinates:

$$A_\varphi = \| A_\varphi^{ij} \| = \left\| \frac{\partial^2 L}{\partial \xi_{\alpha\varphi}^{i} \partial \xi_{\beta\varphi}^{j}} + A_{ij\varphi}^{\alpha\beta} \right\|.$$

The points where this matrix is not regular will be called **extraordinary** points. This notion is independent of the local coordinates, and the set of extraordinary points Z is a closed subset of $\mathfrak{E}$. A submanifold of $\mathfrak{E} - Z$ will be called an **ordinary** manifold.

Theorem III.

1) *Every ordinary extremal manifold of the free problem* $\{\mathfrak{E}, \tilde{\Omega}\}$ *is an integral manifold of* $J_\varphi \bar{\omega}_2$. *If it is, moreover, generic, with respect to* π_2 *then it is projected onto* V_n^p *along an extremal multiplicity of the bound problem* $\{V_n^p, \Omega_\varphi, J_\varphi\}$.

2) *Every submanifold* $\tilde{W}_p$ *of* $\mathfrak{E}$ *with the property that its projection onto* V_n^p *is an extremal multiplicity of the bound problem* $\{V_n^p, \Omega_\varphi, J_\varphi\}$ *is an extremal manifold of the free problem* $\{\mathfrak{E}, \tilde{\Omega}\}$.

Let $\bar{K}$ be the set of sequences $(r) = (r_1, \dots, r_p)$ in which $1 \le r_\alpha \le n$, the r_α are all distinct, and each sequence is always assumed to be distinct from a permutation of the sequence 1, 2, …, p. Let $K \subset \bar{K}$ be the subset of strictly increasing sequences ($r_1 < r_2 < \dots < r_p$). To each sequence $(r) \in \bar{K}$ one associates the increasing sequence $K(r)$ that is obtained by permuting its elements and a number (r) that equals +1 or –1 depending upon whether the permutation is even or odd, respectively. One denotes the coordinates of a point in $\mathbb{R}^k$ by:

$$P_{(r_1 \cdots r_p)} = P_{(r)}((r) + K).$$

Instead of expressing $\tilde{\Omega}_\varphi$ by using the set of semi-basic forms dx^α, ω^j, as basis, as we did in (3), we may use the basis dx^α, dx^i. The form $\tilde{\Omega}_\varphi$ is then written:

$$\tilde{\Omega}_\varphi = -F_\varphi dx_\varphi^1 \wedge \cdots \wedge dx_\varphi^p + \sum_k B_{(r_1 \cdots r_p)\varphi} dx_\varphi^{r_1} \wedge \cdots \wedge dx_\varphi^{r_p},$$

in which the F_φ and the $B_{(r)\varphi}$ are functions of $x_\varphi^r, \xi_{\alpha\varphi}^i$, $\Lambda_{m\varphi}$ ($m \in M'$). It may happen that the sets K and $M' \cap L$ have the same number of elements and that the Jacobian matrix of the function $B_{(r)\varphi}$ with respect to $\xi_{\alpha\varphi}^i$, $\Lambda_{m\varphi}$ is regular at all ordinary points. One then deduces the following:

Theorem IV.

Let P_0 *be an ordinary point of* $\mathfrak{E}$ *and let* $p = \pi_2(P_0)$ *be its projection onto* V_n. *There exists a neighborhood* U_φ ($\varphi \in \Phi$) *of* P_0, *an open neighborhood* $\tilde{U}_\varphi \subset \mathfrak{E} - Z$ *of* P_0, *an open* $Q_{k\varphi} \subset \mathbb{R}^k$, *a function* H_φ *on* $U_\varphi \times Q_{k\varphi}$, *and a differentiable isomorphism:*

$$F_\varphi : U_\varphi \times Q_{k\varphi} \to \tilde{U}_\varphi,$$

that satisfy the following conditions:

1) *If* θ_φ *is the canonical map* $U_\varphi \times Q_{k\varphi} \to U_\varphi$ *then one has that:*

$$\theta_\varphi = \pi_2 \cdot F_\varphi .$$

2) *One has:*

$$\tilde{\Omega} F_\varphi = \sum_k P_{(r_1 \cdots r_p)} dx^{r_1} \wedge \cdots \wedge dx^{r_p} - H\, dx^1 \wedge \cdots \wedge dx^p .$$

3) *The function* H_φ *satisfies first order partial differential equations that express the fact that the p-vector* $\bar{X}$ *of* $\mathbb{R}^n$ *whose components are* $X^{12\dots p} = 1$:

$$X^{(r_1\cdots r_p)} = \frac{\partial^p H}{\partial(r_1,\cdots,r_p)}$$

is simple.

The map F_φ defines a system $\tilde{\varphi}_{P_0}$ of local coordinates $x^r_{\bar{\varphi}}$, $P_{(r)}$, which are called the **canonical coordinates** that are associated with the local coordinate system φ on V_n, and at an ordinary point P_0 the functions $P_{(r)\varphi} : \tilde{U}_\varphi \to \mathbb{R}$, and the function $H_\varphi \cdot F_\varphi^{-1} : \tilde{U}_\varphi \to \mathbb{R}$ are called the **momenta** and the **local Hamiltonian function**, resp., that are associated with $\tilde{\omega}_{P_0}$.

When we apply theorem I we obtain the equations of the extremals in canonical coordinates. In order to facilitate writing them, we introduce the functions:

$$\mathfrak{P}_{(r_1\cdots r_p)} = \xi(r_1,\cdots,r_p)P_{k(r_1\cdots r_p)} \qquad (r) \in \bar{K}$$

in $\mathbb{R}^k$. The equations in question can then be written in the form (the **canonical equations**):

$$\Theta_\varphi^{(r_1\cdots r_p)} = dx_\varphi^{r_1} \wedge\cdots\wedge dx_\varphi^{r_p} - \frac{\partial H_\varphi}{\partial P_{(r_1\cdots r_p)}} dx_\varphi^1 \wedge\cdots\wedge dx_\varphi^p = 0$$

$$\Omega_{r\varphi} \quad = \mathfrak{P}_{(r_2\cdots r_p)} \wedge dx_\varphi^{r_2} \wedge\cdots\wedge dx_\varphi^{r_p} - (p-1)\frac{\partial H_\varphi}{\partial x_\varphi^r} dx_\varphi^1 \wedge\cdots\wedge dx_\varphi^p$$

$$= 0 \qquad ((r) \in \bar{K}).$$

Consider the ideal $\mathfrak{a}_\varphi$ in the algebra $\Lambda(U_\varphi)$ that is generated by the $\Theta_\varphi^{(r_1\cdots r_p)}$, $\Theta_{i\varphi}$; it is, moreover, a sub-ideal of the ideal $\mathfrak{a}'_\varphi$ that is generated by the $\Theta_{\alpha\varphi}$. One may show that any p-dimensional manifold $\tilde{U}_\varphi$ that is integral for $\mathfrak{a}_\varphi$ and generic with respect to the map π_2 is also integral for $\mathfrak{a}'_\varphi$. This comes from the fact that the ideal $\mathfrak{a}_\varphi$ is incomplete in the following sense: the set $\Lambda(\tilde{U}_\varphi)$ of differential forms that vanish on any generic p-dimensional integral manifold for $\mathfrak{a}_\varphi$ constitutes an ideal $\mathfrak{b}_\varphi$ that is strictly greater than $\mathfrak{a}_\varphi$ and has the property that $\mathfrak{b}_\varphi \subset \mathfrak{a}_\varphi$.

This property generalizes the well-known fact of classical mechanics that the "energy equation" is a consequence of the other canonical equations. The p equations:

$$\Theta_{\alpha\varphi} = 0,$$

therefore merit the name of "**energy equations.**"

8. Indicatrix, figuratrix, transversality. – Let $\mathcal{L}_\Omega$ be a global Lagrangian function for the problem $\{V_n^p, \Omega_\varphi, J_\varphi\}$. One calls the submanifold of $\bar{V}_n^{p*}$ that is defined by the equation $\mathcal{L}_\Omega(\bar{X}) = 1$ the **indicatrix** $\mathfrak{I}$ of the problem. One considers $\bar{V}_n^{p*}$ to be the manifold of simple non-zero p-vectors on V_n, which is embedded in the manifold $\bar{D}_n^p$ of arbitrary p-vectors on V_n. The latter has fiber $\mathbb{R}^{\binom{p}{n}}$ and projection $\pi: \bar{D}_n^p \to V_n$.

Assume that the function $\mathcal{L}_\Omega$ is non-zero. There then exists a canonical map $\bar{\eta}: V_n^p \to \mathfrak{I}$ that gives the identity on η. Let P_0 be a point of $\mathfrak{E}$ at which the form $\tilde{\Omega}$ defines a form $\tilde{\Omega}(P_0)$ on the tangent space to V_n at $p_0 = \pi_2(P_0)$, i.e., a linear form (or covector) $\lambda(P_0)$ on the fiber $A(p_0) = \pi(p_0) \subset \bar{D}_n^{p'}$. Similarly, one defines a differentiable map λ of $\mathfrak{E}$ into the manifold $\bar{D}_n^{p'}$ of p-covectors on V_n, which is a fiber bundle over V_n whose fibers $A'(p_0)$ are isomorphic to the dual of and whose projection is $\pi' : \bar{D}_n^p \to V_n$. The manifold $\mathfrak{I}$ is a regular submanifold of $\mathbb{R}^{\binom{p}{n}}$ of dimension $p(n-p)$.

Let $\mathbf{p}_0$ be a point of $\mathfrak{I}$ that has the projection $\pi(\mathbf{p}_0) = p_0$. The hyperplanes in the fiber $A(p_0)$ over p_0 that are tangent to $I(p_0) = A(p_0) \cap \mathfrak{I}$ at $\mathbf{p}_0$ and do not pass through the origin consitute a linear family $L(\mathbf{p}_0)$ that is isomorphic to $\mathbb{R}^{m'}$. Each of these hyperplanes defines a linear form H_α on $A(p_0)$ with the property that the statement "$\bar{X}_p \in \alpha$" is equivalent to the statement "$<X_p, \Omega_\varphi \cdot J_\varphi> = 1$." The set of all of these hyperplanes that is defined when $\mathbf{p}_0$ varies over $\mathfrak{I}$ is a submanifold $\mathfrak{F} \subset \bar{D}_n^{p'}$ that is called the **figuratrix** of the problem $\{V_n^p, \Omega_\varphi, J_\varphi\}$. The $L(\mathbf{p}_0)$ constitute isomorphism classes, but they are not necessarily disjoint. One says that F is **pseudo-fibered** by the **pseudo-fibers** $L(\mathbf{p}_0)$.

Let $\mathbf{p}_0'$ be a point of $\mathfrak{F}$ that projects onto $P_0 = \pi'(\mathbf{p}_0')$, and let $L(\mathbf{p}_0)$ be a pseudo-fiber that passes through $\mathbf{p}_0'$ ($L(\mathbf{p}_0) \subset A(\mathbf{p}_0)$, $\mathbf{p}_0 \in \mathfrak{F}$). The hyperplane in the fiber $A'(\mathbf{p}_0) \cap \mathfrak{F}$ that is tangent to $F(\mathbf{p}_0) = A'(\mathbf{p}_0) \cap \mathfrak{F}$ at each point $\mathbf{p}' \in L(\mathbf{p}_0)$ defines a linear form on $A'(\mathbf{p}_0)$, i.e., a point $\mathbf{p}'$ of $A(\mathbf{p}_0)$ that is either $\mathbf{p}_0$ or the origin. In the first case, $\mathbf{p}'$ is called **ordinary**; in the second case, it is called **extraordinary**.

Remark. – Here, one sees an apparent essential difference between the cases $p = 1$ and $p = n - 1$ and the case of an arbitrary p. In the first case the indicatrix and the figuratrix are n–1-dimensional hypersurfaces in an n-dimensional vector space. The duality between these hypersurfaces may be expressed by a pointwise correspondence. In the general case the indicatrix **I** is a $p(n-p)$-dimensional hypersurface in a Grassmann cone Γ at every point of V_n, and the figuratrix **F** is a hypersurface of dimension $\binom{n}{p} - 1$ in a vector space of dimension $\binom{n}{p}$. The duality between **I** and **F** is expressed by a correspondence between, on the one hand, points of **I** and hyperplanes that are tangent to

F, and, on the other hand, hyperplanes that are tangent to **I** (in the $\binom{n}{p}$–dimensional vector space that is spanned by Γ) and points of **F**. On the whole, there are as many hyperplanes tangent to **F** as there are points of **I** and as many points of **F** as there are hyperplanes tangent to **I**. If **F** were $\binom{n}{p}$–1-dimensional and **I** were $p(n-p)$-dimensional then this would result in the existence of submanifold L on **F** (which turn out to be linear) along which the tangent hyperplane is fixed. In other words **F** is a "developable ruling," and the tangent hyperplane is indeterminate at certain points of each generatrix, which form an "edge of regression" on **F**.

Each covector $\mathbf{p}'$ on V_n at p_0, i.e., each point **p** of $\bar{D}_n^{p'}$ in the fiber $\pi'(\mathbf{p}'_0) = \mathbf{p}_0$, may be identified with a p-form $\Omega(\mathbf{p}')$ on the tangent space to V_n at p_0, and the map $\mathbf{p}' \to \Omega(\mathbf{p})$ is a semi-basic differential p-form on $\bar{D}_n^{p'}$.

Theorem V.

The map λ: $\mathfrak{E} \to \bar{D}_n^{p'}$ *maps* $\mathfrak{E}$ *onto the figuratrix* $\mathfrak{F}$ *and takes an ordinary (extraordinary) point of* $\mathfrak{E}$ *onto an ordinary (extraordinary) point of* $\mathfrak{F}$*. The fibers of* $\mathfrak{E}$ *are taken to the pseudo-fibers of* $\mathfrak{F}$*.* λ *is a differentiable isomorphism in a neighborhood of each ordinary point, and one has* $\bar{\Omega} = \Omega\lambda$*. In a fiber* $A' \subset \bar{D}_n^{p'}$ *that passes through a point* $\mathbf{p} \in \mathfrak{F}$ *the hyperplane that is tangent to* $\mathbf{F} = \mathfrak{F} \cap \mathrm{A}'$ *at is indeterminate if* $\mathbf{p}'$ *is extraordinary; in the other case it is identified with the point* $\mathbf{p} \in \mathbf{I}$ ([8]) *with the property that* $\mathbf{p}' \in L(\mathbf{p})$.

For any point $P \in \mathfrak{E}$ the element $\lambda(P) \in \mathfrak{F}$ is said to be **transversal** to P. $\lambda(P)$ also defines a homogeneous $(n-p)$-vector $\mu(P)$ (i.e., an n-p-vector that is given up to a factor) that is tangent to V_n at $\pi_2(P)$. This $(n-p)$-vector is also said to be **transversal** to P. By abuse of language, one also says that X_p = and $\lambda(P)$ or $\mu(P)$ are **transversal**, but the correspondence $X_p \to \lambda(P)$ or $\mu(P)$ is not unambiguous.

One verifies that the hypothesis $\mathcal{L}_\Omega(\bar{X}_p) \neq 0$ ($X_p = \eta(\bar{X}_p)$) implies the existence of one and only one element $P_0 \in \tilde{\omega}_2^1(X_p)$ such that $\lambda(P_0)$ or $\mu(P_0)$ is simple and the latter is thus identified with an $(n-p)$-dimensional contact element $X_{n-p} = \nu(X_p)$ that is tangent to ν_n at $\pi(X_p)$. That element $X_{n-p} = \nu(X_p)$ is called the **Caratheodory transversal contact element** at X_p. $\lambda(P_0)$ is nothing but the intersection of the pseudo-fiber $L(\bar{\eta}(X_p))$ with the Grassmann cone of simple p-vectors of $A'(\pi(X_p))$ in $A'(\pi(X_p))$. From this, one deduces that the if P_0 is ordinary then the map $X_p \to \nu(X_p)$ is locally two-to-one and differentiable.

---

[8] $\mathbf{I} = \mathbf{I}(\pi'(\mathbf{p}')) = \mathfrak{F} \cap A(\pi'(\mathbf{p}'))$.

9. Geodesic fields and complete figures. – The problem $\{E, \tilde{\Omega}\}$ is a problem on a fibered manifold since the form $\tilde{\Omega}$ is semi-basic with respect to the projection π_2: $\mathfrak{E} \to V_n$. It is useful to examine the particular case proposition 4 for which each slice $\bar{c}_t$ of the homotopy $\bar{c}$ is an integral chain of $J_\varphi\,\tilde{\omega}_2$. First, one has the following property, in which the word "multiplicity" is taken to mean "p-dimensional submanifold of $\mathfrak{E}$," which makes the projection onto $\bar{V}_n^p$ is a multiplicity in the sense of sec. 6.

Proposition 9.

Let s be a geodesic field, relative to $\tilde{\Omega}$, on the manifold $\mathfrak{E}$, which is considered to be fibered over the base V_n. Any multiplicity (or any generic integral chain of $J_\varphi\,\tilde{\omega}_2$) that is incorporated into the field s is extremal.

In order to define the excess function there is good reason to consider the inverse image F of the diagonal of $V_n \times V_n$ in the fibered manifold $\bar{\mathfrak{E}} \times \bar{\mathfrak{E}} \times \mathbb{R}^r$ with base $V_n \times V_n$ (see sec. 5). However, $\mathfrak{E}$ is provided with a canonical map $\gamma: \bar{\mathfrak{E}} \to \bar{V}_n^p$ that makes the fiber space the inverse image and has the projection $\delta: \bar{\mathfrak{E}} \to \bar{V}_n^p$. One lets $\bar{\mathfrak{E}}^{**}$ denote the set of all $e \in \bar{\mathfrak{E}}^*$ such that $\tilde{\omega} \cdot \delta(e) = \bar{\eta} \cdot \gamma(e)$, where $\bar{\mathfrak{E}}^* = \bar{\gamma}(\bar{V}_n^{p*})$. It is clear that σ_c is identified with a p-chain of $\bar{\mathfrak{E}}^{**}$ for any generic p-chain c that is integral for $J_\varphi\,\tilde{\omega}_2$. Conversely, $s\pi_2 c$ is not, in general, an integral chain of $J_\varphi\,\tilde{\omega}_2$, and $\sigma s\pi_2 c$ does not correspond to a chain of $\bar{\mathfrak{E}}^{**}$, but to a chain of $\bar{\mathfrak{E}}^*$. We therefore consider the inverse image F^* of the diagonal of $V_n \times V_n$ in $\bar{\mathfrak{E}}^{**} \times \bar{\mathfrak{E}}^* \times \mathbb{R}^p$ and identify the chain $\tilde{\sigma}_{\bar{c}_1}$ with a chain of F^*. The function $\mathfrak{c}^*_{\tilde{\Omega}} = \mathfrak{C}_{\bar{\Omega}}\big|_{F^*}$, which is the restriction of $\mathfrak{C}_{\bar{\Omega}}$ to F^*, is called the **Weierstrass excess function** for the problem $\{V_n^p, \Omega_p, J_\varphi\}$.

Proposition 10.

Let s be a geodesic field of E, relative to $\tilde{\Omega}$, and let $\bar{c}$ be a semi-restricted homotopy of a p-chain c of E that satisfies the following conditions:

1) $\bar{c}_1$ *is generic and integral for* $J_\varphi\,\tilde{\omega}_2$.
2) *c is incorporated into the field s.*

Under these conditions one has:

$$\int_{\bar{c}_1} \tilde{\Omega} - \int_c \tilde{\Omega} = \int_{\sigma(\bar{c}_1)} \mathfrak{C}^*_{\Omega}\, dt^1 \wedge \cdots \wedge dt^p \ .$$

Each geodesic field s over an open $U \subset V_n$ defines not only a point $s(x) \in \mathfrak{E}$ at each point $x \in U$, but also a contact p-element $X(x) = \tilde{\Omega}_2 s(x)$ and a homogeneous $(n-p)$-vector $Y(x) = \mu s(x)$, which both originate at x. The pair of fields in U, $(X(x), Y(x))$, is called the **complete figure** that is associated with the geodesic field s.

In the case $p = 1$ the field $X(x)$ is a vector field on U, which defines a foliation $\mathfrak{I}_1$ with one-dimensional leaves. The transversal field $Y(x)$ is a field of contact $(n-1)$-elements that is completely integrable, by virtue of the condition: $(d\tilde{\Omega})s = 0$. It therefore defines a foliation of U, $\mathfrak{F}_1$, that has $(n-1)$-dimensional leaves that are transversal to the leaves of $\mathfrak{I}_1$ ([9]).

Whenever p is arbitrary the field $X(x)$ is a field of contact p-elements that is not, in general, completely integrable, and, similarly, the field $Y(x)$ is not a field of contact $(n-p)$-elements; however, this is the case when the form $\tilde{\Omega}$ is simple. The condition $(d\tilde{\Omega})s = 0$ then implies that $Y(x)$ is completely integrable, and it therefore defines a foliation of U, $\mathfrak{F}_p$, by $(n-p)$-dimensional leaves. One then says that the geodesic field s is a **Caratheodory field**. When the field $X(x)$ is completely integrable (with $Y(x)$ arbitrary) it defines a foliation of U, $\mathfrak{T}_p$, by extremal p-dimensional leaves (i.e., one for which the corresponding multiplicities are extremal) that are transversal to the field $Y(x)$. One then says that the field s is a **Mayer field**. If the geodesic field s is both Caratheodory and Mayer then it defines two foliations, $\mathfrak{T}_p$ and $\mathfrak{F}_p$, into transversal leaves of dimension p and $n-p$, respectively. The complete figure that corresponds to them is called the **Caratheodory complete figure.**

Proposition 11.

> *When* $p = 1$, *any geodesic field is both Caratheodory and Mayer.*
> *When* $p = n-1$, *any geodesic field is Caratheodory.*

Upon conferring a result of R. Debever one obtains the following property, which roughly signifies that for $p > 1$ any foliation of V_n into extremal leaves locally defines a Mayer field.

Theorem VI.

Assume that $p > 1$. *Let* $\mathfrak{T}$ *be a foliation of an open* $U \subset V_n$ *by extremal leaves* W_p *(i.e., the corresponding multiplicities* W_p^* *are extremal) and let* s: $U \to V_n^p$ *be the section over* U *that associated the* p*-dimensional contact element that is tangent at* x *to the leaf that passes through each point of* $x \in U$. *Let* $U \subset U'$ *be an open set in which the foliation* $\mathfrak{T}$ *induces a fibration* Ψ *into fibers that are isomorphic to a* p*-dimensional ball. There then*

[9] In the problem that occur in optics the leaves of $\mathfrak{I}$ constitute a sheaf of trajectories of light rays, and the leaves of $\mathfrak{F}$ represent the successive positions of the corresponding wavefront.

exists a Mayer field s: $U' \to \mathfrak{E}$ *such that* $\tilde{\omega}_2 \cdot s = s$, *i.e., such that the associated foliation is nothing but the one that is induced by* $\mathfrak{T}$.

10. The Hamilton-Jacobi equation. – Let s be a geodesic field over an open $U \subset V_n$, let x_0 be a point of U such that $P_0 = s(x_0)$ is ordinary, let $\tilde{U}_\varphi$ be a canonical neighborhood of P_0 such that $U_\varphi = \pi_2(\tilde{U}_\varphi) \subset U$ is isomorphic to a ball and let x^r, $P(r)$, and H be the canonical coordinates and the corresponding Hamiltonian function, respectively. There then exists a differential (p-1)-form Σ in U_φ such that $d\Sigma = \tilde{\Omega}s$, and, if we denote the coefficients of $dx^{r_1} \wedge \cdots dx^{r_p}$ in $d\Sigma$ by $\dfrac{\partial \Sigma}{\partial x^{(r_1 \cdots r_p)}}$ then one sees that Σ satisfies the partial differential equation:

$$\frac{\partial \Sigma}{\partial x^{1\cdots p}} + H\left(x^r, \frac{\partial \Sigma}{\partial x^{(r_1\cdots r_p)}}\right) = 0 \qquad (r_1, \ldots, r_p) \in K, \tag{4}$$

which we call the **Hamilton-Jacobi equation** – generalized to a canonical open set $\tilde{U}_\varphi$. (One recovers the classical equation when $p = 1$.) Conversely, let Σ be a solution of that equation such that $(x^r, \partial\Sigma/\partial x^m) \in U \times \mathbb{R}^k$ are the coordinates of a point of $\tilde{U}_\varphi$ for each point of Σ. The map σ: $U_\varphi \to U \times \mathbb{R}^k$ that is defined by $\sigma(x^r) = (x^r, \partial\Sigma/\partial x^m)$ defines a geodesic field $s = F_\varphi \cdot \sigma$ on U_φ.

Definition. – One calls a differential (p–1)-form Σ on U_φ that depends differentiably on a point $\alpha = (\alpha_m)$ of an open set $Q \subset \mathbb{R}^k$ (in other words, it is a semi-basic (p–1)-form on $U_\varphi \times Q$) a **complete integral** of the Hamilton-Jacobi equation (4) if it satisfies the following conditions:

1) The points $(x^r, \partial\Sigma/\partial x^m)$ belong to $U_\varphi \times Q_{k\varphi}$.
2) For every $\alpha \in Q$ the form $\Sigma(\alpha)$ is a solution to (4).
3) At every point of $U_\varphi \times Q$ the $k \times k$ matrix $\left\| \dfrac{\partial^2 \Sigma}{\partial x^m \partial \alpha_m} \right\|$ is non-singular.

Let Σ be such a complete integral. The map:

$$\theta: (x^r, \alpha_m) \mapsto \left(x^r, \frac{\partial \Sigma}{\partial x^m}(x^1, \alpha_n)\right)$$

is a local isomorphism, and therefore defines local coordinates (x^r, α_m) on a neighborhood U_0 of every point $P_0 \in F_\varphi \cdot \theta(U_\varphi \times Q) \subset \tilde{U}_\varphi$. If one lets d_α denote the exterior derivative – but only with respect to the α variables – of a form on $U_\varphi \times Q$ then one has:

$$d_\alpha\Sigma = \sum \beta_m \wedge d\alpha_m \, ,$$

in which the β_m are semi-basic (p–1)-forms. For forms on $(F_\varphi \cdot \theta)^{-1}(U_0)$ one has:

$$d\Sigma = \Omega + \sum \beta_m \wedge d\alpha_m \qquad \text{with} \qquad \Omega = \tilde{\Omega}\, F_\varphi \cdot \theta.$$

Theorem VII (generalized Jacobi theorem).

With the preceding notations, every extremal manifold $W_p \subset U_0$ *satisfies the equations* $d\beta_m = 0$. *When* $p = 1$ *it satisfies the equations* $d\alpha_m = 0$, *as well.*

Proposition 12.

Any extremal p-chain in U_0 *satisfies the equation:*

$$\sum \beta_m \wedge d\alpha_m = 0.$$

Therefore, if $p = 1$ *for any extremal chain then one has:*

$$\int_c \tilde{\Omega} = \int_d \Sigma \qquad \text{where} \qquad d = (F_\varphi \cdot \theta)^{-1}(\partial c)\,. \tag{5}$$

Definition. – A foliation $\mathfrak{G}$ of $\mathfrak{E} - Z$ by p-dimensional manifolds is called **geodesic** if every leaf of $\mathfrak{G}$ is a generic extremal of the problem $\{\mathfrak{E}, \tilde{\Omega}\}$. A **$\mathfrak{G}$-primitive** of is a ($p$–1)-form Σ^* such that for any p-chain c that is situated in a leaf of $\mathfrak{G}$ one has:

$$\int_c \tilde{\Omega} = \int_{\partial c} \Sigma^* \, . \tag{6}$$

Theorem VIII.

Let $\mathfrak{G}$ *be a geodesic foliation of* $\mathfrak{E} - Z$ and let U *be an open set of* $\mathfrak{E} - Z$ *in which* $\mathfrak{G}$ *defines a foliation that is isomorphic to the foliation of a p-dimensional ball by balls in* $\mathbb{R}^{n-p} \times \mathbb{R}^k$. *There then exists a* $\mathfrak{G}$*-primitive for* $\tilde{\Omega}$ *in U.*

Remark. – A complete integral is semi-basic, but one does not impose the analogous condition on $\mathfrak{G}$-primitives; the notions coincide in the case where $p = 1$. Furthermore, any complete integral of a $\mathfrak{G}$-primitive plays a universal role, in the sense that (5) and (6) are true for any extremal.

BIBLIOGRAPHY

[1] G. A. BLISS, *Lectures on the Calculus of Variations*, Chicago, 1944.

[2] H. BOERNER, "Über die Extremalen und geodätischen Felder in Variationsrechnung der mehrfacher Integralen," Math. Ann. **112** (1936), 187-220.
"Über die Legendresche Bedingung und die Feldtheorie in der Variationsrechnung der mehrfacher Integralen," Math. Z., **46** (1940), 720-741.

[3] N. BOURBAKI, *Eléments de mathématique,* particularly Book II, *Algébre*, ch. III, Paris, Hermann (1948).

[4] C. CARATHEODORY, "Über die Variationsrechnung bei mehrfacher Integralen," Acta Szeged., **4** (1929), 193-216.

[5] *Variationsrechnung und partielle Differentialgleichungen erster Ordnung*, Teubner, Leipzig and Berlin, (1935).

[6] E. CARTAN, *Leçons sur les invariants intégraux*, Paris, Hermann, 1922.

[7] H. CARTAN, "Notions d'algébre différentielle," Coll. de Topologie du C.B.R.M., Brussels, 1950, 15-27.

[8] "Idéaux et modules de functions analytiques de variables complexes," Bull. Soc. Math. France, **78** (1950), 29-64.

[9] "Faisceaux sur un espace toologique," Séminaire de Topologie algébrique, Paris, E.N.S. 1950-51 (mimeographed notes).

[10] C. CHEVALLEY, *Theory of Lie Groups*, Princeton, 1946, esp. ch. 3 and 5.

[11] R. DEBEVER, "Les champs de Mayer dans le calcul des variations des intégrals multiples," Bull. Acad. R. Belg., **23** (1937), 809-815.

[13] P. DEDECKER, "Sur les intégrales multiples du calcul des variations," III$^{\rm rd}$ Congrés Nat. Sciences,Belgique, Brussels, 1950, 29-35.
"Equations différentielles extérieures et calcul de variations," Colloque du Topologie de Strasbourg, 1951-52 (mimeographed notes).

[14] Th. De DONDER, *Théorie invariantive du calcul des variations,* Paris, Gauthier-Villars, 1935.

[15] Ch. EHRESMANN, "Sur la théorie des espaces fibrés," Colloque de topologie algébrique, Paris, C.N.R.S. (1947), 3-15.

[16] J. HADAMARD, "Sur une question du calcul des variations," Bull. Soc. Math. France, **30** (1902), 253-256.

[17] "Sur quelques questions du calcul des variations," ibid., **33** (1905), 73-80.

[18] D. HILBERT, "Zur Variationsrechnung," Oeuvres, Springer, 1935, 38-55.

[19] E HÖLDER, "Die Infinitesimalen Berührungstransformationen der Variationsrechnung," Jahresb. D. M. Verein, **49** (1939), 162-178.

[20] E. KÄHLER, *Einführung in die Theorie der Systeme von Differentialgleichungen*, Teubner, Leipzig, 1934.

[21] A. KAWAGUCHI, “On areal spaces, I, II, III,” Tensor, 1, pp. 14-45, 67-88, 89-103.

[22] Th. LEPAGE, “Sur les champs géodésiques du calcul des variations,” Bull. Acad. R. Belg., **22** (1936), pp 716-729.
“Sur les champs géodésiques des intégrales multiples,” ibid., **27** (1941), 27-46.
“Champs stationaires, champs géodésiques et formes intégrables,” ibid., **28** (1942), 73-92 and 247-265.

[23] G. REEB, “Quelques propriétés des géodésiques d’un espace de Finsler et des variétés minima d’un espace de Cartan,” Colloque de Topologie de Strasbourg, February 1951 (mimeographed notes).

[24] J. P. SERRE, “Homologie singuliére des espaces fibrés, applications,” Ann. Math., **54** (1951), 425-505.

[25] N. E. STEENROD, *Topology of Fiber Bundles,* Princeton, 1951.

[26] L. VAN HOVE, “Sur les champs de De Donder-Weyl et leur construction par la méthode des caractéristiques,” Bull. Acad. R. Belg., **31** (1945), 278-285.
“Sur les champs de Caratheodory et leur construction par la méthode des caractéristiques,” ibid., 625-638.

[27] V. VOLTERRA, “Sulle equazioni differenziali che provengono de questioni di calcolo delle variazioni,” Rend. Acc. R. Lincei (1890).

[28] V. WAGNER, “Geometry of a space with an areal metric and its applications to the calculus of variations,” Mat. Sbornik, **19** (61) (1946), 341-406.

[29] H. WEYL, “Geodesic fields in the calculus of variations for multiple integrals,” Ann. Math., **36** (1935), 607-629.