Finding Alternative Musical Scales

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Abstract. We search for alternative musical scales that share the main advantages of classical scales: pitch frequencies that bear simple ratios to each other, and multiple keys based on an underlying chromatic scale with tempered tuning. We conduct the search by formulating a constraint satisfaction problem that is well suited for solution by constraint programming. We find that certain 11-note scales on a 19-note chromatic stand out as superior to all others. These scales enjoy harmonic and structural possibilities that go significantly beyond what is available in classical scales and therefore provide a possible medium for innovative musical composition.

1 Introduction

The classical major and minor scales of Western music have two characteristics that make them a fertile medium for musical composition: pitch frequencies that bear simple ratios to each other, and multiple keys based on an underlying chromatic scale with tempered tuning. Simple ratios allow for rich and intelligible harmonies, while multiple keys greatly expand possibilities for complex musical structure. While these traditional scales have provided the basis for a fabulous outpouring of musical creativity, expressive power, and structural sophistication over several centuries, one might ask whether alternative scales with the same favorable characteristics—simple ratios and multiple keys—could unleash even greater creativity.

We take a step toward answering this question by undertaking a systematic search for musically appealing alternative scales. We restrict ourselves to diatonic scales, whose adjacent notes are separated by a whole tone or semitone. We conduct the search by defining a constraint satisfaction model that, for each suitable diatonic scale, seeks to assign relatively simple ratios to intervals in the scale. The ratios must be amenable to tuning based on equal temperament. To the extent that such an assignment of ratios is possible, the scale is a potential candidate for musical use.

Constraint programming is well adapted to this problem because it naturally expresses a recursive condition requiring that each note bear simple ratios with some other notes, but not necessarily with the tonic. The constraint programming model also solves in a reasonable amount of time.

We find that while the classical 7-note scales deserve the attention they have received, certain 11-note scales based on a 19-note chromatic stand out as possibly even more attractive, based on the criteria developed here. To our knowledge, this is a new result.

After a brief review of previous work, we present a rationale for preferring the simple ratios and multiple keys that characterize the classical scales. We then discuss how these characteristics can be precisely formulated as criteria for nonstandard scales, and we state a constraint programming model that formulates the criteria. We then present computational results, focusing on scales based on the 12-note and 19-note chromatics. We exhibit several particular scales that composers may wish to investigate.

2 Previous Work

Composers have experimented with a number of alternative scales in recent decades. One of the most discussed is the Bohlen-Pierce scale, which consists of 9 notes on a 13-tone tempered chromatic scale [4, 9]. The scale spans a twelfth, rather than the traditional octave. It treats notes that lie a twelfth apart as equivalent, much as traditional scales treat notes an octave apart as equivalent. Composers Richard Boulanger, Ami Radunskaya and Jon Appleton have written pieces using the Bohlen-Pierce scale [18]. In this paper we focus instead on scales that span the traditional octave, due to the ear's strong tendency to identify tones an octave apart, and the interesting possibilities that remain to be explored among these scales. Pierce [17] also experimented with a scale that divides the octave into 8 equal intervals, but we will find this scale to be unappealing due to the lack of simple pitch ratios.

A number of composers have written music that uses the quarter-tone scale, in which the octave is divided into 24 equal intervals. Some of the better-known examples are Béla Bartók, Alban Berg, Ernest Bloch, Pierre Boulez, Aaron Copeland, George Enescu, Charles Ives, and Henry Mancini. We will find, however, that quarter tones do not offer significant musical advantages, at least according to the criteria developed here.

Benson [2] reports that several composers have experimented with "super just" scales that use only perfect ratios. These include Harry Partch (43-tone scale), Wendy Carlos (12 tones), Lou Harrison (16 tones), Wildred Perret (19 tones) [16], John Chalmers (a similar 19-tone scale), and Michael Harrison (24 tones). These scales are not fitted to a tempered chromatic scale as are the scales we discuss here and therefore lack the advantage of providing multiple keys.

Combinatorial properties of scales, keys and tonality have been studied by Balzano [1], Noll [11–13], and others [6, 7, 15, 22]. The composer Olivier Messiaen studied "modes of limited transposition" (scales with fewer keys than notes in the underlying chromatic) [10].

Sethares [20] formulates an optimization problem for finding an instrumental timbre (i.e. relative strength of upper harmonics) that maximizes the degree to which the notes of a given scale sound consonant with the tonic when played on that instrument. The object is to design an instrument that is most suitable for a given scale, rather than to find possible scales.

To our knowledge, no previous study conducts a systematic search for scales with simple pitch ratios and multiple keys, nor formulates an optimization or constraint satisfaction problem for conducting such a search.

3 Characteristics of Standard Scales

We first provide a rationale for preserving the main characteristics of standard scales: intervals that correspond to simple frequency ratios, and multiple keys based on tempered tuning.

3.1 Simple Ratios

A *harmonic partial* of a tone (or a harmonic, for short) is an equal or higher tone whose frequency is an integral multiple of the frequency of the original tone. Two tones whose frequencies bear a simple ratio have many harmonics in common, and this helps the ear to recognize the interval between the tones. If the frequency ratio is a/b (where $a > b$ and a, b are coprime), every ath harmonic of the lower tone coincides with every bth harmonic of the upper tone. For example, if $a/b = 3/2$ as in a perfect fifth, every third harmonic of the lower tone coincides with every other harmonic of the upper tone. This coincidence of harmonics is aurally important because a tone produced by almost any acoustic instrument is accompanied by many upper harmonics (or perhaps only odd harmonics, as in the case of a clarinet). The ear therefore learns to associate a given interval with the timbre produced by a certain coincidence of harmonics, and this distinctive timbre makes the interval easier to recognize. In particular, the octave interval tends to be perceived as a unison, because the upper note adds nothing to the harmonic series: every harmonic of the upper note is a harmonic of the lower.

This ease of recognition benefits both harmony and counterpoint, which might be viewed as the two principal mechanisms of Western polyphonic music. The benefit to harmony is clear. It is hard to distinguish one tone cluster from another if the pitch frequencies have no discernible ratios with each other, while if the ratios are simple, a given tone cluster generates a series of harmonics that reinforce each other in a recognizable pattern. Harmony can scarcely play a central role in music if listeners cannot distinguish which chord they are hearing. In addition, harmony adds immeasurably to the composer's expressive palette. Because each chord has its own peculiar timbre, shifting from one set of frequency ratios to another can create a wide variety of effects the listener can readily appreciate, as does moving from 4:5:6 to 10:12:15 (major to minor triad) or from 8:10:12:15 to 12:15:18:20 (major seventh to a "softer" major sixth chord). The expressive use of harmony has been a key element of music at least since J. S. Bach and became especially important for impressionist and jazz composers.[1]

[1] Simple ratios also tend to produce intervals that are consonant in some sense, although consonance and dissonance involve other factors as well. One theory is that the perception of dissonance results from beats that are generated by upper

Recognizable intervals are equally important for counterpoint, because without them, simultaneous moving voices are perceived as cacophony. Voices that create recognizable harmonic relationships, on the other hand, can be perceived as passing tones from one recognizable chord to another, thus making counterpoint intelligible. This is confirmed by Schenkerian analysis, which interprets Western music as consisting largely of underlying major and minor triads connected by passing tones [5, 14].

3.2 Multiple Keys

Multiple keys enable a signature trait of Western musical structure: the ability to begin in a tonic key, venture away from the tonic into exotic keys, and eventually return "home" to the tonic with an experience of satisfaction and closure. Multiple keys are implemented by embedding the corresponding 7-note scales within a single 12-note "chromatic" scale with tempered tuning. For example, one can play a major scale rooted at any tone of the chromatic scale by sounding the 1st, 3rd, 5th, 6th, 8th, 10th, and 12th notes of the chromatic scale beginning at that tone. This results in 12 distinct major keys.

It is remarkable that the frequency ratios that define classical scales are closely matched by the pitches in a tempered chromatic scale. The pitches are "tempered" in the sense that they are adjusted so that no key is too far out of tune. Various types of temperament have been used historically, but the modern solution is to use equal temperament, in which the kth pitch of the chromatic scale has a frequency ratio of $2^{(k-1)/12}$ with the first pitch. Table 1 shows tuning errors that result for the major diatonic scale. For example, the fifth note of the scale is slightly flat when played on a tempered scale, and the third note is sharp. None of the errors is greater than 0.9%, or about 16 cents.[2]

Temperament was originally adopted to allow a musical instrument with fixed tuning (such as a piano or organ) to play in all keys. But it has an equally important function in musical composition. It allows one to move into a different key by changing only a few notes of the tonic key, where more "distant" keys share fewer notes with the tonic. For example, the most closely related keys, the dominant and subdominant (rooted at the fifth and fourth note) share 6 of the 7 notes of the tonic key. This allows the composer to exploit a wide range of possible relationships when moving from one key to another, making the musical texture richer and more interesting.

harmonics that are close in frequency [18–21]. We will occasionally refer to simple ratios as resulting in "consonant" intervals, but this is not to deny the other factors involved.

[2] We use the tempered pitch as a base for the percentage error because it is the same across all scales and so permits more direct comparison of errors. A cent is 1/1200 of an octave, or 1/100 of a semitone. Thus if two tones differ by c cents, the ratio of their frequencies is $2^{c/1200}$. An error of +0.9% is equivalent to +15.65 cents, and an error of −0.9% to −15.51 cents.

Table 1. Relative pitch errors of the equally tempered major diatonic scale, as a percentage of tempered tuning. Positive errors indicate sharp tuning, negative errors flat tuning.

Note	Perfect ratio	Tempered ratio	Error %	Error cents
1	1/1	1.00000	0.000	0
2	9/8	1.12246	−0.226	−3.91
3	5/4	1.25992	+0.787	+13.69
4	4/3	1.33484	+0.113	+1.96
5	3/2	1.49831	−0.113	−1.96
6	5/3	1.68179	+0.899	+15.64
7	15/8	1.88775	+0.675	+11.73

4 Requirements for Alternative Scales

Given the advantages of simple ratios and multiple keys, we will attempt to generate alternative scales with these same characteristics. In general, a scale will have m notes on a chromatic scale of n notes. The equally tempered chromatic pitches should result in intervals with something close to simple ratios.

4.1 Keys and Temperament

The first decision to be made is the tolerance for inaccurate tuning in the tempered scale. The only reliable guide we have is two centuries of experience with the equally tempered 12-tone chromatic. It is famous for producing flat fifths, but the error is much greater for major thirds and sixths, which are sharp. The tempered major third is in fact quite harsh, although we have learned to tolerate it, and the error is magnified in the upper partials. It therefore seems prudent to limit the relative error to the maximum error in the traditional major scale, namely $\pm 0.9\%$, or between -15.51 and $+15.65$ cents.

There are $\binom{n}{m}$ scales of m notes on n chromatic pitches, but many of these scales are aesthetically undesirable. We can begin by considering only diatonic scales, whose adjacent notes are no more than two chromatic tones (semitones) apart. Diatonic scales are easier to perform, and restricting ourselves to them helps keep the complexity of the search within bounds.[3]

A diatonic scale can be represented by a binary tuple $s = (s_1, \ldots s_m)$, where $s_i + 1$ is the number of semitones between note i and note $i + 1$. Because there are n semitones altogether, s must contain $m_0 = 2m - n$ zeros and $m_1 = n - m$ ones. This means that there are $\binom{m}{m_0} = \binom{m}{m_1}$ diatonic scales to consider.

We also adopt the aesthetic convention that semitones should be distributed fairly evenly through the scale rather than bunched up together. One approach is to require the scales to have Myhill's property, discussed by Noll [12]. However,

[3] This restriction excludes the classical harmonic minor scale, in which notes 6 and 7 are separated by three semitones, but the harmonic minor scale can be viewed as a variant of a natural minor scale in which note 7 is raised a semitone for cadences.

because few scales satisfy this strong property, we require that the scales have a minimum number of semitone and whole-tone adjacencies. That is, the number of pairs (s_i, s_{i+1}) in which $s_i = s_{i+1}$ should be minimized subject to the given m and n, where s_{m+1} is cyclically identified with s_1. If $m_0 \geq m_1$, the number k_0 of pairs of adjacent zeros can be as few as $m_0 - m_1$, and the number k_1 of adjacent ones can be zero. The reasoning is similar if $m_1 \geq m_0$. We therefore require

$$k_0 = m_0 - \min\{m_0, m_1\}, \quad k_1 = m_1 - \min\{m_0, m_1\}$$

It is not hard to show that the number of diatonic scales satisfying this requirement is

$$\binom{\max\{m_0, m_1\}}{\min\{m_0, m_1\}} + \binom{\max\{m_0, m_1\} - 1}{\min\{m_0, m_1\} - 1} \tag{1}$$

For example, among 7-note scales on a 12-note chromatic, we have $(m_0, m_1) = (2, 5)$, $(k_0, k_1) = (0, 3)$, and $\binom{5}{2} + \binom{4}{1} = 14$ suitable scales.

The number of keys generated by a given scale s depends on the presence of any cyclic repetition in s. Let Δ be the smallest offset that results in the same 0/1 pattern; that is, Δ is the smallest positive integer such that $s_i = s_{i+\Delta}$ for $i = 1, \ldots, m$, where $s_{m+1}, \ldots, s_{2m}$ are respectively identified with $s_1, \ldots, s_m$. Then there are

$$\Delta + \sum_{j=1}^{\Delta} s_j$$

distinct keys. For the classical major scale scale $s = (1, 1, 0, 1, 1, 1, 0)$, we have $\Delta = 7$, and there are $7 + \sum_{j=1}^{7} s_j = 12$ keys. When $\Delta < m$, we have a "mode of limited transposition" [10]. For example, the whole tone scale favored by Debussy has $s = (1, 1, 1, 1, 1, 1)$ and $\Delta = 1$, yielding only $1 + s_1 = 2$ keys, which have no notes in common.

4.2 Simple Ratios

In the previous section, we geneated scales by considering subsets of notes in a chromatic scale. For each such scale, we now wish to determine whether relatively simple ratios can be assigned to the notes of the scale that are within 0.9% of the tempered pitches. It does not seem necessary that every note be consonant with the tonic, because many of the harmonies that occur in music do not involve the tonic. Yet every note should at least be consonant with another note of the scale, to allow it to take part in harmony at some point.

We therefore propose that possible ratios be obtained by *generators*, which are simple ratios that a given note can bear with some other note of the scale (these are not generators in the formal sense of group theory). Since we identify notes an octave apart, we consider notes in a two-octave range. Thus if $r_1, \ldots, r_p$ are the generators and f_i the frequency of note i, we require for each note

$i \in \{1, \ldots, m\}$ that

$$\frac{f_i}{f_j} = r_q \text{ or } \frac{2f_j}{f_i} = r_q \text{ or } \frac{f_j}{f_i} = r_q \text{ or } \frac{2f_i}{f_j} = r_q,$$
$$\text{for some } j \in \{1, \ldots, m\} \setminus \{i\}, \text{ some } q \in \{1, \ldots, p\}$$

This requirement is insufficient, however, because it allows for subsets of notes that are consonant with others in the same subset but are extremely dissonant with notes in other subsets. To avoid this outcome, we make the requirement recursive, beginning with the tonic. That is, a note is acceptable if it bears a simple ratio with the tonic, or if it bears a simple ratio with another acceptable note. This can result in notes that are rather dissonant with the tonic, but they will always be consonant with notes that closely precede it in the recursion.

To express this in notation, we let $\pi = (\pi_1, \ldots, \pi_m)$ be a permutation of $1, \ldots, m$, where π_i is interpreted as the ith note defined in the recursion. Then frequencies $f_1, \ldots, f_m$ are acceptable if and only if $f_1 = 1$ and there is a permutation π with $\pi_1 = 1$ such that for each $i \in \{2, \ldots, m\}$, $1 < f_{\pi_i} < 2$ and

$$\frac{f_{\pi_i}}{f_{\pi_j}} = r_q \text{ or } \frac{2f_{\pi_j}}{f_{\pi_i}} = r_q \text{ or } \frac{f_{\pi_j}}{f_{\pi_i}} = r_q \text{ or } \frac{2f_{\pi_i}}{f_{\pi_j}} = r_q,$$
$$\text{for some } j \in \{1, \ldots, i-1\}, \text{ some } q \in \{1, \ldots, p\} \tag{2}$$

Whenever $f_{\pi_i}/f_{\pi_j} = r_q$, we also have $2f_{\pi_j}/f_{\pi_i} = 2/r_q$. Thus there is no need to consider both r_q and $2/r_q$ as generators. That is, we need only consider reduced fractions with odd numerators. We will order the generators by decreasing simplicity, beginning with the smallest denominators, and for each denominator, beginning with the smallest numerator. The first several generators, in order of decreasing simplicity, are

$$\tfrac{3}{2}, \tfrac{5}{3}, \tfrac{5}{4}, \tfrac{7}{4}, \tfrac{7}{5}, \tfrac{9}{5}, \tfrac{7}{6}, \tfrac{11}{6}, \tfrac{9}{7}, \tfrac{11}{7}, \tfrac{13}{7}, \tfrac{9}{8}, \tfrac{11}{8}, \tfrac{13}{8}, \tfrac{15}{8}, \tfrac{11}{9}, \tfrac{13}{9}, \tfrac{17}{9} \tag{3}$$

5 Constraint Programming Model

Constraint programming is naturally suited to formulate the problem described above, because it readily accommodates the variable indices π_i that occur in the expressions f_{π_i} of the recursive formulation. To state the model, we write each frequency ratio f_i as a fraction a_i/b_i in lowest terms. We set $f_1 = 1$, so that f_i is the frequency ratio of note i with the tonic. In the model below, we treat π_i, a_i, and b_i as integer variables. Constraint (a) ensures that π is a permutation. Constraint (b) initializes the recursion. Constraint (c) requires that a_i/b_i be a valid ratio in lowest terms, where "coprime" is a pre-defined predicate. Constraint (d) reduces symmetry by requiring that ratios be indexed in increasing order. Constraints (e) and (f) enforce condition (2), where G is the set of generators. Constraint (g) requires that temperament lie within tolerance ϵ (= 0.009), where t_i indexes the chromatic tone corresponding to scale note i. Thus $t_1 = 1$ and $t_i = t_{i-1} + s_{i-1} + 1$ for $i = 2, \ldots, m$. The domains (h) place an upper bound M on the denominators b_i, to limit the search and avoid intervals that are unreasonably dissonant.

$$
\begin{array}{lr}
\mathrm{alldiff}(\pi_1,\ldots,\pi_m) & (a)\\
\pi_1 = a_1 = b_1 = 1 & (b)\\
1 < \dfrac{a_i}{b_i} < 2,\ \ \mathrm{coprime}(a_i,b_i),\ i=1,\ldots,m & (c)\\
\dfrac{a_{i-1}}{b_{i-1}} < \dfrac{a_i}{b_i},\ i=2,\ldots,m & (d)\\
\bigvee_{j<i}\Big[(\pi_i > \pi_j) \Rightarrow \Big(\dfrac{a_{\pi_i}/b_{\pi_i}}{a_{\pi_j}/b_{\pi_j}} \in G \ \vee\ \dfrac{2a_{\pi_j}/b_{\pi_j}}{a_{\pi_i}/b_{\pi_i}} \in G\Big)\Big],\ i=2,\ldots,m & (e)\\
\bigvee_{j<i}\Big[(\pi_i < \pi_j) \Rightarrow \Big(\dfrac{a_{\pi_j}/b_{\pi_j}}{a_{\pi_i}/b_{\pi_i}} \in G \ \vee\ \dfrac{2a_{\pi_i}/b_{\pi_i}}{a_{\pi_j}/b_{\pi_j}} \in G\Big)\Big],\ i=2,\ldots,m & (f)\\
\dfrac{|a_i/b_i - 2^{(t_i-1)/n}|}{2^{(t_i-1)/n}} \le \epsilon,\ i=1,\ldots,m & (g)\\
\pi_i \in \{1,\ldots,m\},\ a_i \in \{1,\ldots,2M\},\ b_i \in \{1,\ldots,M\},\ i=1,\ldots,m & (h)
\end{array}
\qquad (4)
$$

Conditions of the form $\alpha \in G$ in (e) and (f) are formulated by writing the constraint $\sum_{g\in G}(\alpha = g) \ge 1$. Fractions are shown in the constraints for readability, but they are removed in the model given to the solver, for example by writing $a/b < c/d$ as $ad < bc$.

6 Computational Results

The search algorithm was implemented in IBM OPL Studio 12.6.2. The OPL script language was used to search tempered scales s, for given values of n and m. The number of scales examined is given by (1). For each scale s, the model (4) was solved by the CP Optimizer to find acceptable ratios that are within tolerance of the tempered scale.

A key decision is what set G of generators to use. We found that in several cases, there were no solutions for the simplest generators. We therefore used a rather large set of generators for all scales, namely those in (3), which typically resulted in many solutions. Since the first solutions found tend to have the simplest ratios, we terminated the process after finding 50 solutions. Distinct solutions $(a_1/b_1,\ldots,a_m/b_m)$ were found by re-solving the problem with constraints that exclude the solutions already found. The solver generally obtained each solution in well under a minute, perhaps only a few seconds, depending on the number of chromatic tones. When the solver could no longer find a solution, it ran several hours without proving infeasibility. We therefore set the maximum computation time for finding each solution at 5 minutes, on the assumption that this suffices to find any remaining solution if it exists.

We focused on solutions $(a_1/b_1,\ldots,a_m/b_m)$ that can be obtained from relatively small generators. Since a given solution can typically be obtained from several distinct sets of generators, we computed for each solution and each scale note i the simplest generator that could derive it from another note. That is, we computed for each note i the simplest of the following ratios that fall in the range [1,2], over all $j \neq i$:

$$\frac{a_i/b_i}{a_j/b_j},\ \frac{2a_j/b_j}{a_i/b_i},\ \frac{a_j/b_j}{a_i/b_i},\ \frac{2a_i/b_i}{a_j/b_j}$$

where simplicity is measured by the size of the denominator when the fraction is in reduced terms. We will call this resulting ratio the *minimal generator* for note i. The minimal generator need not be among the generators actually used to obtain the scale in the solution of the constraint programming model (4).

Each solution obtained for a given scale s represents one way the ear might interpret the frequency ratios between the tempered notes of s and the tonic. The existence of a solution with relatively simple ratios and relatively simple minimal generators indicates that scale s is a possible candidate for musical use.

6.1 Scales on a 12-note Chromatic

We began by analyzing scales on the classical 12 chromatic tones, since they can be performed on traditional instruments. The results for 7-note scales appear in Table 2, which shows the number of solutions (a, b) found for each of the 14 possible scales.[4] Since there are multiple solutions for each scale, the table displays a solution in which the ratios are simplest (sometimes there are 2 or 3 solutions in which the ratios are more or less equally simple). It also shows the minimal generators for each scale. Most of these scales correspond to the classical Greek modes and/or modern major and minor scales, as indicated in the table. Interestingly, the classical modes are precisely the scales that can be obtained from the single generator 3/2.[5]

We also investigated nonclassical scales with 6, 8 or 9 notes (Table 3). The only 6-note scale is the whole-tone scale, whose musical possibilities are limited. There are only two 8-note scales, each of which has three keys. The first of the two might be viewed as superior, because it contains both the major third and the fifth, neither of which occurs in the second. However, the second has a half-step leading tone to the tonic (i.e., $s_8 = 0$), which may be viewed as desirable because it allows for stronger cadences. There are thirty 9-note scales, and these tend to contain a large number of consonant ratios, giving them a distinctive sound. Table 3 displays the 10 scales that begin with a whole tone. Scales 22 and 23 seem especially appealing for composition due to their distribution of semitones and simple ratios. They are identical, except that one has a major sixth and one a dominant seventh interveral. The author wrote an extended work for organ using scale 23 [8].

4 We follow the convention of numering the scales in the order of the tuples s treated as binary numbers.

5 For the Dorian, a solution with generators 3/2 and 5/3 is shown because it results in simpler ratios. The single generator 3/2 results in ratios 9/8, 32/27, 4/3, 3/2, 27/16, 16/9.

Table 2. The fourteen 7-note scales on a 12-note chromatic. The number of solutions obtained is shown for each scale, followed by one selected solution with relatively simple ratios and minimal generators. The solutions are generated with a maximum denominator of $M = 32$. All scales have 12 keys.

Scale	Solns	Ratios with tonic	Minimal generators	
1. 0101111	27	$\frac{1}{1}\ \frac{16}{15}\ \frac{6}{5}\ \frac{5}{4}\ \frac{45}{32}\ \frac{8}{5}\ \frac{16}{9}$	$\frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{4}\ \frac{9}{8}\ \frac{3}{2}\ \frac{5}{3}$	
2. 0110111	10	$\frac{1}{1}\ \frac{18}{17}\ \frac{6}{5}\ \frac{4}{3}\ \frac{24}{17}\ \frac{8}{5}\ \frac{16}{9}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$	Locrian mode
3. 0111011	18	$\frac{1}{1}\ \frac{16}{15}\ \frac{6}{5}\ \frac{4}{3}\ \frac{3}{2}\ \frac{8}{5}\ \frac{16}{9}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$	Phrygian mode
4. 0111101	26	$\frac{1}{1}\ \frac{16}{15}\ \frac{6}{5}\ \frac{4}{3}\ \frac{3}{2}\ \frac{5}{3}\ \frac{16}{9}$	$\frac{3}{2}\ \frac{5}{3}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}$	
5. 1010111	6	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{4}{3}\ \frac{45}{32}\ \frac{8}{5}\ \frac{16}{9}$	$\frac{3}{2}\ \frac{5}{4}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{4}\ \frac{3}{2}\ \frac{3}{2}$	
6. 1011011	6	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{4}{3}\ \frac{3}{2}\ \frac{8}{5}\ \frac{16}{9}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$	Aeolian mode (natural minor)
7. 1011101	10	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{4}{3}\ \frac{3}{2}\ \frac{5}{3}\ \frac{16}{9}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}$	Dorian mode
8. 1011110	27	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{4}{3}\ \frac{3}{2}\ \frac{5}{3}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{5}{3}$	melodic minor
9. 1101011	14	$\frac{1}{1}\ \frac{9}{8}\ \frac{5}{4}\ \frac{4}{3}\ \frac{3}{2}\ \frac{8}{5}\ \frac{16}{9}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{9}{8}$	
10. 1101101	9	$\frac{1}{1}\ \frac{9}{8}\ \frac{5}{4}\ \frac{4}{3}\ \frac{3}{2}\ \frac{5}{3}\ \frac{16}{9}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$	Mixolydian mode
11. 1101110	17	$\frac{1}{1}\ \frac{9}{8}\ \frac{5}{4}\ \frac{4}{3}\ \frac{3}{2}\ \frac{5}{3}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$	Ionian mode (major)
12. 1110101	10	$\frac{1}{1}\ \frac{9}{8}\ \frac{5}{4}\ \frac{45}{32}\ \frac{3}{2}\ \frac{5}{3}\ \frac{16}{9}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{4}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$	
13. 1110110	16	$\frac{1}{1}\ \frac{9}{8}\ \frac{5}{4}\ \frac{45}{32}\ \frac{3}{2}\ \frac{5}{3}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$	Lydian mode
14. 1111010	34	$\frac{1}{1}\ \frac{9}{8}\ \frac{5}{4}\ \frac{45}{32}\ \frac{8}{5}\ \frac{5}{3}\ \frac{15}{8}$	$\frac{5}{3}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{4}\ \frac{3}{2}\ \frac{3}{2}$	

6.2 Scales on a 19-note Chromatic

We now consider nonclassical chromatic scales. An obvious question is how many chromatic notes result in attractive scales. One initial screening is to investigate which chromatics contain tones with several simple ratios with the tonic (within tolerance), because these ratios then become available for the scales. Table 4 shows the simple ratios that occur in various chromatic scales. The 19-tone scale stands out as clearly superior. It is the the only scale that strictly dominates the classical 12-tone scale, containing its simple ratios plus three more. The 24-note scale (quarter tones) obviously contains all the simple ratios of the classical scale, but no more, and so there is no compelling reason to move to quarter tones. We therefore concentrate on the 19-note chromatic scale.

In particular, we study 11-note diatonic scales, which contain 3 semitones, or one more semitone than the classical 7-note scales. Since the 19 chromatic tones are already rather closely spaced, it seems desirable to limit the number of semitones in the scale. We therefore rule out 12-note scales, which contain 5 semitones, meaning that almost half of the intervals are semitones. On the other hand, 10-note scales have only one semitone and are therefore nearly whole-tone scales and of limited musical interest. Eleven-note scales seem a good compromise.

Table 3. The 6-note whole-tone scale, two 8-note scales, and 10 of the 30 9-note scales on a 12-note chromatic. Solutions are generated with maximum denominator $M = 32$.

Scale	Solns	Keys	Ratios with tonic	Minimal generators
1. 111111	6	2	$\frac{1}{1}\ \frac{9}{8}\ \frac{5}{4}\ \frac{45}{32}\ \frac{8}{5}\ \frac{16}{9}$	$\frac{5}{4}\ \frac{5}{4}\ \frac{5}{4}\ \frac{5}{4}\ \frac{5}{4}\ \frac{9}{5}$
1. 01010101	>50	3	$\frac{1}{1}\ \frac{16}{15}\ \frac{6}{5}\ \frac{5}{4}\ \frac{45}{32}\ \frac{3}{2}\ \frac{5}{3}\ \frac{16}{9}$	$\frac{3}{2}\ \frac{5}{3}\ \frac{5}{3}\ \frac{3}{2}\ \frac{9}{8}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}$
2. 10101010	>50	3	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{4}{3}\ \frac{45}{32}\ \frac{8}{5}\ \frac{5}{3}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}$
21. 100001010	>50	12	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{5}{4}\ \frac{4}{3}\ \frac{45}{32}\ \frac{8}{5}\ \frac{5}{3}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$
22. 100010010	>50	12	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{5}{4}\ \frac{4}{3}\ \frac{3}{2}\ \frac{8}{5}\ \frac{5}{3}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$
23. 100010100	>50	12	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{5}{4}\ \frac{4}{3}\ \frac{3}{2}\ \frac{8}{5}\ \frac{16}{9}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$
24. 100100010	>50	12	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{5}{4}\ \frac{45}{32}\ \frac{3}{2}\ \frac{8}{5}\ \frac{5}{3}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$
25. 100100100	>50	4	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{5}{4}\ \frac{45}{32}\ \frac{3}{2}\ \frac{8}{5}\ \frac{16}{9}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{9}{5}\ \frac{3}{2}$
26. 100101000	>50	12	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{5}{4}\ \frac{45}{32}\ \frac{3}{2}\ \frac{5}{3}\ \frac{16}{9}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{9}{8}\ \frac{3}{2}$
27. 101000010	>50	12	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{4}{3}\ \frac{45}{32}\ \frac{3}{2}\ \frac{8}{5}\ \frac{5}{3}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}$
28. 101000100	>50	12	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{4}{3}\ \frac{45}{32}\ \frac{3}{2}\ \frac{8}{5}\ \frac{16}{9}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$
29. 101001000	>50	12	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{4}{3}\ \frac{45}{32}\ \frac{3}{2}\ \frac{5}{3}\ \frac{16}{9}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}$
30. 101010000	>50	12	$\frac{1}{1}\ \frac{9}{8}\ \frac{6}{5}\ \frac{4}{3}\ \frac{45}{32}\ \frac{8}{5}\ \frac{5}{3}\ \frac{16}{9}\ \frac{15}{8}$	$\frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}$

Table 4. Simple ratios (indicated by heavy black dots) that occur in chromatic scales having 6 to 24 notes.

Ratio	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
3/2	·	·	·	·	·	·	●	·	·	·	·	●	·	●	·	·	●	·	●
4/3	·	·	·	·	·	·	●	·	·	·	·	●	·	●	·	·	●	·	●
5/3	·	·	●	·	·	●	●	·	·	●	●	·	·	●	●	·	●	●	●
5/4	●	·	·	●	·	·	●	·	·	●	●	·	●	●	·	●	·	·	●
7/4	·	·	·	·	●	●	·	·	·	●	●	·	·	·	●	●	·	·	·
6/5	·	·	·	·	·	●	·	·	·	●	·	·	·	●	·	·	●	●	·
7/5	·	·	·	·	·	·	·	·	·	·	·	·	·	●	·	●	·	●	·
8/5	●	·	·	●	·	·	●	·	·	●	●	·	●	●	·	●	●	·	●
9/5	·	●	·	·	·	·	·	●	●	·	·	·	·	●	●	●	·	·	·

There are 77 11-note diatonic scales on a 19-note chromatic that satisfy our criteria. Since we selected the 19-note chromatic due to its inclusion of many simple ratios, it is reasonable to concentrate on scales in which most of these simple ratios occur. Table 5 displays, for each of the 77 scales, the largest subset of simple ratios that occur in at least one solution. Thirty-seven different subsets of ratios appear in the scales, corresponding to the columns of Table 5. They partition the scales into 37 equivalence classes, labeled A–Z and a–k.

As it happens, all of the classes are dominated by the four classes A, E, P and W (indicated by boldface in Table 5), in the sense that the simple ratios that

Table 5. Occurrences of simple ratios in 11-note scales on a 19-note chromatic. Each column corresponds to a class of scales as indicated in the key at the bottom.

Ratio	**A**	B	C	D	**E**	F	G	H	I	J	K	L	M	N	O	**P**	Q	R	S	T	U	V	**W**	X	Y	Z	a	b	c	d	e	f	g	h	i	j	k
3/2	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
4/3	●	●	●	●	●	●	●	●	●	●	●	●	·	·	·	·	·	·	·	·	·	·	●	●	●	●	●	●	●	●	●	·	·	·	·	·	·
5/3	●	●	●	●	·	·	·	·	·	·	·	·	●	●	●	●	●	●	·	·	·	·	●	●	●	●	●	·	·	·	·	●	●	●	●	·	·
5/4	●	●	●	●	●	●	●	●	●	●	·	·	●	●	●	·	·	·	●	●	·	·	●	●	●	·	·	●	●	●	·	●	●	·	·	●	·
7/4	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
6/5	·	·	·	·	●	●	·	·	·	·	●	●	·	·	·	●	●	●	·	·	●	●	●	●	·	●	●	●	·	·	·	·	·	●	●	·	●
7/5	●	●	●	·	●	·	●	●	·	·	·	·	●	●	●	●	●	●	●	●	●	●	·	·	·	·	·	·	·	·	·	●	●	●	●	●	●
8/5	●	·	·	●	●	●	●	●	●	●	●	·	●	·	·	●	·	·	●	●	●	·	·	·	·	·	·	●	●	·	●	·	·	·	·	●	●
9/5	●	●	·	●	·	·	●	·	●	·	·	●	●	●	·	●	●	·	●	·	●	●	●	·	·	●	·	·	●	·	·	●	·	●	·	·	·

A - 72	K - 12,43	U - 57	e - 13,29,44
B - 69,70,71	L - 28	V - 42	f - 60,61
C - 68	M - 65,66	**W** - 26,27	g - 59
D - 74,75	N - 63,64	X - 10,11,25	h - 18,35,36,50,51,54
E - 7,8	O - 62	Y - 5,6	i - 17,34,49
F - 22,23	**P** - 40,41,55,56	Z - 15,31,32,46,47	j - 58
G - 73	Q - 20,21,38,39,53	a - 14,30,45	k - 16,33,48
H - 2	R - 19,37,52	b - 9,24	
I - 76	S - 67	c - 77	
J - 3	T - 1	d - 4	

occur in any class also occur in one of these four. We therefore concentrate on these classes, which collectively contain 9 scales. Table 3 displays two solutions for a selected scale in each of the four classes. The first solution shown is one with the smallest minimal generators (i.e., one for which the largest minimal generator is smallest). The second solution is one with the simplest ratios with the tonic. The two solutions are very similar but indicate alternative ways the ear can interpret the more complicated pitch ratios.[6]

Scale 72 (class A) contains the most simple ratios with the tonic, including a fifth, fourth, major third, major sixth, minor sixth, and two additional intervals with ratios 7/5 and 9/5. Scale 7 (class E) lacks the fourth and the 9/5 ratio, but it contains a minor third. It also lacks a half-step leading tone. Scale 56 (class P) contains as many simple ratios as scale 7, but it lacks the major third, which might be regarded as a weakness. Scale 27 (class W) lacks the fifth, perhaps a greater weakness.

The key structure of the 11-note scales contrasts significantly with that of the classical scales, as indicated in Table 7. The table shows the distance of each key from the tonic, where distance is measured by m minus the number of notes in common (recall that m is the number of notes in the scale). In the classical case, the two most closely related keys start on the two most consonant intervals, the fourth and fifth. In scale 72, the closest key starts on the major sixth, while the keys starting on the fourth and fifth are among the most distant. In scale 7, the closest key starts a step below the tonic, while in scales 27 and 56, it starts a step above the tonic. In scale 72, the second closest key starts a step above the tonic. This means that in all of these 11-note scales, one can wander further from

[6] A complete list of all 50 solutions found for each of the 77 scales is available at web.tepper.cmu.edu/jnh/music/scales11notes19.pdf and as electronic supplementary material published online with this article.

Table 6. Four of the 77 11-note scales on a 19-note chromatic. Each scale has 19 keys. At least 50 solutions were obtained for each scale, of which 2 are shown. The solutions were generated with maximum denominator $M = 64$.

Scale	Class	Ratios with tonic	Minimal generators
7. 01101011111	E	$\frac{1}{1}\ \frac{25}{24}\ \frac{9}{8}\ \frac{6}{5}\ \frac{5}{4}\ \frac{4}{3}\ \frac{7}{5}\ \frac{3}{2}\ \frac{8}{5}\ \frac{12}{7}\ \frac{25}{18}$	$\frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{7}{4}\ \frac{3}{2}$
		$\frac{1}{1}\ \frac{36}{35}\ \frac{9}{8}\ \frac{6}{5}\ \frac{5}{4}\ \frac{4}{3}\ \frac{7}{5}\ \frac{3}{2}\ \frac{8}{5}\ \frac{12}{7}\ \frac{13}{17}$	$\frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{4}{2}\ \frac{7}{4}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{13}{7}$
27. 10101111110	W	$\frac{1}{1}\ \frac{15}{14}\ \frac{9}{8}\ \frac{6}{5}\ \frac{5}{4}\ \frac{4}{3}\ \frac{10}{7}\ \frac{54}{35}\ \frac{5}{3}\ \frac{9}{5}\ \frac{27}{14}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{5}{4}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{4}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{4}$
		$\frac{1}{1}\ \frac{16}{15}\ \frac{9}{8}\ \frac{6}{5}\ \frac{5}{4}\ \frac{4}{3}\ \frac{10}{7}\ \frac{14}{9}\ \frac{5}{3}\ \frac{9}{5}\ \frac{35}{18}$	$\frac{3}{2}\ \frac{5}{4}\ \frac{5}{4}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{7}{4}\ \frac{5}{4}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{4}$
56. 11011110110	P	$\frac{1}{1}\ \frac{15}{14}\ \frac{7}{6}\ \frac{6}{5}\ \frac{9}{7}\ \frac{7}{5}\ \frac{3}{2}\ \frac{8}{5}\ \frac{5}{3}\ \frac{9}{5}\ \frac{27}{14}$	$\frac{3}{2}\ \frac{5}{3}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}$
		$\frac{1}{1}\ \frac{13}{12}\ \frac{7}{6}\ \frac{6}{5}\ \frac{9}{7}\ \frac{7}{5}\ \frac{3}{2}\ \frac{8}{5}\ \frac{5}{3}\ \frac{9}{5}\ \frac{35}{18}$	$\frac{3}{2}\ \frac{13}{7}\ \frac{5}{3}\ \frac{3}{2}\ \frac{7}{5}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{5}{3}$
72. 11110110110	A	$\frac{1}{1}\ \frac{16}{15}\ \frac{7}{6}\ \frac{5}{4}\ \frac{4}{3}\ \frac{7}{5}\ \frac{3}{2}\ \frac{8}{5}\ \frac{5}{3}\ \frac{9}{5}\ \frac{35}{18}$	$\frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{5}{3}$
		$\frac{1}{1}\ \frac{15}{14}\ \frac{7}{6}\ \frac{5}{4}\ \frac{4}{3}\ \frac{7}{5}\ \frac{3}{2}\ \frac{8}{5}\ \frac{5}{3}\ \frac{9}{5}\ \frac{27}{14}$	$\frac{3}{2}\ \frac{7}{5}\ \frac{5}{3}\ \frac{3}{2}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{5}{3}\ \frac{3}{2}\ \frac{5}{3}\ \frac{9}{5}$

Table 7. Key structure of selected scales, showing distance of each key from the tonic. The interval m3^{rd} is a minor third.

Classical major scale																			
Note	1	1♯	2	2♯	3	4	4♯	5	5♯	6	6♯	7							
Interval			2^{nd}		3^{rd}	4^{th}		5^{th}		6^{th}		7^{th}							
Distance	0	5	2	3	4	1	5	1	4	3	2	5							
Scale 23 of 9 notes on 12-note chromatic																			
Note	1	1♯	2	3	4	5	5♯	6	7	7♯	8	9							
Interval			2^{nd}	m3^{rd}	3^{rd}	4^{th}		5^{th}	m6^{th}		m7^{th}	7^{th}							
Distance	0	3	3	2	2	2	3	2	2	2	3	3							
Scale 7 of 11 notes on 19-note chromatic																			
Note	1	2	2♯	3	3♯	4	5	5♯	6	7	7♯	8	8♯	9	9♯	10	10♯	11	11♯
Interval				2^{nd}		m3^{rd}	3^{rd}		4^{th}			5^{th}		m6^{th}					
Distance	0	8	3	5	5	4	5	5	4	5	5	4	5	5	4	5	5	3	8
Scale 27 of 11 notes on 19-note chromatic																			
Note	1	1♯	2	3	3♯	4	5	5♯	6	6♯	7	7♯	8	8♯	9	9♯	10	10♯	11
Interval				2^{nd}		m3^{rd}	3^{rd}		4^{th}						6^{th}				
Distance	0	8	3	5	4	6	3	6	4	5	5	4	6	3	6	4	5	3	8
Scale 56 of 11 notes on 19-note chromatic																			
Note	1	1♯	2	2♯	3	4	4♯	5	5♯	6	6♯	7	7♯	8	9	9♯	10	10♯	11
Interval						m3^{rd}						5^{th}		m6^{th}	6^{th}				
Distance	0	8	3	5	6	2	7	3	6	4	4	6	3	7	2	6	5	3	8
Scale 72 of 11 notes on 19-note chromatic																			
Note	1	1♯	2	2♯	3	3♯	4	4♯	5	6	6♯	7	7♯	8	9	9♯	10	10♯	11
Interval							3^{rd}		4^{th}			5^{th}		m6^{th}	6^{th}				
Distance	0	8	3	5	6	2	7	3	6	4	4	6	3	7	2	6	5	3	8

the tonic (up to a point) by taking steps up or down, as opposed to following the cycle of fourths or fifths as in the traditional scales. The table also shows scale 23 discussed earlier, whose key structure is again very different. All of the alternate keys have a distance 2 or 3 from the tonic key. Some other 11-note scales have keys with a distance 1 from the tonic key; namely, scales 9, 13, 14, 30, 34, 35, 50, 53, 54, 64, and 66.

We can also contrast the harmonic structure of the 11-note scales with that of the classical major scale. The basic triads in the classical scale are the major triad, with ratios 4:5:6, and the minor triad 10:12:15. The primary quadrads are the major seven chord 8:10:12:15, the minor seven 10:12:15:18, and the all-important dominant seven 36:45:54:64. The rather dissonant dominant seven chord is not so much inspired by harmonic considerations as by the ubiquitous passing tone from the fifth to the third in cadences, which creates a seven chord with the dominant triad.

The 11-note scales differ harmonically in two major respects: the disappearance of the dominant seven, and the addition of exotic harmonies with simple ratios. For definiteness, we focus on scale 72, which contains the largest collection of simple ratios. While the dominant seven chord 36:45:54:64 occurs in some nonstandard scales (such as 9-note scales 23, 25 and 26 in Table 3), it does not occur in scale 72. This suggests that cadences could look very different than in classical scales.

Like the classical major scale, scale 72 contains the major and minor triads (notes 1-4-7 and 5-8-12) as well as the minor seven chord (9-12-15-18), although it lacks the major seven chord. It presents several new harmonies with simple ratios as well. There are three triads that might be viewed as compressed minor triads, and that extend nicely to quadrads. One has ratios 5:6:7 that extend to 5:6:7:9 (notes 9-12-14-18), a second has ratios 6:7:8 that extend to 6:7:8:10 (notes 1-3-5-9), and a third has ratios 7:8:10 that extend to 7:8:10:12 (notes 3-5-9-12). The scale also has a quadrad that is similar to a dominant seven chord (notes 5-9-12-15), except that it has a flatter seventh and much simpler ratios 4:5:6:7.

A final question is whether the "tensions" that are widely used in jazz harmony have a parallel in 11-note scales. Tensions are usually formed by adding notes that are a major ninth above notes of an existing chord [3]. As an example, a major seven chord 1-3-5-7 is extended to 1-3-5-7-9-11♯-13. There does in fact seem to be a parallel to tensions in scale 72, except that they are formed by adding notes whose ratio to the next lower note is 6/5. In this way, we can extend the major triad 1-4-7 to 1-4-7-13-15-18-21, with all notes within the same key (the next note 24♭ moves to a different key). The ratios are exact, except that we must slightly adjust the tension ratio 54/25 of note 13 to 32/15, which is the ratio for this note implied by one of the two solutions of Table 6.

7 Conclusion

We developed a method for systematically generating alternative diatonic scales that share two important characteristics of classical 7-note scales: intervals that correspond to simple ratios, and multiple keys based on a tempered chromatic scale. We defined these characteristics mathematically, and in particular we recursively defined suitable pitch ratios as those that can be obtained from other ratios using a small set of "generators." This approach was partially vindicated by the fact that within the classical 12-note chromatic, the scales that can be obtained entirely from the simplest generator (3/2) are precisely the classical Greek modes, which include the modern major and natural minor scales.

We found our criteria to be well suited for formulation in a constraint satisfaction model and therefore used a constraint programming solver to search for acceptable scales. We considered tempered chromatic scales having from 6 to 24 tones and observed that two of them stand out as superior with respect to the number of simple ratios they contain: the classical 12-note chromatic and the 19-note chromatic. This allowed us to narrow the range of search by concentrating on scales based on these two chromatics.

We first studied scales on the 12-tone chromatic having 6, 7, 8, and 9 notes and identified two 9-note scales that, aside from the classical modes, seem particularly appealing. We focused most of our effort, however, on exploring scales on the 19-note chromatic. Scales with 11 notes appear to be the most promising, and 9 of these 77 scales most deserve attention due to the number of simple ratios they contain. We found these scales to provide significant musical resources that are not available in classical scales, including a contrasting and more complex key structure, as well as a number of new harmonies.

In particular, the most attractive 11-note scale contains, in addition to the classical major and minor triads, three triads and four quadrads with simple ratios that do not appear in traditional scales. These provide many new possiblities for harmonic texture. The scale contains no dominant seven chord, which suggests that it would inspire very different chord progressions than the classical major and minor scales. In addition, it supports complex tensions that are analogous to but harmonically different from those commonly occurring in jazz arrangements.

We conclude that this and the other 11-note scales we singled out could take music in an interesting new direction, and we suggest them to composers as possibly worthy of experimentation. Such experiments would presumably rely primarily on electronic synthesizers, due to the difficulty of building acoustic instruments that support nonstandard scales. It is essential, however, not to generate tones as sine waves or other simplified wave forms. The tones should carry a full complement of upper harmonics that mimic those that would be generated by acoustic instruments, because otherwise the exotic intervals and harmonies of these scales cannot be easily recognized or appreciated.

Appendix

A Chorale and Fugue for organ [8] uses scale 23 on 9 notes. The chorale cycles through the tonic (A) and the two most closely related keys (C♯, F). The cadences illustrate that dominant seven chords need not play a role, even though they occur in the scale. Rather, the cadences use two leading tones and pivot on the tonic, often by moving from the lowered submediant. The chorale is followed by a double fugue that again cycles through the three keys A, C♯, F. The first subject enters on these pitches but without a key change. The second subject (bar 96) illustrates the expanded possibilities for suspensions and pivots.

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