University of California at Berkeley | CE 130
Department of Civil and Environmental Engineering | Section 2
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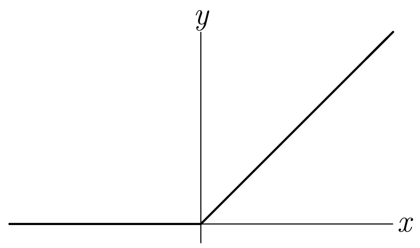
Singularity functions

Ramp function

The **Macauley bracket** $<\cdot>$ defines the **ramp function**

$$<x> = \begin{cases} 0, & x \leq 0, \\ x, & x > 0. \end{cases}$$



Obviously.

$$<x-a> = \begin{cases} 0, & x \leq a, \\ x-a, & x > a, \end{cases}$$

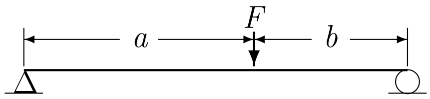
$$<x-a>^n = \begin{cases} 0, & x \leq a, \\ (x-a)^n, & x > a \end{cases}$$

for $n > 0$, and

$$\frac{d}{dx}<x-a>^{(n+1)} = (n+1)<x-a>^n,$$

so that

$$\int <x-a>^n \, dx = \frac{1}{n+1}<x-a>^{(n+1)}.$$



For example, in a beam loaded as shown above, the bending moment is given by

$$M(x) = \begin{cases} \dfrac{Fbx}{L}, & x < a \\ \dfrac{Fa(L-x)}{L}, & x > a \end{cases}$$

But

$$\frac{Fa(L-x)}{L} = \frac{Fbx}{L} - F(x-a);$$

therefore the moment can be given by the **single expression**

$$M(x) = \frac{Fbx}{L} - F{<}x-a{>}.$$

This can now be integrated twice to give a single expression for the deflection:

$$EIv(x) = \frac{Fbx^3}{6L} - \frac{F}{6}{<}x-a{>}^3 + C_1 x + C_2.$$

The boundary condition $v(0) = 0$ yields $C_2 = 0$, while $v(L) = 0$ gives

$$C_1 = -\frac{Fb}{6L}(L^2 - b^2),$$

leading to

$$v(x) = \frac{F}{6EI}\left\{\frac{b}{L}[x^3 - (L^2 - b^2)x] - {<}x-a{>}^3\right\}.$$

Step function

The derivative of the ramp function is

$$\frac{d}{dx}{<}x{>} = \begin{cases} 0, & x < 0, \\ 1, & x > 0, \end{cases}$$

known as the **Heaviside step function**. There are various notations for it: $H(x)$, $U(x)$, $1(x)$ and others (the notation ${<}x{>}^0$ is ambiguous because it gives 0^0 for negative x). Let us use $H(x)$. Note that $H(0)$ is not defined; it is connventional to give it the value $\frac{1}{2}$.

The shear in the preceding example is given by

$$V(x) = \frac{F}{L}\left[b - LH(x-a)\right],$$

that is, Fb/L for $x < a$ and $-Fa/L$ for $x > a$.

The step function can be used to represent loading that is distributed over a part of a beam. For example, a uniform load of intensity w distributed over the middle third of the beam is given by

$$q(x) = -w[H(x - L/3) - H(x - 2L/3)].$$

The deflection can be obtained by integrating four times, resulting in the single expression

$$v(x) - \frac{w}{24EI}[<x - L/3>^4 - <x - 2L/3>^4] + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3x + C_4,$$

and the constants $C_1, \ldots, C_4$ can be determined from the boundary conditions at $x = 0$ and $x = L$.

Delta function

In order to represent the loading in the case of a concentrated load, we must be able to differentiate the step function: if the shear is given by

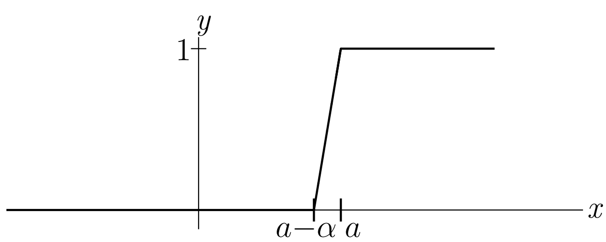
$$V(x) = \frac{F}{L}\left[b - LH(x-a)\right],$$

then the loading is

$$q(x) = \frac{d}{dx}V(x) = -F\frac{d}{dx}H(x).$$

Clearly the derivative of $H(x)$ is zero for $x \neq 0$, but at $x = 0$ it is infinite, so it is not really a function (mathematicians call it a *distribution*).

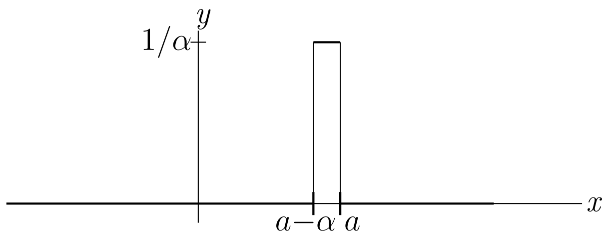
Let's consider the following approximation to the step function, say $H_\alpha(x)$:



This is given by

$$H_\alpha(x-a) = \frac{1}{\alpha}[<x-a+\alpha> - <x-a>].$$

Note that, in the limit as $\alpha \to 0$, this is just the derivative of $<x-a>$. But let's differentiate $H_\alpha(x-a)$ before taking the limit:



$$\frac{d}{dx} H_\alpha(x-a) = \frac{1}{\alpha}[H(x-a+\alpha) - H(x-a)].$$

Formally, this becomes the derivative of $H(x-a)$ in the limit as $\alpha \to 0$. What happens is that the rectangle in the picture gets narrower and taller, but its area remains 1. This limit is known as the **Dirac delta function** and is almost universally denoted $\delta(x-a)$, except that in Mechanics of Materials textbooks the strange notation $<x-a>_*^{-1}$ is used; the subscript asterisk is there to tell us that the superscipt -1 is not really the power -1.

The loading on the beam with a concentrated load at $x = a$ can now be written as

$$q(x) = -F\delta(x-a).$$

Note that it doesn't matter what the delta function "really" means; all that matters is that when we integrate it we get the step function, and so on.

By the same token we can define the derivative of the delta function, $\delta'(x-a)$ (denoted $<x-a>_*^{-2}$ in the book). Suppose we have a downward concentrated force of magnitude C/α at $x = a$, and an upward one of the same magnitude at $x = a - \alpha$, forming the clockwise couple C, independent of α; in the limit as $\alpha \to 0$, this becomes the **concentrated** couple C. Now, mathematically, the loading is given by

$$q(x) = \frac{C}{\alpha}[\delta(x-a+\alpha) - \delta(x-a)],$$

and in the limit as $\alpha \to 0$ this **formally** becomes C times the derivative of $\delta(x-a)$, which we can write as $C\delta'(x-a)$.