SECOND ORDER ELLIPTIC EQUATIONS AND BOUNDARY VALUE PROBLEMS

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1. Introduction

While the theory of higher order elliptic differential equations with smooth coefficients grew up rapidly in the ten years since Gårding's paper [5] on the Dirichlet problem appeared, the development of the theory of equations with discontinuous coefficients has been very slow even for single second order equations.

The fact that solely a differential operator with smooth coefficients can be considered locally as a small perturbation of an operator with constant coefficients makes a significant difference. Consequently the techniques of Fourier transform and of singular integral operator, although extremely useful for equations with constant coefficients, fail in the case of equations with discontinuous coefficients.

The main reason for studying equations with discontinuous coefficients arises from nonlinear equations; however, there are several linear boundary value problems which escape the general theory of differential equations with smooth coefficients (for instance, the so-called transmission problem [26]).

The theory of equations with discontinuous coefficients is quite different in the case of two variables than it is for more variables. For two variables the theory commenced with a paper by C. B. Morrey [15] appearing in 1938 and developed with Nirenberg's results [21] in 1954 (see also [1]). For the applications of these results to nonlinear equations see the book of Miranda [14]. The theory of equations in two variables is closely connected with the beautiful and well developed theory of quasi-conformal mappings. See for expositions Courant-Hilbert [3] (the supplement to Chapter IV by L. Bers) and I. Vekua [29].

In more than two variables the theory was completely wrapped in mystery until a few years ago when the De Giorgi [4]–Nash [20] theorem was proved. This theorem is related to the equation in divergence form

$$(a_{ij}u_{x_i})_{x_j}=0 \tag{1.1}$$

with measurable and bounded coefficients when the condition

$$\nu^{-1}|\xi|^2\leqslant a_{ij}\xi_i\xi_j\leqslant\nu|\xi|^2 \quad (\nu\geqslant 1) \tag{1.2}$$

is satisfied; it states that any weak solution in $H^1(\Omega)$ is locally Hölder continuous.

This theorem led to many new results about the theory of linear and nonlinear second order elliptic equations in divergence form.

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A theory of more general equations in the form

$$a_{ij}u_{x_i x_j} + b_i u_{x_i} = 0 \tag{1.3}$$

with discontinuous coefficients is still lacking.

I want to limit myself to a review of some results for linear equations in divergence form and to mention only some applications to very special nonlinear problems. More general results are described in the talk by Nirenberg on nonlinear problems. Other questions on elliptic equations are described in an expository paper by Gilbarg [7].

2. Some notation

If Ω is a domain of the Euclidean space E^n, $\partial\Omega$ denotes its boundary and $\bar{\Omega}$ its closure. If $\xi \equiv (\xi_1, \xi_2, \ldots, \xi_n)$ is a vector in E^n we put $|\xi|^2 = \xi_1^2 + \ldots + \xi_n^2$. We shall say that a function $u(x)$ belongs to $C^m(\bar{\Omega})$ if u is continuous together with all the partial derivatives of order $\leqslant m$ in $\bar{\Omega}$ ($C^\infty(\Omega) = \bigcap_{m=0}^{\infty} C^m(\Omega)$). We shall denote by $C_\lambda^0(\bar{\Omega})$ the class of the functions which are Hölder continuous with exponent λ $(0 < \lambda \leqslant 1)$ in $\bar{\Omega}$, i.e.

$$\sup_{\substack{x', x'' \in \Omega \\ x' \neq x''}} \frac{|u(x') - u(x'')|}{|x' - x''|^\lambda} < +\infty .$$

A function of $C_1^0(\bar{\Omega})$ is also called a Lipschitz function in Ω.

Domains and boundary values of class C_λ^m are defined as usual.

The completion of $C^m(\bar{\Omega})$ with respect to the norm

$$\sum \|D^j u\|_{L^p(\Omega)} \quad (p \geqslant 1),$$

where $D^j u$ denotes any of the jth derivatives and the sum is extended to all derivatives of order $\leqslant m$ will be denoted by $H^{m,p}(\Omega)$, or more simply by $H^m(\Omega)$ when $p = 2$. By $H_0^{m,p}(\Omega)$ $[H_0^m(\Omega)]$ we shall denote the closure in $H^{m,p}(\Omega)$ $[H^m(\Omega)]$ of the set of the functions of $C^m(\Omega)$ vanishing near $\partial\Omega$. $u(x) \in H_{\text{loc}}^{m,p}(\Omega)$ $[H_{\text{loc}}^m(\Omega)]$ if $u(x) \in H^{m,p}(\Omega')$ for any compact subdomain Ω' of Ω. Consider the differential operator

$$M(u) = (a_{ij} u_{x_i})_{x_j}, \tag{2.1}$$

where a_{ij} are measurable and bounded functions defined in Ω and suppose it is uniformly elliptic, i.e.

$$\nu |\xi|^2 \leqslant a_{ij} \xi_i \xi_j \quad (\nu > 0). \tag{2.2}$$

A function $u(x)$ of $H^1(\Omega)$ is an M-subsolution [M-supersolution] in Ω, if

$$\int_\Omega a_{ij} u_{x_j} \phi_{x_j} dx \leqslant 0 [\geqslant 0] \tag{2.3}$$

for all ϕ of $H_0^1(\Omega)$ such that $\phi \geqslant 0$ a.e. in Ω.

A function $u(x)$ which belongs to $H_{\text{loc}}^1(\Omega)$ and which satisfies (2.3) for every non-negative C^∞ function ϕ with compact support in Ω, will be called a local M-subsolution [or M-supersolution].

Consequently one defines the solutions and the local solutions.

3. The maximum principle

One of the best known results for second order elliptic equations is the Hopf maximum principle [8], (see also [3], p. 326) which holds for equations in the general form (1.3) even if the coefficients are supposed only to be bounded, but for smooth solutions. Different forms of the maximum principle which hold for very weak solutions, but for equations with smooth coefficients, have been found by E. Calabi [2] and W. Littman [12]. A balanced form of the maximum principle holds for M-subsolutions or for local M-subsolutions when the operator M has the form (2.1) and satisfies (2.2).

More generally we shall consider a continuous function $u(x)$ which belongs to $H^1_{\text{loc}}(\Omega)$ and satisfies the inequality

$$\int_\Omega a_{ij} u_{x_i} \phi_{x_j} dx \leqslant \int_\Omega f_j \phi_{x_j} dx, \tag{3.4}$$

$$\left[\int_\Omega a_{ij} u_{x_i} \phi_{x_j} dx \geqslant \int_\Omega f_j \phi_{x_j} dx\right] \tag{3.4'}$$

for every non-negative C^∞ function with compact support in Ω, where a_{ij} are measurable and bounded functions in Ω satisfying (2.2) and where $f_j \in L^p(\Omega)$ with $p > n$.

(3.4) [(3.4′)] means that u satisfies locally the differential inequality

$$M(u) \geqslant f_{jx_j} \quad [\leqslant]. \qquad (3.5)\ [(3.5')]$$

Then we have the following weak form (for a strong form see § 4) of the maximum principle:

THEOREM I. *If M is an elliptic operator satisfying* (2.2), *if* $u(x) \in C^0(\bar{\Omega}) \cap H_{\text{loc}}(\Omega)$ *and satisfies locally* (3.5) [(3.5′)] *with f_j in $L^p(\Omega)$ ($p > n$), then there exists a constant $C(p,n)$ depending only on p and n such that*

$$\max_\Omega u(x) \leqslant \max_{\partial\Omega} u(x) + \frac{C(p,n)}{\nu} \sum \|f_j\|_{L^p(\Omega)} (\text{mes}\ \Omega)^{1/n - 1/p}, \tag{3.6}$$

$$\left[\min_\Omega u(x) \geqslant \min_{\partial\Omega} u(x) - \frac{C(p,n)}{\nu} \sum \|f_j\|_{L^p(\Omega)} (\text{mes}\Omega)^{1/n-1/p}\right]. \tag{3.6'}$$

We cannot give here the proof of the theorem, but we want to mention that the proof is based on the same idea introduced by Marcinkiewicz and used extensively by Zygmund [31], Hörmander [9] and others which consists in looking for information about the behaviour of the function

$$\phi(k) = \text{mes}\{x \mid u > k\}.$$

Here we can prove that for $h > k > \max_{\partial\Omega} u$ the function ϕ satisfies an inequality of the type

$$\phi(h) \leqslant \frac{\text{const}}{(h-k)^\alpha} [\phi(k)]^\beta, \tag{3.7}$$

where α, β are positive constants and $\beta > 1$ for $p > n$.

By iteration, one obtains from (3.7) that $\phi(d)=0$ where d is the right term of (3.6).

Exactly the same proof shows that the (3.6) [(3.6′)] holds for M-subsolutions [M-supersolutions] in Ω also if they are not continuous. The meaning of $\max_{\partial\Omega}$ and $\min_{\partial\Omega}$ has to be taken in a weak sense. Let $u(x)$ be a function belonging to $H^1(\Omega)$; $u(x)$ is bounded from above on $\partial\Omega$ by a constant Φ if there exists a sequence of functions $u_m \in C^1(\bar{\Omega})$ such that $u_m \leqslant \phi$ on $\partial\Omega$ and u_m tends to u in $H^1(\Omega)$. The smallest of such numbers Φ is denoted by $\max_{\partial\Omega} u$.

The same method of proof, as described above, can be used also for more general linear equations in order to obtain *a priori* bounds for the L^q-norm of the solution in terms of the L^p-norm of f_j and the L^2-norm of the solutions themselves; here $1/q = 1/p - 1/n$ if $p < n$ and $q = +\infty$ if $p > n$ (see Stampacchia [23, 24, 27*a*] and Maz'ya [13]). Ladyzenskaya and Uralt'seva [10, 11] and myself [27] used the same method in order to prove the boundedness of the solutions of some regular integrals of the calculus of variations.

About the Theorem I, Weinberger [30] proved that the constant $C(p,n)$ in (3.6) and in (3.6′) is given by

$$C(p,n) = \omega_n^{-1/n}\left(\frac{p-1}{p-n}\right)^{1-1/p} n^{1/n-1/p},$$

where ω_n is the measure of the unit n-sphere. Equality occurs in (3.6) and (3.6′) for Laplace's equation when Ω is a sphere.

4. Harnack's inequality and the Hölder continuity in the interior

The extension of the classic theorem by Harnack on the positive harmonic functions to the solution of more general elliptic equations has been given for $n=2$ by Bers and Nirenberg [1] and in general by Serrin [22] supposing, when $n>2$, the coefficients continuous.

Recently Moser [18] was able to prove the Harnack inequality for local solutions of (1.1) when the condition (1.2) is satisfied. The quoted paper by Bers and Nirenberg is based on a famous theorem by Lebesgue on the continuity of monotone functions with finite Dirichlet integrals. Such a result does not hold for functions in more variables. The Moser proof is based on a very different argument, and makes use of a special case of a theorem by John and Nirenberg [9*a*] which partially takes the place of Lebesgue's theorem.

The Moser theorem is the following

THEOREM II. *If u is a positive solution in Ω of equation* (1.1) *and* (1.2) *is satisfied, and if Ω' is a compact subdomain of Ω, then*:

$$\max_{\Omega'} u \leqslant c \min_{\Omega'} u, \tag{4.1}$$

where c depends on Ω', Ω and ν.

As a consequence of the Harnack inequality we can deduce the strong form of the maximum principle.

Consider now a positive solution of the equation

$$(a_{ij}u_{x_j})_{x_i} = f_{jx_j}, \tag{4.2}$$

where (1.2) is satisfied and $f_j \in L^p(\Omega)$ with $p > n$.

Using Moser's theorem and the maximum principle, as stated in Theorem I, we can get the following consequence. Let x_0 be any point of a compact subdomain Ω' of Ω and let

$$M(r) = \max_{|x-x_0|\leqslant r} u, \quad \mu(r) = \min_{|x-x_0|\leqslant r} u \quad (0 < r < \tfrac{1}{2}d),$$

where d denotes the distance from Ω' to $\partial\Omega$. Then, there exists a constant c depending on Ω', Ω, ν, such that

$$M(r) \leqslant c(\mu(r) + \sum \|f_j\|_{L^p(\Omega)} r^{1-(n/p)}). \tag{4.3}$$

From (4.3) it is possible to deduce easily that for any solution of (4.2) the following inequality holds

$$\omega\left(\frac{r}{2}\right) \leqslant \theta\omega(r) + C\sum \|f_j\|_{L^p(\Omega)} r^{1-(n/p)}, \tag{4.4}$$

where $\omega(r) = M(r) - \mu(r)$ is the oscillation of u in $|x - x_0| \leqslant r$ and where $\theta < 1$.

From (4.4) follows that any solution of (4.2) is Hölder continuous in the interior of Ω. When $f_j \equiv 0$ this result proves the De Giorgi–Nash theorem.

A direct proof of this theorem which does not use the Harnack theorem has been given by Moser [19].

5. Boundary value problems

The extension at the boundary of the latter result has been obtained independently by C. B. Morrey [16], Ladyzenskaya and Uralt'seva [10] and the author [25]. It states (in a special case)

THEOREM III. *If $u(x) \in H_0^1(\Omega)$ and satisfies the equation* (4.2) *where* (1.2) *holds and $f_j \in L^p(\Omega)$ with $p > n$ and if Ω satisfies a suitable condition $\mathcal{R}$, then $u(x)$ is Hölder continuous in $\bar{\Omega}$.*

The condition $\mathcal{R}$ on $\partial\Omega$ is the following [25]:

Let $B(y,r)$ be the sphere with center in y and radius r, there exist two constants K and r_0 such that for all $v \in C^1(B(y,r))$ vanishing on $\complement\Omega \cap B(y,r)$

$$|v(x)| \leqslant K \int_{B(y,r)} \frac{|\text{grad } v|}{|x-t|^{n-1}} dt$$

for $x \in B(y,r)$, $y \in \partial\Omega$, $r < r_0$.

As a special case, condition $\mathcal{R}$ is satisfied if there exist two constants α, r_0 such that, for $y \in \partial\Omega$

$$\text{mes}\{\complement\Omega \cap B(y,r)\} \geqslant \alpha \text{ mes } B(y,r) \quad (r < r_0)$$

(see [16] and [10]).

Using Hilbert space approach and the regularity theorem just mentioned we deduce that there exists one solution $u(x)\in C^0_\lambda(\bar{\Omega})\cap H^1(\Omega)$ of the Dirichlet problem for the equation (4.2) where (1.2) holds, $f_j\in L^p(\Omega)$ with $p>n$ and for the boundary values

$$u-v\in H^1_0(\Omega),$$

provided that $v\in C^0_\lambda(\bar{\Omega})\cap H^{1,q}(\Omega)$ $(q>n)$ and Ω satisfies the condition $\mathcal{R}$. We remark that if Ω is smooth the condition on v can be $v\in H^{1,q}(\Omega)$ with $q>n$.

Making use of the maximum principle of § 3 and the Harnack inequality of § 4 we can extend the Perron method from Laplace equation to the more general equation (1.1) satisfying (1.2). Using the preceding theorem, the existence of a barrier at the boundary can be proved if the condition $\mathcal{R}$ on Ω is satisfied.

Then *the Dirichlet problem for equation* (1.1) *satisfying* (1.2) *with arbitrarily assigned continuous boundary values has one solution* $u(x)$ *in* $C^0(\bar{\Omega})\cap H^1_{loc}(\Omega)$ *if the condition* $\mathcal{R}$ *on* Ω *is satisfied.*

We shall say that the points of $\partial\Omega$ are regular points for the operator M if the Dirichlet problem for arbitrarily assigned continuous boundary values is possible in $C^0(\bar{\Omega})\cap H^1_{loc}(\Omega)$.

The condition $\mathcal{R}$ assures that the points of $\partial\Omega$ are regular for the operator M.

We do not know if the class of domains which are regular for Laplace's operator coincides with the one of domains which are regular for the operator M given by (1.1) as is the case for operators with smooth coefficients [7a, 21a, 28](1).

6. Some special cases of non-linear problems

The De Giorgi–Nash theorem and its extensions led to very interesting results on the differentiability of weak solutions of nonlinear equations or of minimizing functions of multiple integrals of the calculus of variations.

The first results in this direction were found by De Giorgi [4] for problems of the form

$$I(u)=\int_\Omega f(p)dx=\min \quad (p=\text{grad } u),$$

supposing that

$$\nu^{-1}|\xi|^2\leqslant f_{p_ip_j}(p)\xi_i\xi_j\leqslant \nu|\xi|^2 \quad (\nu\geqslant 1).$$

More general results have been obtained by C. B. Morrey [17] and Ladyzenskaya and Uralt'seva [10, 11] for general multiple integrals

(1) *Added in proof.* Recently H. F. Weinberger, W. Littman, and the author proved (Regular points for elliptic equations with discontinuous coefficients, to appear in *Ann. Scuola Norm. Sup., Pisa*) that a domain is regular for the operator M if and only if it is regular for the Laplace operator.

$$\int_\Omega F(x,u,p)\,dx.$$

These results require that F satisfy a condition of the type

$$\nu^{-1}(1+|p|^2)^\tau|\xi|^2 \leqslant F_{p_ip_j}(x,u,p)\xi_i\xi_j \leqslant \nu(1+|p|^2)^\tau|\xi|^2$$

with $\tau > -\frac{1}{2}$.

To begin with we remark that the classical variational problem of the minimal surface, where $F=\sqrt{1+|p|^2}$ is not included in these results. Recently it has been proved [27] that there exists a Lipschitz function in $\bar{\Omega}$ solution of the variational problem for the integral

$$I(u)=\int_\Omega f(p)\,dx=\min, \tag{6.1}$$

provided that f is strictly convex, i.e.

$$f_{p_ip_j}(p)\xi_i\xi_j > 0 \quad \text{for } \xi \neq 0,$$

and Ω is strictly convex and the boundary values are sufficiently smooth. The solution is analytic if the data are analytic. A similar theorem, using a different approach has been proved, independently by Gilbarg [6].

Such a statement may fail when F depends on the function u too. For the special integrals

$$I(u)=\int_\Omega \{f(p)+G(x,u)\}\,dx, \tag{6.2}$$

supposing that

$$f_{p_ip_j}(p)\xi_i\xi_j \geqslant \nu(1+|p|^2)^\tau|\xi|^2 \quad (\nu>0, -\tfrac{1}{2}<\tau\leqslant 0)$$

and Ω strictly convex has been proved [27] the existence of a Lipschitz function minimizing $I(u)$ on the class of all Lipschitz functions vanishing on $\partial\Omega$ provided that suitable conditions on $G(x,u)$ are satisfied. Also here the solutions are smooth if the data are sufficiently smooth.

The assumption on the function $G(x,u)$ can be, for instance, the following:

$$\lim_{|u|\to+\infty}\frac{uG_u(x,u)}{|u|^\alpha} > -\nu\Lambda(\alpha,\Omega), \tag{6.3}$$

where $\alpha=2(\tau+1)$ and $\Lambda\ (\alpha,\Omega)$ is given by

$$\Lambda(\alpha,\Omega)=\inf_{u\in H_0^{1,\alpha}(\Omega)}\frac{\int_\Omega|\text{grad } u|^\alpha\,dx}{\int_\Omega|u|^\alpha\,dx}.$$

In particular $\Lambda(2,\Omega)$ is the first eigenvalue of the boundary value problem: $\Delta u+\lambda u=0$ in Ω, $u=0$ on $\partial\Omega$.

Application of these theorems to the boundary value problems for the Euler equations of (6.1) and (6.2) leads to new results.

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