Characteristic function chi-squared distribution:

Let us assume that we have the following function where X is normally distributed:

$$X \sim N(0, \sigma^2)$$
$$Y = X^2$$

We can state the following:

$$F_y(y) = P(Y \leq y) = P(X^2 \leq y) = P\left(|X| \leq \sqrt{y}\right) = F_x\left(\sqrt{y}\right) - F_x(-\sqrt{y})$$

By taking the derivative of the cumulative distribution function of Y, we get:

$$\frac{d}{dy}F_y(y) = \frac{d}{dy}\left(F_x\left(\sqrt{y}\right) - F_x\left(-\sqrt{y}\right)\right) = \frac{1}{2\sqrt{y}}P_x\left(\sqrt{y}\right) + \frac{1}{2\sqrt{y}}P_x(-\sqrt{y})$$

As the PDF of the stochastic variable X is even (It is symmetric due to the fact that it has a normal distribution), we can state:

$$= \frac{P_x(\sqrt{y})}{\sqrt{y}}$$

This results in:

$$\frac{P_x(\sqrt{y})}{\sqrt{y}} = \frac{1}{\sigma\sqrt{2\pi y}}e^{-\frac{y}{2\sigma^2}} \ with \ y \geq 0$$

We now can find the characteristic function through the moment generating function:

$$M_y(it) = E\left[e^{itY}\right] = \frac{1}{\sigma\sqrt{2\pi}}\int_0^\infty \frac{1}{\sqrt{y}}e^{-\frac{y}{2\sigma^2}}e^{ity}dy$$

We now make the following substitutions:

$$a = \frac{1}{\sigma\sqrt{2\pi}}$$
$$c = \sqrt{2}\sigma$$
$$= a\int_0^\infty \frac{1}{\sqrt{y}}e^{-\frac{y}{c^2}}e^{ity}dy = a\int_0^\infty \frac{1}{\sqrt{y}}e^{-y(\frac{1}{c^2}-it)}dy$$

We use the substitution rule:

$$\alpha = \sqrt{y}$$
$$\frac{d\alpha}{dy} = \frac{1}{2\sqrt{y}}$$
$$2\sqrt{y}\, d\alpha = dy$$
$$= 2a\int_0^\infty e^{-\alpha^2(\frac{1}{c^2}-it)}d\alpha = 2a\int_0^\infty e^{-\alpha^2(\frac{1-ic^2t}{c^2})}d\alpha$$

Making the following substitution:

$$\beta = \frac{c}{(1-ic^2t)^{\frac{1}{2}}}$$
$$= 2a\int_0^\infty e^{-\frac{\alpha^2}{\beta^2}}d\alpha$$

Using the substitution rule:

$$\varphi = \frac{\alpha}{\beta}$$
$$\frac{d\varphi}{d\alpha} = \frac{1}{\beta}$$
$$\beta\, d\varphi = d\alpha$$
$$= 2a\beta\int_0^\infty e^{-\varphi^2}d\varphi$$

We can clearly see that the integrand is symmetric. By extending the upper and lower limit of this Riemann integral to ∞ and $-\infty$ and taking the $\frac{1}{2}$ of the result we can find the answer. This extension leads to the well known Gaussian integral. For more information I would like to redirect to my website http://www.planetmathematics.com where one can find a document about this integral.

$$= 2a\beta\frac{1}{2}\int_{-\infty}^{\infty} e^{-\varphi^2}d\varphi = a\beta\sqrt{\pi}$$

Resubstituting a, β and c results in:

$$= \frac{1}{\sigma\sqrt{2\pi}}\frac{\sqrt{2}\sigma}{(1-i2\sigma^2t)^{\frac{1}{2}}}\sqrt{\pi} = \frac{1}{(1-i2\sigma^2t)^{\frac{1}{2}}}$$

Normally when we see the definition of the chi-square distribution we have the following:

$$Y = X_1^2 + X_2^2 + \cdots + X_n^2$$

Where X_i is independent and identically distributed. One advantage of Fourier transforming (calculating the characteristic function) of a summation of independent stochastic variables is that their respective characteristic functions can be multiplied in the Fourier Domain (or convolution in the spatial domain).

Proof:

$$M_y(it) = E\left[e^{itY}\right] = E\left[e^{it\sum_{i=1}^{n} X_i^2}\right] = E\left[e^{itX_1^2} \cdots e^{itX_n^2}\right]$$

Due to the independence of the stochastic variables and the properties of the expectance operator:

$$E\left[e^{itX_1^2} \cdots e^{itX_n^2}\right] = E\left[e^{itX_1^2}\right] \cdots E\left[e^{itX_n^2}\right] = \frac{1}{(1 - i2\sigma^2 t)^{\frac{N}{2}}}$$

Hence the characteristic function of the central chi-square distribution.