

Shell structures

Chris J K Williams

University of Bath

1 What is a shell?

Structures can be classified in many ways according to their shape, their function and the materials from which they are made [Adriaenssens et al., 2014].

The most obvious definition of a shell might be through its geometry. A structure or structural element may be a fully three-dimensional solid object, or it might have some dimensions notably smaller than others. A beam is straight and it is relatively long in comparison to its cross-section. Thus it is defined by a straight line. An arch is defined by a curved line and a plate by a plane.

A shell is a structure defined by a curved surface. It is thin in the direction perpendicular to the surface, but there is no absolute rule as to how thin it has to be. It might be curved in two directions, like



Figure 1: Giant clam shell



Figure 2: Chapel Lomas de Cuernavaca by Felix Candela

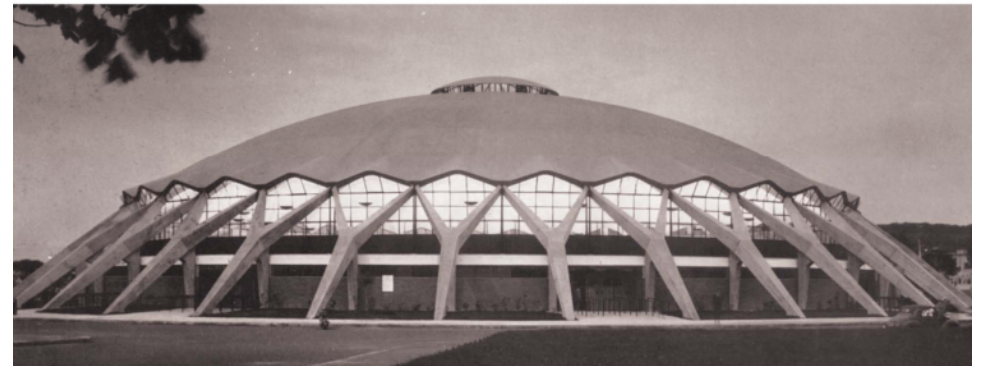
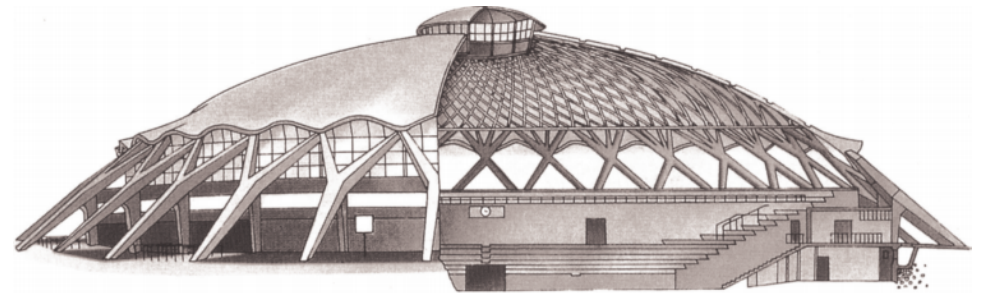


Figure 3: Palazzetto dello Sport by Pier Luigi Nervi

a dome or a cooling tower, or it may be cylindrical and curve only in one direction.

This definition would clearly include birds' eggs, sea shells (figure 1) and concrete shells, such as Felix Candela's Chapel Lomas de Cuernavaca (figure 2) and Nervi's Palazzetto dello Sport (figure 3). It would also include ships, monocoque car bodies and aircraft fuselages (coque is one of the French words for a shell), drinks cans, glasses cases (figure 4), all sorts of objects.

But this definition would also include tension structures like sails, balloons and car tyres. If one wanted to exclude tension structures, one might stipulate that shells have to work in both tension and compression, but that would exclude masonry vaults that can only work in compression [Block, 2009, Heyman, 1995]. Most people would describe masonry vaults such as the hôtel de ville in Arles (figure 6) or the fan vaulting of Bath Abbey (figure 5) as shell structures.

However the word 'shell' has the implication of something relatively rigid, and this article is about such structures. We therefore need to have a separate category of tension structures to include sails and balloons as well as piano strings and fishing nets. Then we have six possible types of structure:



Figure 4: Glasses case



Figure 5: Fan vaulting of Bath Abbey. Photo: Adrian Pingstone



Figure 6: L'hôtel de ville d'Arles. Photo: Jacqueline Poggi



Figure 7: Colander



Figure 8: sieve

1. Tension structures: strings, nets and fabric structures
2. Straight line elements: beams, columns
3. Curved line elements: arches, rings
4. Plates: slabs, walls
5. Shells: timber, concrete, metal or masonry
6. Fully three dimensional lumps of material

A colander (figure 7) is a curved surface structure. It contains holes for draining food, but these holes do not stop it being shell. It is a continuous surface with a relatively small area removed. A sieve (figure 8) is very similar, except that the surface is made from a large number of initially straight wires which are woven into a flat sheet and then bent into a hemisphere. It is also a shell, a gridshell - see section 5. This is very much like assembling lots of straight line elements to form trussed arch such as that of the Viaduc de Garabit (figure 9).

Clearly there is some similarity between a sieve and a spider web - they are both lattice-like and are intended to catch things. The spider web is essentially flat and made up of straight elements and when the wind blows it bows outwards like a sail and becomes curved. It



Figure 9: Viaduc de Garabit, Gustave Eiffel, Maurice Koechlin, Léon Boyer. Photo: J. Thurion (Belgavox)



Figure 10: Spiderweb

therefore adjusts its shape to the loading, which is the characteristic of tension structures. The sieve may be in tension, compression or a mixture of the two. Where it is in compression deflections lead to the structure becoming less able to carry the load, possibly leading to buckling. Columns carry loads via axial forces, but bending stiffness is required to stop buckling, and so it is with shells, although with shells buckling is resisted by a combination of bending and in-plane action.

2 How do shells work?

Shells use all the modes of structural action available to beams, struts, arches, cables and plates, plus another mode that we might call 'shell action', which we will now try and pin down.

Structural elements that approximate to lines (beams, arches and cables) or to surfaces (plates and shells) all share the same property: they are much easier to bend than to stretch. We use the word 'stretch' to mean change in length, possibly getting shorter, a 'negative stretch'.

Clearly a cable will stretch when we apply a tension to it. A column will undergo a negative stretch when we apply a compression to it.

But if we apply more load it will buckle and it will get shorter through bending, rather than axial strain.

A parabolic arch or cable can carry a uniform vertical load per unit horizontal length using only axial compression or tension. The component of load perpendicular to the cable is balanced by the axial force multiplied by the curvature. Thus load in KN/m is balanced by a force in KN multiplied by the curvature in 1/metres. Note that curvature is defined as $1/(\text{radius of curvature})$.

Other loads will cause bending moment in the arch or deflection of the cable. The arch bending moment is the product of the thrust and its eccentricity from the axis.

Flat plates and plane stress

In order to understand curved arches, we first learn about straight beams. Similarly to understand shells we first need to think about something simpler. We could start with arches and go from curved lines to curved surfaces. Or we could start with plates and go from flat surfaces to curved surfaces. Both approaches can be helpful, but let us start with plates.

A flat plate can be loaded by forces in its own plane, figure 11, or out

of plane, figure 12, in which forces are shown in blue and bending and twisting moments are shown in red. The term plane stress is used for in plane loading and it appears in all sorts of situations, for example the bending of an I-beam. Clearly the beam is loaded perpendicular to its axis, but most of the stress in the web and flanges are in the plane or the steel plates. Out of plane loading of a plate or slab produces plate bending, and as we have already noted it is much easier to bend a plate than to stretch it.

In figure 11 we have introduced the components of membrane stress, σ_x , σ_y , τ_{xy} and τ_{yx} . Membrane stress is a central concept in shell theory and corresponds to the axial stress in an arch - as opposed to the bending stress. Membrane stress is usually quoted as a force per unit length crossing an imaginary cut, rather than force per unit area. Equilibrium of moments about the normal tells us that $\tau_{xy} = \tau_{yx}$ and this applies even if the plate is undergoing angular acceleration (the moment of inertia of an element tends to zero faster than the moment due to stress as the size of the element tends to zero).

τ_{xy} and τ_{yx} are shear stresses in the plane of the plate. We also get shear stress perpendicular to the plate due to plate bending. These are not labeled in figure 12 because the notation for plate bending is rather confusing.

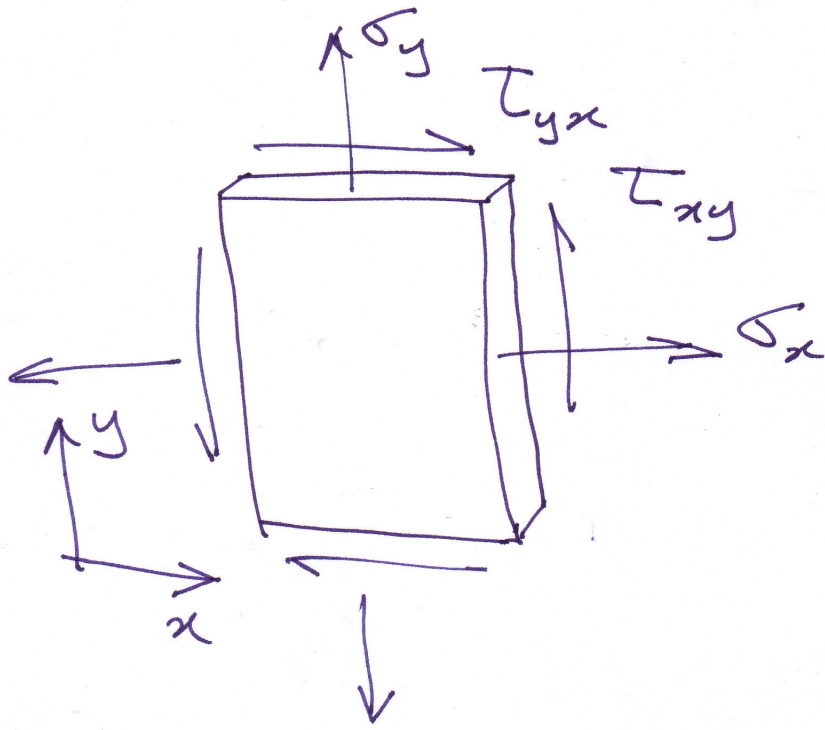


Figure 11: Plane stress

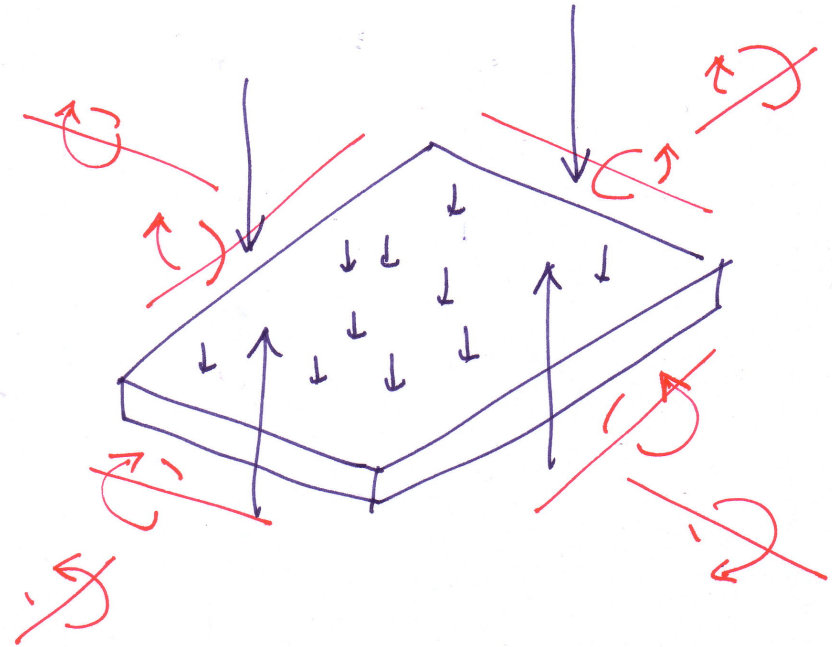


Figure 12: Plate bending

Thus for plane stress we have three unknown stresses, σ_x , σ_y and $\tau_{xy} = \tau_{yx}$. We have two equations of equilibrium that even though q_x and q_y are zero, the plate can still be loaded at its edges.

$$\begin{aligned}\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} &= q_x \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} &= q_y\end{aligned}\quad (1)$$

in the x and y directions respectively. q_x and q_y are the loads per unit area applied to the plate, in its own plane, for example the own weight of a wall. Thus we have three unknown stresses and only two equations of equilibrium so that plane stress is **statically indeterminate**.

If q_x and q_y are both zero, the stresses can be written in terms of the Airy stress function ϕ ,

$$\begin{aligned}\sigma_x &= \frac{\partial^2 \phi}{\partial y^2} \\ \sigma_y &= \frac{\partial^2 \phi}{\partial x^2} \\ \tau_{xy} = \tau_{yx} &= -\frac{\partial^2 \phi}{\partial x \partial y}\end{aligned}\quad (2)$$

so that they **automatically** satisfy the equilibrium equations, 1. Note

If the plate is elastic we can solve for ϕ and hence the stresses by using the stress - strain relationships,

$$\begin{aligned}\varepsilon_x &= \frac{1}{E} (\sigma_x - \nu \sigma_y) \\ \varepsilon_y &= \frac{1}{E} (\sigma_y - \nu \sigma_x) \\ \gamma_{xy} &= \frac{2(1 + \nu) \tau_{xy}}{E}\end{aligned}\quad (3)$$

(in which E is Young's modulus and ν is Poisson's ratio) together with the compatibility equation,

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = 0.\quad (4)$$

The compatibility equation comes from the fact that our three strains ε_x , ε_y and γ_{xy} are due to only two components of displacement, in the x and y directions.

We finally end up with just one equation,

$$\nabla^4 \phi = \frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0\quad (5)$$

which is known as the biharmonic equation. Even though it looks complicated, it actually behaves very well and is not difficult to solve [Timoshenko and Goodier, 1970].

The membrane theory of shells

In the membrane theory of shells we still have three components of membrane stress, exactly as in plane stress. But we now have three equations of equilibrium. Two of them are in the directions tangent to the shell, again exactly as in the case of plane stress. The third equation is perpendicular to the tangent to the shell surface. The load is balanced by the membrane stresses multiplied by the curvature. Here the load would be in kN/m², the membrane stress in kN/m and the curvature in 1/m.

Thus we have three unknown stresses and three equations of equilibrium so that shells should be statically determinate. **Unfortunately we have three partial differential equations of equilibrium in three unknown membrane stresses and whether or not these equations have a solution depends upon the shape of the shell and the boundary conditions.** This is a very difficult area of mathematics and it is often impossible to say whether a solution exists or not, let alone find one.

The simplest way to express this mathematically is using plane coordinates. The horizontal equilibrium equations, 1, still apply if the stress components are redefined as the horizontal component of membrane stress per unit horizontal length. In particular if a shell is only loaded in the vertical direction, the horizontal equilibrium equations are still satisfied by use of the Airy stress function, equations 3. Then equilibrium in the vertical direction is simply

$$w = \frac{\partial^2 \phi}{\partial x^2} \frac{\partial^2 z}{\partial y^2} - 2 \frac{\partial^2 \phi}{\partial x \partial y} \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial y^2} \frac{\partial^2 z}{\partial x^2} \quad (6)$$

in which z is the height of the shell and w is the load per unit plan area, both assumed to be known functions of x and y . This equation may not look any more complicated than the biharmonic equation, 5, but depending upon the shape of the shell and the boundary conditions, it may be impossible to solve for ϕ . Equation 6 is 'exact' in that it does not assume that the slope of the shell is small.

If the shell is the wrong shape or it doesn't have enough boundary support, it may be a mechanism as far as the membrane theory is concerned and be able to undergo **inextensional deformation**, that is deformation in which the shell is bent without stretching.

The dome and cooling tower in figures 13 and 15 are statically determinate and cannot undergo inextensional deformation. The cooling

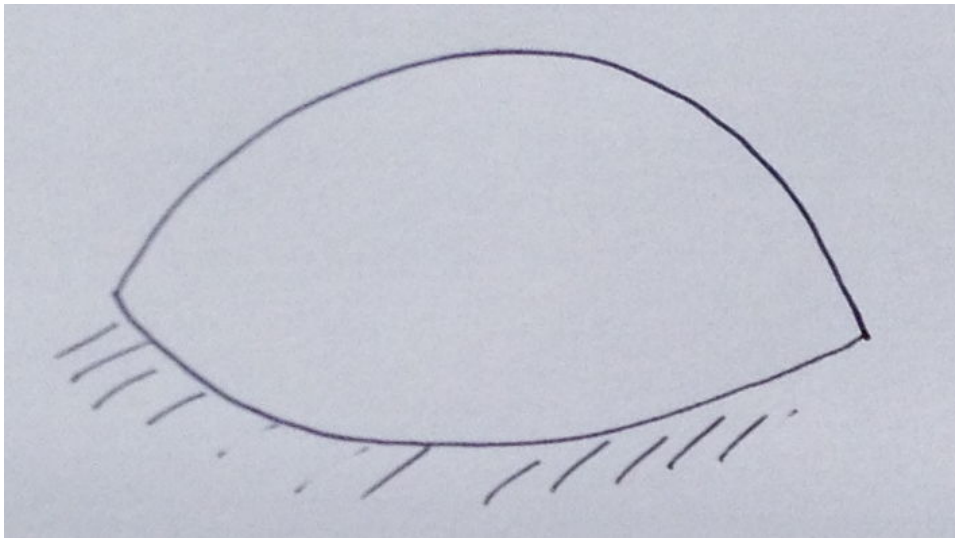


Figure 13: Dome: positive Gaussian curvature, statically determinate, inextensional deformation not possible



Figure 14: Dome with hole: positive Gaussian curvature, a mechanism, inextensional deformation possible

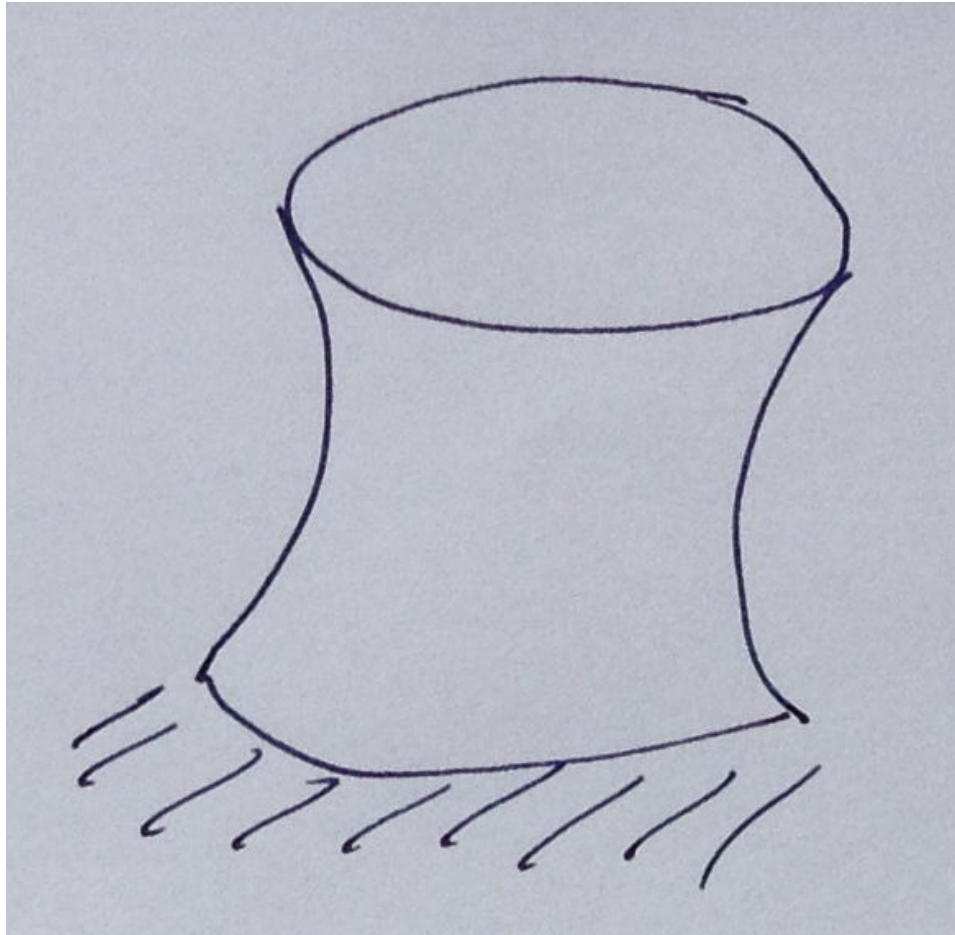


Figure 15: Cooling tower: negative Gaussian curvature, statically determinate, inextensional deformation not possible

tower has a big hole at the top, but putting a bit hole in a dome produces a structure which is a mechanism, figure 14. The holes in the colander do not effect things so much because they are small.

The difference between the dome and the cooling tower is that the dome is synclastic and has positive Gaussian curvature whereas the cooling tower is anticlastic and has negative Gaussian curvature. 'We may divide curved surfaces into Anticlastic and Synclastic. A saddle gives a good example of the former class; a ball of the latter. The outer portion of an anchor-ring is synclastic, the inner anticlastic.' (William Thomson Kelvin and Peter Guthrie Tait *Treatise on natural philosophy* Vol. 1 1867 Oxford: Clarendon Press). Here saddle refers to that on a horse and an anchor-ring is a torus.

The Gaussian curvature [Eisenhart, 1909, Struik, 1988] is the product of the two principal curvatures on a surface and they are of opposite signs on a cooling tower. Mathematically surface curvature is a second order symmetric tensor, as is membrane stress. Thus we can draw Mohr's circle for curvature in the same way that we do for stress.

Gauss's Theorem (Theorema Egregium) tells us that the Gaussian curvature of a surface can be calculated by only measuring lengths on a surface, and therefore inextensional deformation **does not**

change the Gaussian curvature. A **developable surface** is a surface with zero Gaussian curvature which can be laid out flat. Examples include cylinders and cones.

The Cohen-Vossen theorem from differential geometry tells us that it is not possible to deform a closed convex surface such as an egg without changing lengths on the surface, in other words inextensional deformation is impossible. However part of an egg can be deformed inextensionally, explaining why it is so much more flexible. The stiffness of the part can be regained by glueing it to a support.

The bending theory of shells and buckling

If a shell can undergo inextensional deformation, then it will have to rely on bending stiffness as well as membrane action in order to carry loads. However, even if a shell has the correct shape and is properly supported it must have bending stiffness to prevent buckling if there are any compressive membrane stresses. **Thus for efficiency we want our shell to work primarily by membrane action, which is what shell action means, but we know that we must also have bending stiffness to resist buckling and inextensional deformation.**

Shell buckling is particularly nasty because shell structures are so efficient. Almost no deflection occurs and then suddenly there is total collapse. Paradoxically, the less efficient the shell, in terms of shape, triangulation of the surface and boundary support, the better it behaves in buckling. This is because bending action of shells requires much more deflection than membrane action and therefore small irregularities in shell geometry and other initial imperfections have less effect.

Experiments show that a properly supported shell working primarily by membrane action can never support anything like the theoretical 'eigenvalue' or 'linear' buckling load even when the utmost care is taken to eliminate initial imperfections.

The analysis of shell buckling by hand calculations is effectively impossible, even eigenvalue analysis of a spherical shell is very difficult, and as we have said gives wildly optimistic answers. This means that there is no option but to use computer analysis, but this is quite an esoteric area, and even though many programs offer shell buckling, the results should be treated with a great deal of circumspection. The imperfection sensitivity of shells means that non-linear buckling analysis has to be used. There is still a place for physical model tests for shell buckling.

3 How much do you need to know to design a shell?

In the previous section we tried to describe how shells work in a relatively qualitative manner. It should by now be clear that it is difficult to derive the equations (particularly for the bending theory) and usually impossible to solve them, except for the membrane theory for very simple shapes. The theory of shell structures is described in such classic works as Novozhilov [1959], Flügge [1960], Green and Zerna [1968], Calladine [1988] and is very mathematical. Green and Zerna use the tensor notation, which is difficult to learn, but enables the equations to be written far more elegantly. The tensor notation for shells is essentially the same as that used in the general theory of relativity [Dirac, 1975].

Thus in practice one has three possible approaches:

1. Simple hand calculations 'informed' by the classical theory of shells
2. Numerical analysis using a computer
3. Physical testing

Numerical analysis almost invariably uses the finite element method in the form of shell elements for continuous shells or beam elements for grid shells. The derivation of the finite element equations does not really depend very much on the theory of shells, except for being able to work in a curvilinear coordinates.

The shape functions of the finite element method produce algebraic equations. These equations may be linear or non-linear according to the material behaviour and whether one is concerned with buckling. The equations might be solved using an 'implicit' method involving inversion of a stiffness matrix or an 'explicit' method like Dynamic Relaxation or Verlet integration.

However, as we have already noted, the structural behaviour of shells can be so complicated that numerical predictions may be inaccurate and so there is still a place for physical testing. Physical testing of 'sketch models' can also give a qualitative insight that cannot be obtained from a numerical analysis.

4 Funicular shells

The word 'funicular' comes from the Latin for a 'slender rope'. Figure 16 shows funicular polygons, the rope automatically adjusts its shape

to carry the loads **without any bending moment**. The rope is a mechanism which moves to carry a particular load case.

We have seen that if we are lucky a shell can carry **any load** by membrane action only. However, if our shell has the wrong shape or is not properly supported, it will only be able to carry certain loads for which it is funicular. These are loads which do not excite an inextensional mode of deformation. The concept of funicular loads applies to all shells that are capable of inextensible deformation, but it is particularly relevant to masonry shells for which funicular loads produce no bending moment or tensile stress [Block, 2009].

Tension structures, like the fabric in the Arnolfini Wedding (figure 17), adjust their shape under load to become funicular. Compression structures move in the opposite direction, and that is what causes buckling.

Reversing the loads on a pure tension structure produces a pure compression structure, a fact used by Gaudí for the crypt of the Colònia Güell (figures 18 and 19) and Frei Otto for the Mannheim Multihalle (figures 30 to 32).

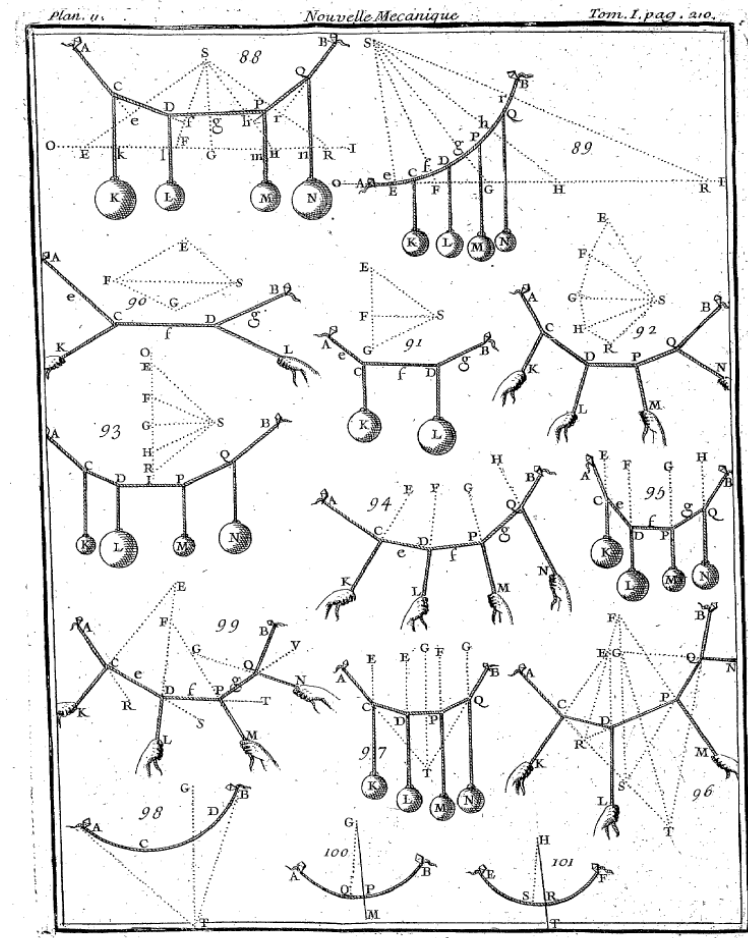


Figure 16: Funicular polygons from *Nouvelle mécanique, ou Statique, dont le projet fut donné en M. DC. LXXXVII. Tome 1*, by Pierre Varignon, Jombert (Paris)-1725



Figure 17: Jan van Eyck, the Arnolfini Wedding



Figure 18: Cripta de la Colònia Güell by Antoni Gaudí

Funicular arches and cables



Figure 19: Frei Otto's reconstruction of Gaudí's hanging model for the cripta de la Colònia Güell

A rope or chain hanging under its own weight forms a catenary (from the Latin for a chain). Thus the catenary is a particular case of a funicular curve. The catenary is of some relevance to the design of shells, so it is worth deriving the mathematical form here.

If a cable is only carrying vertical loads then the horizontal component H of tension in the cable,

$$H = T \cos \lambda = \text{constant}, \quad (7)$$

where T is the tension in the cable and λ is the slope. The vertical component V of tension

$$V = T \sin \lambda = H \tan \lambda = H \frac{dy}{dx}. \quad (8)$$

If the loading, w , is constant per unit arc length, s , then

$$\begin{aligned}
 w &= \frac{dV}{ds} = \frac{dx}{ds} \frac{dV}{dx} = \cos \lambda \frac{dV}{dx} \\
 &= \frac{1}{\sqrt{1 + \tan^2 \lambda}} \frac{dV}{dx} = \frac{1}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} H \frac{d^2y}{dx^2}.
 \end{aligned} \tag{9}$$

This can be integrated to give

$$\frac{dy}{dx} = \sinh \left(\frac{x}{c} \right) \tag{10}$$

in which $c = \frac{w}{H}$ and we have left out the constant of integration because it just moves the curve sideways. Integrating again,

$$\frac{y}{c} = \cosh \left(\frac{x}{c} \right) - 1 \tag{11}$$

in which the constant of integration is chosen so that the curve goes through the origin. This is the catenary, the upper curve in figure 20, while the lower curve is the parabola, the funicular curve for when the load is constant per unit horizontal length, as is the case for a suspension bridge.

It can be seen that the two curves are identical when their slope is low and they only peel apart when the load per unit horizontal length on the catenary increases with slope. The catenary is one of the few

curves where there is a simple relationship between x and y and the arc length along the curve, s , starting from the bottom,

$$s = \int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = c \sinh \left(\frac{x}{c} \right). \tag{12}$$

It is relatively easy to find the funicular load for a given shape of cable or funicular shape for a given load, either by doing a simple physical experiment, or mathematically. Having found the shape it can be inverted, or turned upside down, to find the best shape for the equivalent compression structure or arch.

If an arch is carrying a funicular load, there will be no bending moment in it, which is equivalent to saying that the line of thrust is along the axis of the arch. If a non-funicular load is added, it will produce bending moments and cause a deviation of the line of thrust.

The concept of funicular load applies particularly to structures that have to carry one dominant load case, perhaps their self-weight or some permanent load due to water or soil. Arch bridges such as the Gaoliang Bridge (figure 21) have to carry the extra weight of the masonry and fill over the support, together with the horizontal thrust from the fill. This means that more curvature is required towards the supports than would be the case for a catenary.

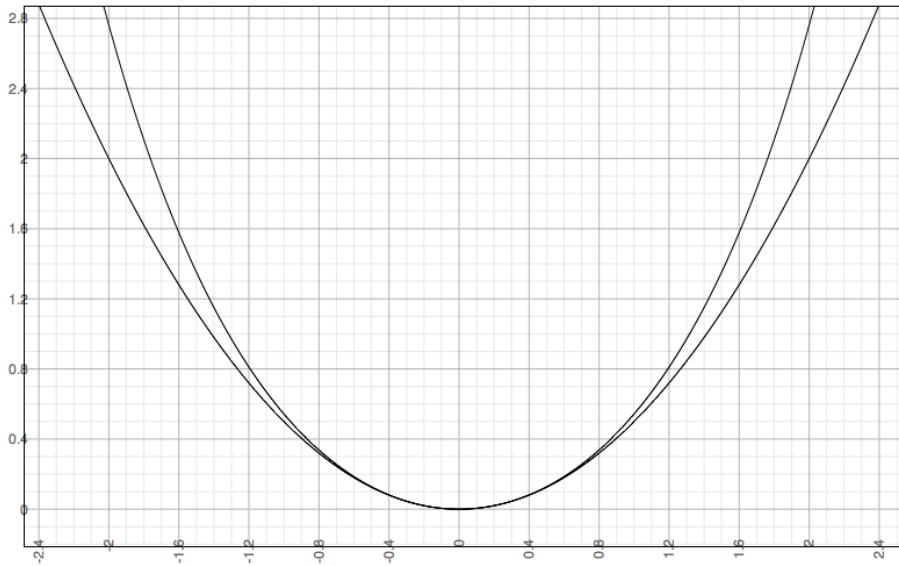


Figure 20: Catenary (upper curve) and parabola



Figure 21: Gaoliang Bridge of The Summer Palace. Photo: Hennessy

Now let us suppose that we want to make a circular arch of varying thickness so its own weight is funicular. If t is the thickness of the arch, R is its radius and ρg is its weight per unit volume,

$$\begin{aligned} \rho g t &= -\frac{1}{R} \frac{dV}{d\lambda} = -\frac{H}{R} \frac{d}{d\lambda} (-\tan \lambda) \\ &= \frac{H}{R} \sec^2 \lambda \end{aligned} \quad (13)$$

so that

$$t = \frac{H}{\rho g R \cos^2 \lambda} \quad (14)$$

in which $\frac{H}{\rho g R}$ is a constant with the units of length. Note that H and V are now forces per unit width and H is positive for a compression. Figure 22 shows how the shell gets thicker as it approaches the vertical. The stress in the arch

$$\sigma = \frac{H}{t \cos \lambda} = \rho g R \cos \lambda, \quad (15)$$

which reduces away from the top, the opposite of what happens with a catenary.

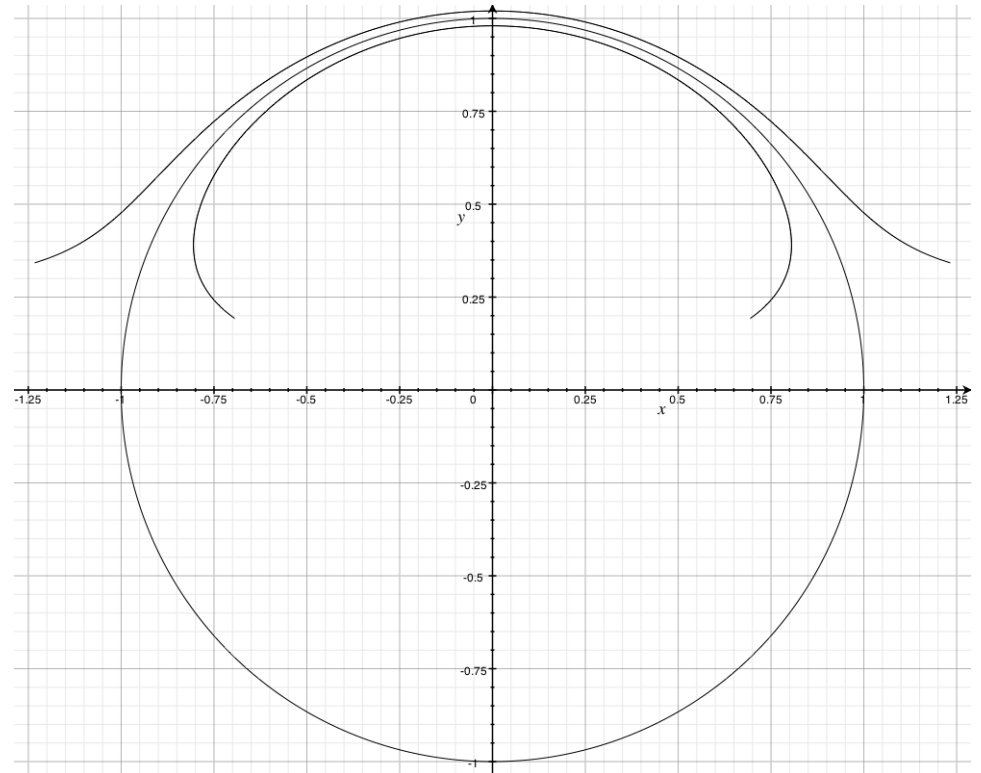


Figure 22: Funicular circular arch

Another possibility is to say that the compressive stress σ should be constant. If that is the case,

$$\begin{aligned} \rho g t &= \frac{dV}{ds} = H \frac{d}{ds} (-\tan \lambda) \\ &= -H \sec^2 \lambda \frac{d\lambda}{ds} \end{aligned} \quad (16)$$

and

$$H = \sigma t \cos \lambda. \quad (17)$$

Thus

$$\begin{aligned} \frac{\rho g ds}{\sigma d\lambda} &= -\frac{1}{\cos \lambda} = \frac{\cos \lambda}{1 - \sin \lambda} - \frac{\sin \lambda}{\cos \lambda}, \\ \frac{\rho g dx}{\sigma d\lambda} &= -1, \\ \frac{\rho g ds}{\sigma d\lambda} &= \tan \lambda. \end{aligned} \quad (18)$$

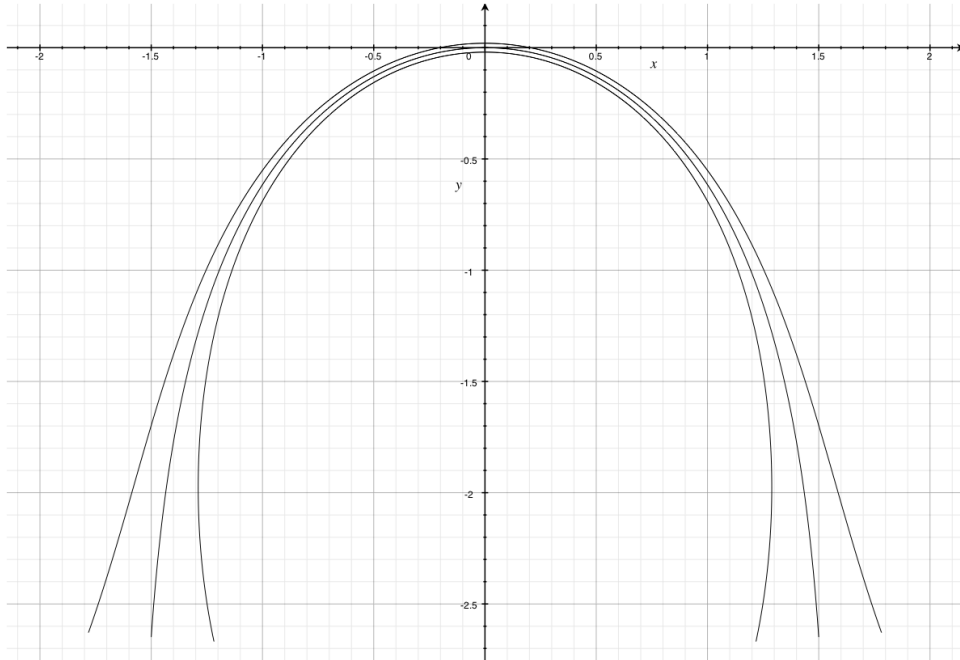


Figure 23: Constant stress arch

These can be integrated to give

$$\begin{aligned}
\frac{\rho g}{\sigma} s &= \log_e \left(\frac{\cos \lambda}{1 - \sin \lambda} \right), \\
\frac{\rho g}{\sigma} x &= -\lambda, \\
\frac{\rho g}{\sigma} y &= \log_e (\cos \lambda), \\
t &= \frac{t_0}{\cos \lambda} = t_0 e^{\left(-\frac{\rho g y}{\sigma} \right)}, \tag{19}
\end{aligned}$$

where t_0 is the thickness at the top of the arch.

Figure 23 shows a constant stress arch and the Taq Kasra in figure 24 is of a similar shape. Let us now think about scale. If the arch is made from a weak masonry we might have $\rho g = 25 \times 10^3 \text{ N/m}^2$ and $\sigma = 0.5 \text{ MPa} = 0.5 \times 10^6 \text{ N/m}^2$. Therefore $\frac{\sigma}{\rho g} = 20 \text{ m}$ which means that if we decided to use the part of the arch in figure 23 between -1.25 and 1.25 on the horizontal axis, the span would be 50 m. If we used concrete at a stress of 25 MPa, the corresponding span will be 1250 m, or 1.25 km.

Figure 25 shows a comparison of the catenary, circular and constant stress arches. They all have the same thickness and curvature (and therefore stress) at the top. However the catenary will have stress increased by a factor of 2.5 at the supports. So the catenary is not a particularly good shape for an arch or cylindrical shell, unless practi-



Figure 24: The Taq Kasra from the ancient city of Ctesiphon in Iraq

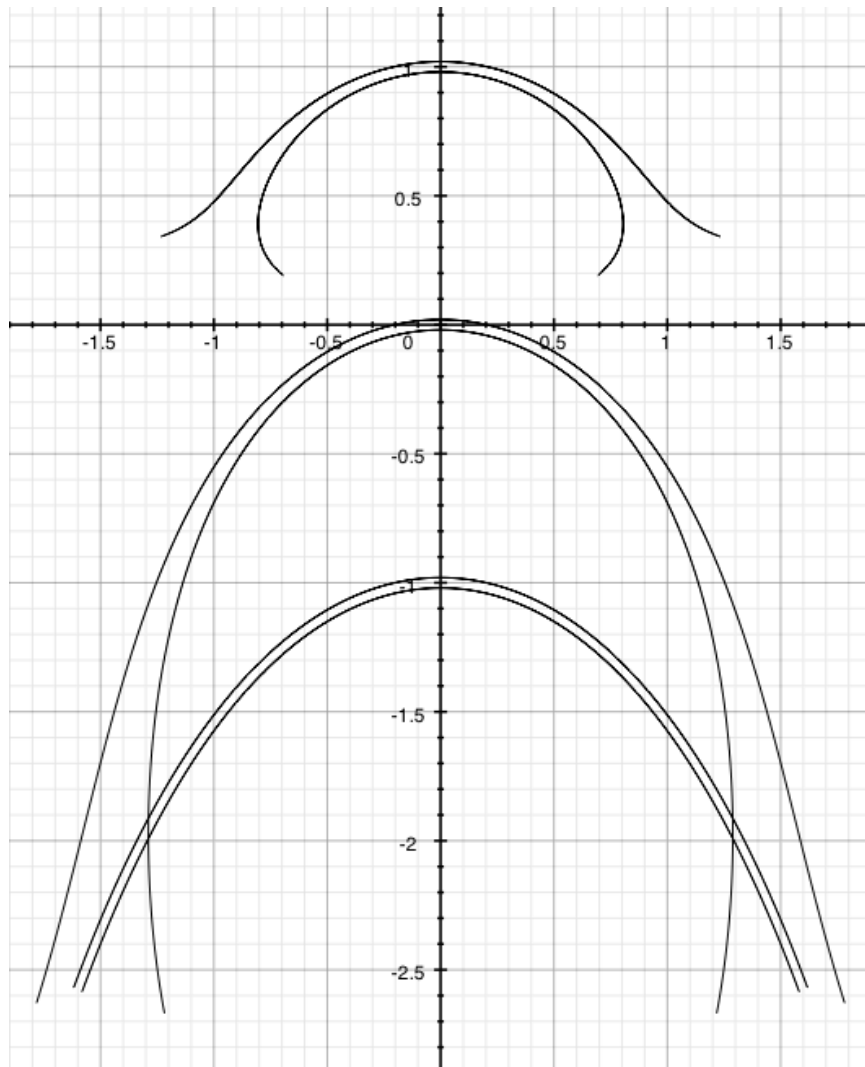


Figure 25: Three arches, circular (top), constant stress (middle) and catenary (bottom)

cal considerations mean that it has to have a constant thickness.

Uniform stress shell

The equivalent of the uniformly stressed arch is the uniformly stressed shell of revolution. Let us imagine a shell of variable thickness, t , which is only loaded by its own weight, ρg per unit volume, and that there is a uniform compressive stress σ (force per unit area) in the material.

The equilibrium equation in the radial direction tangent to the surface is

$$\rho g r t \frac{dz}{dr} = -\sigma t + \frac{d}{dr} (r t \sigma) = -r \sigma \frac{dt}{dr} \quad (20)$$

and therefore

$$\frac{\rho g}{\sigma} = -\frac{1}{t} \frac{dt}{dz} \quad (21)$$

which can be integrated to give

$$t = t_0 e^{-\frac{\rho g z}{\sigma}} \quad (22)$$

This is exactly the same result that we obtained for the uniformly stressed arch, which is a special case of the uniformly stressed shell. In fact this result applies for any plan shape of vertically loaded uniformly stressed shell.

We can now use the equilibrium in the normal direction to find the shape of the shell,

$$\sigma \left[\frac{\frac{d^2 z}{dr^2}}{\left(1 + \left(\frac{dz}{dr}\right)^2\right)^{\frac{3}{2}}} + \frac{\frac{dz}{dr}}{r \sqrt{1 + \left(\frac{dz}{dr}\right)^2}} \right] + \frac{\rho g}{\sqrt{1 + \left(\frac{dz}{dr}\right)^2}} = 0 \quad (23)$$

or

$$\frac{\frac{d^2 z}{dr^2}}{1 + \left(\frac{dz}{dr}\right)^2} + \frac{1}{r} \frac{dz}{dr} + \frac{\rho g}{\sigma} = 0. \quad (24)$$

The quantity in the square brackets is the sum of the principal curvatures of the surface.

This equation probably cannot be solved analytically and figure 26

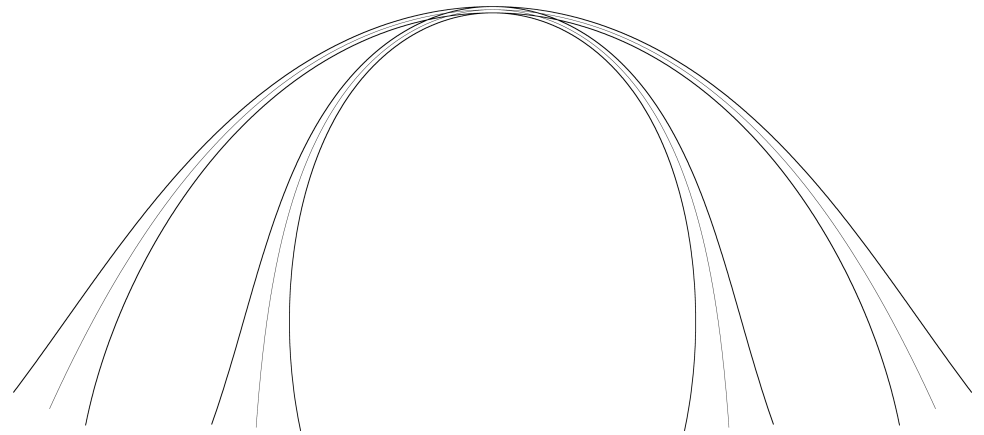


Figure 26: Constant stress shells – shell of revolution has larger span than cylindrical shell

shows a numerical solution. The smaller scale shell in figure 26 is the two dimensional cylindrical shell or arch from figures 23 and 25.

It can be seen that the shell of revolution will span roughly twice as far for the same stress, 100 m for the weak masonry and 2.5 km for concrete – this is if the shell is only carrying its own weight. The thickness doesn't come into the expression for maximum span, but if the shell is too thin other loads will dominate the stresses and also

the shell may buckle.

5 Gridshells

We have already referred to gridshells, that is shells made from a grid, such as the sieve in figure 8. Gridshells are sometimes known as lattice shells, or more rarely as reticulated shells from the Latin, reticulum, a small meshwork bag. They can be classified by material - timber, steel etc. - and by the geometric pattern of the grid, often triangles or quadrilaterals.

The difficulty of joining individual members (figure 27) means that timber gridshells are often made by bending long straight finger or scarf jointed timber members which cross at the nodes. The best known example is Frei Otto's Mannheim Multihalle (figures 30 to 32). It has a sensible shape and it is well supported. However, the timber laths only run in two directions, so that the surface is not properly triangulated to take the three components of membrane stress [Happold and Liddell, 1975]. The shape was initially defined by Frei Otto's hanging model (figure 30), however it was actually built from the numerical model by Büro Linkwitz. The hanging model ensures that the shell is 'funicular' under own weight which is important for resisting creep. However, recent research [Malek, 2012] suggests that introducing corrugations into a gridshell (or indeed any shell) is more important than a funicular shape when it comes to resisting buckling.



Figure 27: Portcullis House gridshell by Hopkins Architects

Corrugations occur in nature (figure 1) and were used by Nervi for the Palazzetto dello Sport (figure 3).

The Mannheim shells are partly braced by diagonal cables and some steel gridshells are braced in the same way, such as the Hippo House at Berlin Zoo 1996 by Jörg Gribl and Schlaich Bergerman und Partner (figure 33).

The steel gridshell for the British Museum Great by Foster + Partners, Buro Happold and Waagner Biro (figure 34) is fully triangulated.

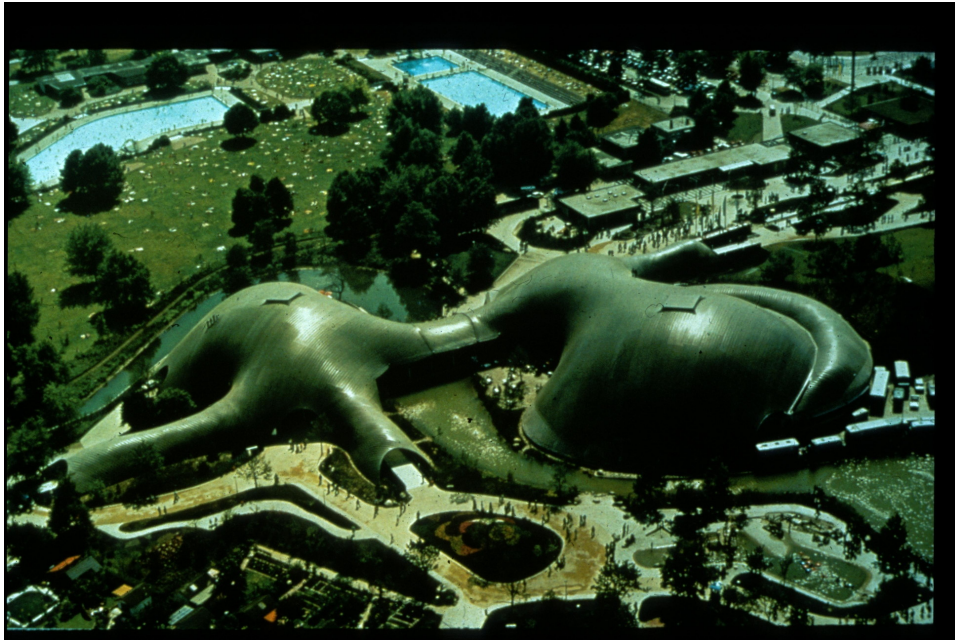


Figure 28: Mannheim gridshell 1974 by Mutschler & Partners, Freo Otto and Ove Arup & Partners - aerial view

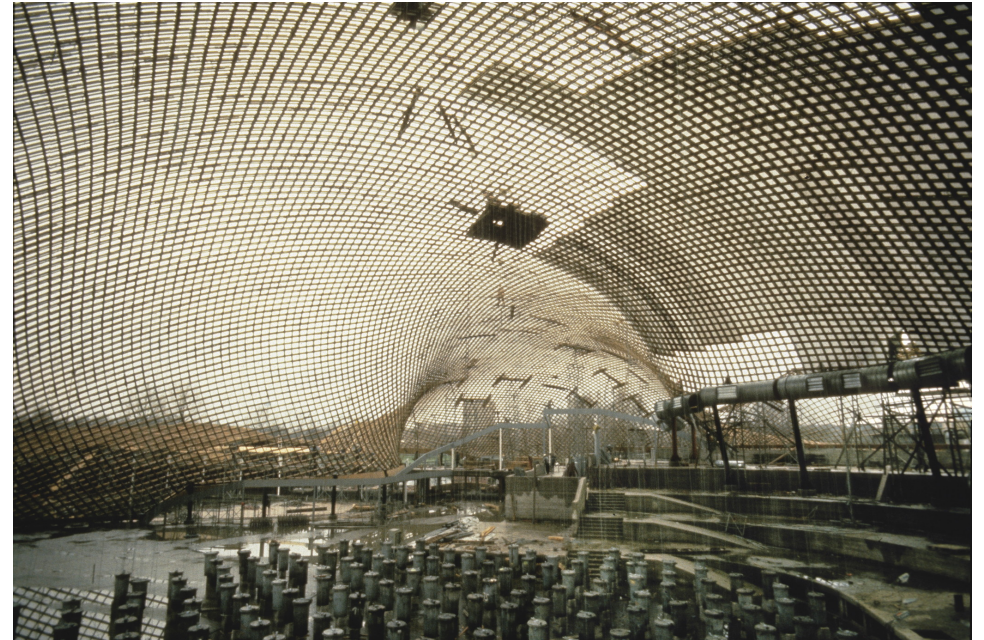


Figure 29: Mannheim gridshell 1974 by Mutschler & Partners, Freo Otto and Ove Arup & Partners - load test

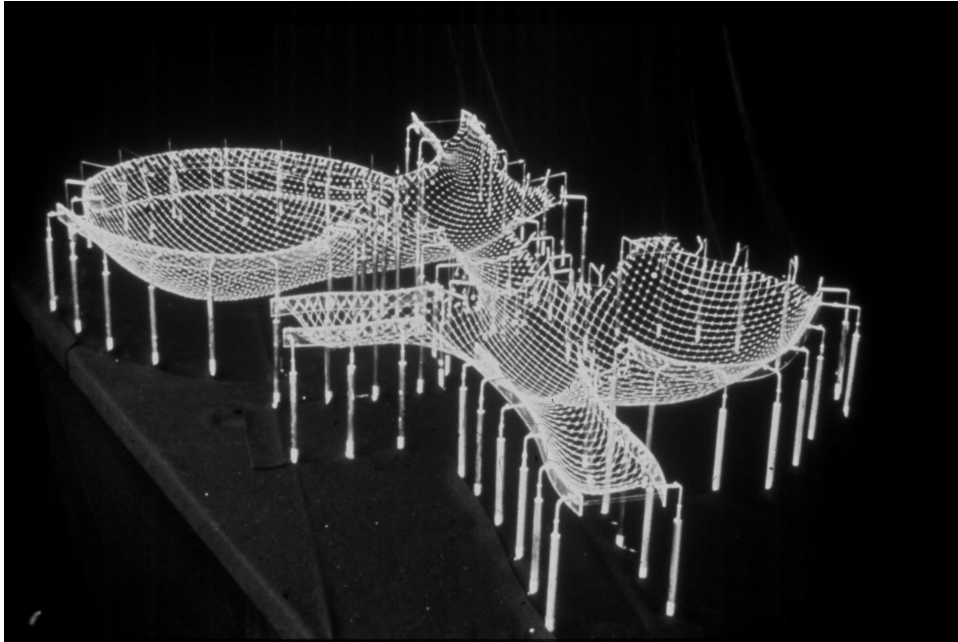


Figure 30: Mannheim gridshell 1974 by Mutschler & Partners, Freo Otto and Ove Arup & Partners - form finding model



Figure 31: Mannheim gridshell 1974 by Mutschler & Partners, Freo Otto and Ove Arup & Partners - erection



Figure 32: Mannheim gridshell 1974 by Mutschler & Partners, Freo Otto and Ove Arup & Partners - detail



Figure 33: Hippo House, Berlin Zoo 1996 by Jörg Gribl and Schlaich Bergerman und Partner.



Figure 34: British Museum Great Court 2000, by Foster + Partners, Buro Happold and Waagner Biro

However it is not properly supported because the existing building cannot take the horizontal thrusts from the roof and it is therefore on sliding bearings. This means that it can only thrust into the corners where the horizontal force is resisted by tension in the edge beams. Thus the shell cannot work by membrane action alone.

6 Conclusion

The uniform stress shell described in the section 4 is some sort of 'optimum', at least for the case when the own weight of the structure dominates.

However in practice there will be all sorts of functional and aesthetic constraints which will mean that the shell will not be structurally optimum. The Aichtal Outdoor Theatre by Michael Balz and Heinz Isler (figure 35) is clearly not 'properly' supported all around its boundary. If it were it would not fulfil the architectural and aesthetic constraints. However the negative Gaussian curvature 'lip' at the free edges reduces the possibility of inextensional modes of deformation and thereof is an optimal design.

Thus we can summarise our discussion as follows:

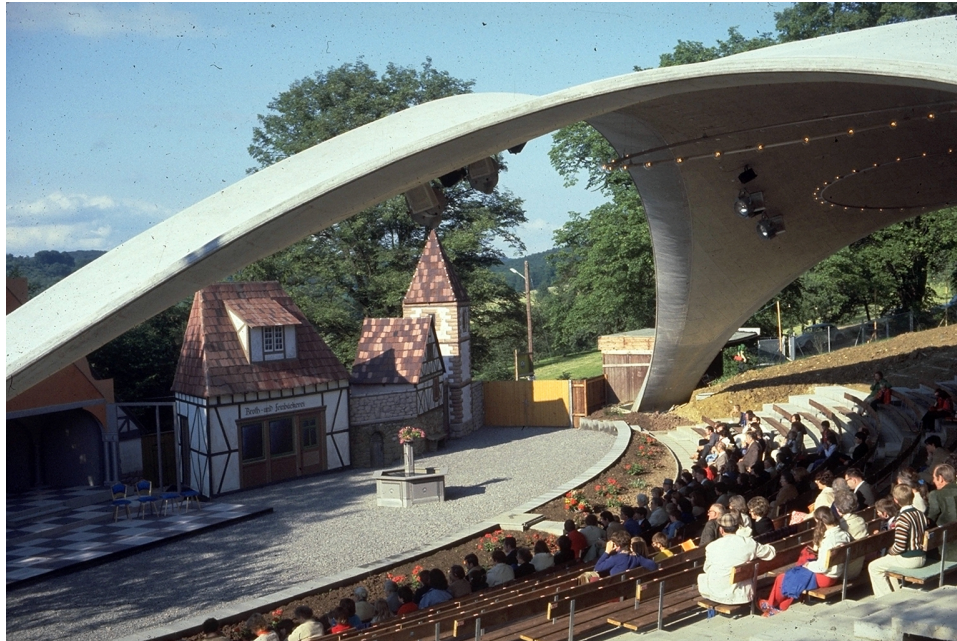


Figure 35: Aichtal Outdoor Theatre by Michael Balz and Heinz Isler

- Shell structures can be represented geometrically by surfaces.
- Shells are relatively rigid and this separates them from tension structures such as nets, balloons and sails.
- Shells work by a combination of membrane and bending action. Membrane action is more efficient but bending action is required to stop buckling and possible inextensible modes of deformation.
- The more efficient the shell, the more sudden the buckling collapse.
- Hand calculations for shells are very difficult or impossible. However some understanding of shell theory will help with choice of shell shape and interpreting computer and model test results.

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