Problem Set 9

Discussion: Nov. 3, Nov. 8, Nov. 10 (on probability and binomial coefficients) The name after the problem is the designated writer of the solution of that problem. (No one is exempted these two weeks)

Discussion Problems

1. (Putnam 1989-B1) A dart, thrown at random, hits a square target. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Express your answer in the form $\frac{a\sqrt{b}+c}{d}$, where a, b, c, d are integers. (`Beth`)

   A dart, thrown at random, hits a square target. Find the probability that the point hit is nearer to the center than to any edge. We begin by determining the shape of the area that is closer to the center than to any edge.

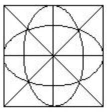
   

   Figure 1: Diagram

   This picture is a rough sketch of the 4 parabolas that give us the area that we need to find. The equations are:

$$\sqrt{(}x^2+y^2) = \frac{1}{2} - x$$
$$\sqrt{(}x^2+y^2) = \frac{1}{2} - y$$
$$\sqrt{(}x^2+y^2) = y + \frac{1}{2}$$
$$\sqrt{(}x^2+y^2) = x + \frac{1}{2}$$

   We need to solve for the point where the parabolas intersect, and setting the right hand side of the first two equations equal, we see it is where x=y. If we plug x in for y in the first equation, we get that

$$y = \frac{1}{2*(\sqrt{(}2)+1)}$$

Solving for y, we find that the equation for the first (top most) parabola is $y = \frac{1}{4} - x^2$. If we integrate this function from $\frac{-1}{2*(\sqrt{(2)}+1)}$ to $\frac{1}{2*(\sqrt{(2)}+1)}$, ie

$$\int_{\frac{-1}{2*(\sqrt{(2)}+1)}}^{\frac{1}{2*(\sqrt{(2)}+1)}} \frac{1}{4} - x^2 dx$$

We get $\frac{2-\sqrt{(2)}}{6}$. If we subtract the area of the rectangle and just leave the curve on top, we get:

$$\frac{2-\sqrt{(2)}}{6} - \frac{3-2\sqrt{(2)}}{2} = \frac{5\sqrt{(2)}-7}{6}$$

This gives us the area of the curved part, and multiplying by 4, we get the areas of all 4 curved sides. This gives us $\frac{10\sqrt{(2)}-14}{3}$. The area of the square inside the parabolas is $4*(\frac{1}{2\sqrt{(2)}+2})^2$ or $3-2\sqrt{(2)}$.

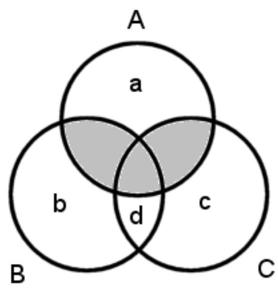
The total area, or the total probability is

$$3-2\sqrt{(2)} + \frac{10\sqrt{(2)}-14}{3} = \frac{4\sqrt{(2)}-5}{3}$$

So a=4, b=2, c=-5, and d=3.

2. How many 4 digit numbers are there (consisting of the digits 0 through 9) with no digit appearing exactly two times. (`Derek`)

3. A one-foot stick is broken at random in two places. What's the average length of the smallest piece? Middle piece? Largest piece? (`Frank`)



Let A, B, and C be the sets of students who solved questions A, B, and C respectively. Consider the regions described in the above diagram. Note that the area shaded in gray is given by $25 - a - b - c - d$ since 25 students solved at least one of the three questions. We have the following constraints:

$$a = b + c \tag{0.1}$$

$$b + d = 2(d + c) \tag{0.2}$$

$$a = 1 + 25 - a - b - c - d \tag{0.3}$$

Rearranging equations 2 and 3, we have:

$$b = d + 2c \tag{0.4}$$
$$2a = 26 - b - c - d \tag{0.5}$$

Substituting 1 into 5, and 4 into the result, we find:

$$\begin{aligned} 2(b+c) &= 26 - b - c - d \\ 3b + 3c + d &= 26 \\ 3b + 3c + (b - 2c) &= 26 \\ 4b + c &= 26 \end{aligned}$$

We know that $a, b, c, d \geq 0$, and from 4 that $d = b - 2c$. Further, $4b + c = 26$. The following table considers all possible values of b and c:

b	4b	c	b-2c
1	4	22	-
2	8	18	-
3	12	14	-
4	16	10	-
5	20	6	-
6	24	2	+

Therefore $b = 6$ and $c = 2$.

4. Two points are picked at random on the unit circle $x^2+y^2 = 1$. What is the probability that the chord joining the two points has length at least 1? (`Erin`)

   Solution: Since we are on the unit circle, we can pick any point on the circumference of the circle and draw the two chords from this point that are of length one. Since the radius of the circle is one it is easy to see that you can draw two equilateral triangles with the chosen point on the circumference serving as one of the common vertices (along with the origin). Since they are equilateral triangles their angles are all 60 degrees. This means that the angle of the two triangles together is 120. Since we are interested in the area outside these two triangles (since we are looking for the probability that the chord is longer than 1) we note that the angle for this area is 240 and since $240/360 = 2/3$, the probability that the chord is longer than 1 is $2/3$.

5. On a $m \times n$ checker board, choose two squares so that they are not in the same row or column. How many different choice do you have? (`Brett`)

   Solution. For each of the mn choices of the first square, there are $(m-1)(n-1)$ choices for the second square. But in order to avoid double counting of pairs you must divide by two. So there are $mn(m-1)(n-1)/2$ ways of choosing two squares so that they are in different rows and columns.

6. (Putnam 1961) Let α and β be given positive real numbers with $\alpha < \beta$. If two points are selected at random from a straight line segment of length β, what is the probability that the distance between them is at least α? (`David Edmonson`)

   Solution. This can be solved by considering which points (x, y) of the square x = 0 to β and y = 0 to β have |x - y| $\geq \alpha$. The acceptable points lie in two right-angled triangles: one formed with vertices at the points (0, α), (0, β), and (β - α, β) and the other formed with vertices at the points (α, 0), (β, 0), and (β, β - α). These two triangles fit together to form a square with side β - α, so the area of the acceptable points is (β - α)^2 out of a total area of the original square, which is β^2. Thus the probability is $\frac{(\beta-\alpha)^2}{\beta^2}$.

7. Pepys wrote Newton to ask which of three events is more likely: that a person get (a) at least 1 six when 6 dice are rolled, (b) at least 2 sixes when 12 dice are rolled, or (c) at least 3 sixes when 18 dice are rolled. What is the answer? (`Ben`)

8. Slips of paper with the numbers from 1 to 99 are placed in a hat. Five numbers are randomly drawn out of the hat one at a time (without replacement). What is the probability that the numbers are chosen in increasing order? (`Tina`)

   Solution:

   There are $\frac{99!}{(99-5)!}$ ways to choose 5 numbers from the 99, taking their order into account. For each set of five numbers, there are 5! = 120 ways to arrange them. Of these 120 ways, for each set of numbers only one of the arrangements will results in their being in increasing order. Therefore, P(five numbers are chosen in increasing order) = $\frac{1}{120}$.

9. In how many ways can n be written as a sum of k nonnegative integers, if the order is taken into account (so that, for example, $10 = 3 + 3 + 4$ and $10 = 3 + 4 + 3$ count as different representations)? (`Lei`)

10. (Putnam 1993-B3) Two real numbers x and y are chosen at random in the interval (0,1) with respect to the uniform distribution. What is the probability that the closest integer to x/y is even? Express the answer in the form $r + s\pi$, where r and s are rational numbers. (`Shelley`)

    Solution

    The probability that x/y falls between two numbers is represented as such: $P(a < x/y < b) \rightarrow P(ay < x < by), 0 \leq a < b < 1$ and P($\frac{x}{b} < y < \frac{x}{a}$), $1 < a < b$

    We take the integral of the first equation ($P(ay < x < by), 0 \leq a < b < 1$):

$$\int_0^1 \int_a^b dx\, dy. \rightarrow \int_0^1 (by - ay)\, dy. = \frac{b-a}{2}$$

    The integral of the second equation ($P(\frac{x}{b} < y < \frac{x}{a}), 1 < a < b$) is:

$$\int_0^1 \int_{\frac{x}{b}}^{\frac{x}{a}} dx\, dy. \rightarrow \int_0^1 (\frac{x}{a} - \frac{x}{b})\, dx. = \frac{1}{2a} - \frac{1}{2b}$$

In order for $\frac{x}{y}$ to be even, the probability must be

$$P(0 < \frac{x}{y} < \frac{1}{2}) + \sum_{n=1}^{\infty} P(2n - 1/2 < \frac{x}{y} < 2n + 1/2)$$

.

Calculating these probabilities, we see it equals

$$\frac{1}{2}(\frac{1}{2}-0)+\sum_{n=1}^{\infty}(\frac{1}{2(2n-\frac{1}{2})}-\frac{1}{2(2n+\frac{1}{2})}) = \frac{1}{4}+\sum_{n=1}^{\infty}(\frac{1}{4n-1}-\frac{1}{4n+1}) = \frac{1}{4}+(1-\frac{\pi}{4}) = \frac{5}{4}-\frac{\pi}{4}$$

11. (Putnam 1992-B2) For nonnegative integers n and k, define $Q(n, k)$ to be the coefficient of x^k in the expansion of $(1 + x + x^2 + x^3)^n$. Prove that

$$Q(n, k) = \sum_{j=0}^{k} \binom{n}{j} \binom{n}{k-2j},$$

where $\binom{a}{b}$ is the standard binomial coefficient. (Reminder: For integers a and b with $a \geq 0$, $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ for $0 \leq b \leq a$, with $\binom{a}{b} = 0$ otherwise.) (`Nicholas`)

12. (Putnam 1958-B3) Real numbers are chosen at random from the interval $[0, 1]$. If after choosing the n-th number the sum of the numbers so chosen first exceeds 1, show that the expected or average value for n is e. (`Richard`)

13. (MIT training problem) Three closed boxes lie on a table. One box (you don't know which) contains a \$1000 bill. The others are empty. After paying an entry fee, you play the following game with the owner of the boxes: you point to a box but do not open it; the owner then opens one of the two remaining boxes and shows you that it is empty; you may now open either the box you first pointed to or else the other unopened box, but not both. If you find the \$1000, you get to keep it. Does it make any difference which box you choose? What is a fair entry fee for this game? (Frank has discussed this problem in his presentation, but here you also need to calculate the expected value (fair entry fee).) (`David Rose`)

1 Solution

It does make a difference which box you choose. If you do not switch, your odds of picking the correct box are simply $\frac{1}{3}$. Now, assume you switch. If you had originally picked an empty box, you now win. The original odds that you picked an empty box are $\frac{2}{3}$, thus your odds of winning are $\frac{2}{3}$ if you switch. A fair price to pay for playing this game is \$500. To see this, assume that you are playing with no strategy. Then, the odds that you switch are $\frac{1}{2}$, and these are also the odds that you don't switch. Thus, your odds of winning are $\frac{1}{3} * \frac{1}{2} + \frac{2}{3} * \frac{1}{2} = \frac{1}{2}$, and $\frac{1000}{2} = 500$.