VECTOR BUNDLES OVER PROJECTIVE CURVES

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Contents

Introduction

These notes are devoted to some recent results in the theory of vector bundles, or, the same, locally free sheaves over singular projective curves. Though there are lots of papers and books devoted to this subject, such as [20, 24] and many others, most of them confine themselves by the questions related to moduli varieries, hence, mainly study stable and semistable sheaves. The aim of my lectures is to give an idea how to deal with all of them, without stability restriction. A dream is, of course, to describe *all* vector bundles. As usually, this dream only can become reality in very few cases. Nevertheless, even they are of importance; moreover, the eleborated technique turns out useful for other problems, including the study of stable vector bundles. I will try to convince the reader of it.

The notes are organized as follows. First I recall the principal definitions, including the relations of vector bundles with locally free sheaves (Section 1); I also deduce from these definitions a well-known description of vector bundels over projective line [16]. For the sake of completeness, I have included Section 2, where reproduce the Atiyah's description of vector bundles over elliptic curves [1]. An excuse for me is that some of his results are reformuleted in a strenghened form using the language of categories (see, for instance, Corollaries 2.11 and 2.12).

Sections 3 and 4 introduce the main tools for investigating vector bundles over *singular* curves: the *sandwich procedure* and *bimodule categories*. First they appeared in the theory of Cohen–Macaulay modules (see, for instance, the survey [11]), but soon proved to be useful in a lot of other questions, especially when one wishes to descend from something non-singular to a subordinate singular object (ring, variety, etc.). In Section 5 we consider a specific class of bimodule problems, the *bunches of chains*. This section is the most technical and, perhaps, many readers will omit the details of proofs here. Nevertheless, I recommend them at least to apprehend the results, since the bunches of chains often appear in various branches of modern mathematics, sometimes quite unforseeen.

In Sections 6 and 7 we apply the developed technique for two classes of singular projective curves where a complete description of vector bundles is achievable: *projective configurations* of types A and $\widetilde{\mathrm{A}}$. The latter include, for instance, nodal cubics. The explicit results of these sections are based on those about bunches of chains, and though in case A they can be obtained by more elementary calculations, I do not believe it possible in case $\widetilde{\mathrm{A}}$, which is more important.

The descriptions of vector bundles for projective line and for projective configurations of type A are, as they say in the represention theory, *of finite type*: if we fix some discrete parameters (ranks and degrees), there are finitely many indecomposable vector bundles (actually, here all of them are line bundles). In cases of elliptic curves and projective configurations of type $\widetilde{\mathrm{A}}$ the situation is much more complicated, since indecomperosable vector bundles can have arbitrary ranks, moreover, there are *families* of vector bundles for fixed sets of discrete parameters. Nevertheless, these cases are *tame* (again in terms of the representation theory), which means that only 1-parameter families can appear. We do not precise the term "tame"; a reader wishing to know exact definitions can find them in [12, 10]. It so happens that all other projective curves are *vector bundle wild*. It means that a description of vector bundles over such a curve contains a description of representations of all finitely generated algebras over the base field. Section 8 presents a formal definition of wildness and a sketch of the proof of the just mentioned result. Thus, we establish what they call *representation type* of projective curves with respect to the classification of vector bundles:

- projective line and projective configurations of type A are *vector bundle finite*;
- elliptic curves and projective configurations of type $\widetilde{\mathrm{A}}$ are *vector bundle tame*;
- all other projective curves are *vector bundle wild*.

In Section 9 we show that the technique of Sections 3 and 4 is useful even in wild cases. Namely, here we apply it to obtain an explicite description of stable vector bundles over the cuspidal cubic, which is, perhaps, one of the simplest examples of vector bundle wild curves. Such explicite descriptions are of interest, for instance, due to their relations with some problems of mathematical physics, especially with Yang–Baxter equations. A concerned reader can find such applications in [8].

Our exposition is rather elementary. For instance, we do not use the methods related to derived categories, which can sometimes simplify and clarify the situation (see, for instance, [7]). However, we must note that the sandwich procedure and bimodule problems (especially bunches of chains!) turned out to be of great use when dealing with derived categories too (see, for instance, [5, 6]). We also do not consider applications, neither already mentioned to Yang–Baxter equations, nor to Cohen–Macaulay modules over surface singularities. The latter can be found in [19, 13, 10].

I am cordially thankful to the Organizing Committee of Escola de Álgebra for a wonderful opportunuty to present these results to a new audience and to the Institute of Mathematics and Staistics of the University of São Paulo for excellent working conditions during my visit, which have made possible writing these notes in a proper (I hope) way.

1. Generalities

First I recall the main definitions concerning vector bundles (see [18, 20]). Let X be an algebraic variety over a filed $\Bbbk$, which we always suppose algebracally closed. A *vector bundle* over X is a morphism of algebraic varieties $\xi : B \to X$, which "*locally looks like a direct product with a vector space.*" It means that there is an open covering $X = \bigcup_{i=1}^m U_i$ and isomorphisms $\phi_i : U_i \times \mathbb{A}^r \xrightarrow{\sim} \xi^{-1}(U_i)$ for some r such that for every pair i, j there is a morphism $\phi_{ij} : U_i \cap U_j \to \mathrm{GL}(r, \Bbbk)$ such that $\phi_i\phi_j^{-1}(x, v) = (x, \phi_{ij}v)$ for every pair $x \in U_i \cap U_j,\ v \in \mathbb{A}^n$. We call the tuple (U_i, ϕ_i, ϕ_{ij}) a *trivialization* of the vector bundle B (or, to be rigorous, $\xi : B \to X$). The integer r is called the *rank* of the vector bundle and denoted by $\mathrm{rk}(B)$. If $r = 1$, they say that B is a *line bundle.*

Let $\xi : B \to X$ and $\xi' : B' \to X$ be two vector bundles of ranks, respectively, r and r', with trivializations, respectively, (U_i, ϕ_i, ϕ_{ij}) and $(U'_i, \phi'_i, \phi'_{ij})$. A *morphism* of vector bundles $f : B \to B'$ is a morphism of algebraic varieties such that for every pair (i, j) there is a morphism $f_{ij} : U_i \cap U'_j \to \mathrm{Mat}(r' \times r, \Bbbk)$ such that $\phi'_i f \phi_j^{-1}(x, v) = (x, f_{ij}v)$ for all $x \in U_i \cap U'_j,\ v \in \mathbb{A}^r$.

Actually, a vector bundle of rank r is defined if we choose an open covering $X = \bigcup_i U_i$ and morphisms $\phi_{ij} : U_i \cap U_j \to \mathrm{GL}(r, \Bbbk)$ such that $\phi_{ii} = \mathrm{id}$ and $\phi_{ij}\phi_{jk} = \phi_{ik}$ on $U_i \cap U_j \cap U_k$. An isomorphic vector bundle is then given by morphisms ϕ'_{ij} such that $\phi'_{ij} = f_i\phi_{ij}f_j^{-1}$ for some morphisms $f_i : U_i \to \mathrm{GL}(r, \Bbbk)$. Thus the set of isomorphism classes of vector bundles of rank r is in one-to-one correspondence with the cohomology set $\mathrm{H}^1(X, \mathrm{GL}(r, \Bbbk))$ [17].

It is convenient to identify $\mathrm{VB}(X)$ with a certain subcategory of $\mathrm{Coh}\, X$, the category of coherent sheaves of $\mathcal{O}_X$-modules. Namely, it is well-known [18, 20] that, given a vector bundle $\xi : B \to X$, one can construct a coherent sheaf $\mathcal{B}$ taking for $\mathcal{B}(U)$ the set of its *sections over* U, i.e. maps $\sigma : U \to B$ such that $\xi\sigma = \mathrm{id}$. It is always a *locally free sheaf of constant rank*, i.e. such that all its stalks $\mathcal{B}_x$ are free $\mathcal{O}_{X,x}$-modules of rank $r = \mathrm{rk}\, B$. On the contrary, let $\mathcal{B}$ be a locally free coherent sheaf of constant rank r. There is an open affine covering $X = \bigcup_{i=1}^m U_i$ such that $\mathcal{B}|U_i \simeq \mathcal{O}_{U_i}^r$ for some r. Fix isomorphisms $\beta_i : \mathcal{O}_{U_i}^r \xrightarrow{\sim} \mathcal{B}|U_i$ and set $\beta_{ij} = \beta_i\beta_j^{-1}|(U_i \cap U_j)$. Let $A_i = \Gamma(U_i, \mathcal{O}_X)$; it is the coordinate ring of the affine variety U_i. Therefore, $A_i[x_1, x_2, \dots, x_r]$ is the coordinate ring of $U_i \times \mathbb{A}^r$. Moreover, β_{ij} is given by a matrix from $\mathrm{GL}(r, A_{ij})$, hence, can be identified with a morphism $U_i \cap U_j \to \mathrm{GL}(r, \Bbbk)$. Obviously, $\beta_{ij}\beta_{jk} = \beta_{ik}$ on $U_i \cap U_j \cap U_k$ and $\beta_{ii} = \mathrm{id}$, so these morphisms define a vector bundle over X. Moreover, these two operations are inverse to each other. Thus, they establish an equivalence of $\mathrm{VB}(X)$ and the category of locally free sheaves. In what follows we identify these two categories; in particular, we identify a vector bundle and the sheaf of its local sections, and use the words “vector bundles” and “locally free sheaves” as synonyms.

If $X = \mathbb{A}^n$, a locally free sheaf of $\mathcal{O}_X$-modules is given by a *projective* $\Bbbk[x_1, x_2, \dots, x_n]$-module. It is known (due to Quillen–Suslin) that every such module is actually *free*, hence, the corresponding sheaf is isomorphic to $r\mathcal{O}_X$ and the corresponding vector bundle is *trivial*, i.e. isomorphic to $X \times \mathbb{A}^r$. (Certainly, for $n = 1$ it follows from the fact that $\Bbbk[x[$ is the ring of principle ideals.) Consider the easiest non-affine case, when $X = \mathbb{P}^1$. Let $(x_0 : x_1)$ are the homogeneous coordinates on $\mathbb{P}^1$ and $X = U_0 \cup U_1$ be its standard affine covering, where $U_i = \{\, (x_0 : x_1) \mid x_i \neq 0 \,\}$; $U_i \simeq \mathbb{A}^1$ and its coordinate ring A_i is identified with $\Bbbk[x_j/x_i]$ $(j \neq i)$. If $\mathcal{F}$ is a locally free sheaf on $\mathbb{P}^1$, its restrictions on both U_i are free: there are isomorphoisms $\beta_i : \mathcal{F}|U_i \simeq r\mathcal{O}_{U_i}$. So we only have to define the gluing $\beta = \beta_1\beta_0^{-1} : r\mathcal{O}_{U_0}|U \simeq r\mathcal{O}_{U_1}|U$, where $U = U_0 \cap U_1$. Obviously, $U \simeq \mathbb{A}^1 \setminus \{0\}$, so it is also affine with the coordinate ring $A = \Bbbk[t, t^{-1}]$, where we may choose $t = x_1/x_0$, so $A_0 = \Bbbk[t]$, $A_1 = \Bbbk[t^{-1}]$. Therefore, β can be considered as a matrix from $\mathrm{GL}(r, A)$. If another locally free sheaf $\mathcal{F}'$ is given by another matrix $\beta' \in \mathrm{GL}(r', A)$, a morphism $f : \mathcal{F} \to \mathcal{F}'$ is given by a pair of matrices (f_0, f_1), $f_i \in \mathrm{Mat}(r' \times r, A_i)$ defining the restrictions of f on U_i. These matrices must satisfy the compatability equation $f_1\beta = \beta' f_0$. In particular, $\mathcal{F} \simeq \mathcal{F}'$ if and only if $r = r'$ and there are matrices $f_i \in \mathrm{GL}(r, A_i)$ such that $\beta' = f_1\beta f_0^{-1}$. Therefore, a complete description of vector bundles over $\mathbb{P}^1$ is indeed an easy corollary of the following claim, which we propose as an exercise (rather analogous to the famous Smith theorem on the diagonalization of a matrix over integers or over polynomials in one variable).

Exercise 1.1. Prove that, given a matrix $\beta \in \mathrm{GL}(r, \Bbbk[t, t^{-1}])$, there are matrices $f_0 \in \mathrm{GL}(r, \Bbbk[t])$, $f_1 \in \mathrm{GL}(r, \Bbbk[t^{-1}])$ such that $f_1 \beta f_0^{-1} = \mathrm{diag}(t^{d_1}, t^{d_2}, \dots, t^{d_r})$.

Corollary 1.2 (Grothendieck, [16])**.** *Every locally free sheaf over $\mathcal{O}_{\mathbb{P}^1}$ decomposes to a direct sum of line bundles $\mathcal{O}_{\mathbb{P}^1}(d)$, the latter defined by the 1×1 matrix (t^{-d}).*

(This notation agrees with the usual notation for twisted sheaves over projective curves.)

Recall [18, Chapter III, §5] that, if X is a projective variety, all morphism spaces $\mathrm{Hom}_X(\mathcal{F}, \mathcal{F}') = \mathrm{H}^0(X, \mathscr{H}om_X(\mathcal{F}, \mathcal{F}'))$, where $\mathcal{F}, \mathcal{F}'$ are coherent sheaves and $\mathscr{H}om_X$ denotes the sheaf of local homomorphisms, are finite dimensional. Especially, if $\mathcal{F}$ is indecomposbale, i.e. its endomorphism ring $\mathrm{End}(\mathcal{F})$ has no idempotents, the latter is *local.* Then the standard arguments [15] show that a decomposition of a cohereht sheaf (in particular, of locally free one) into a direct sum of indecomposables is unique (up to isomorphismm and numeration of the summands).

We also can easily calculate all morphisms between locally free sheaves over $\mathbb{P}^1$ (we will widely use this information later).

Proposition 1.3.

$$\mathrm{Hom}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(d), \mathcal{O}_{\mathbb{P}^1}(d')) = \begin{cases} 0 & \text{if } d > d', \\ \Bbbk[t]_{d'-d} & \text{if } d \leqslant d', \end{cases}$$

where $\Bbbk[t]_m$ denotes the space of polynomials of degree $\leqslant m$. In particular,

$$\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) = \mathrm{Hom}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(d)) = \begin{cases} 0 & \text{if } d < 0, \\ \Bbbk[t]_d & \text{if } d \geqslant 0. \end{cases}$$

Proof. Such a morphism f is given by two polynomials $f_0(t)$, $f_1(t^{-1})$ such that $f_1 x^{-d} = x^{-d'} f_0$, or $f_1(t^{-1}) = x^{d-d'} f_0(t)$. Obviousely, it is impossible if $d > d'$, while if $d \leqslant d'$, $f_0(t)$ can be any polynomial of degree $\leqslant d' - d$ and $f_1(t)$ is uniquely determined by f_0. □

Exercise 1.4. Show that

$$\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}(d)) = \begin{cases} 0 & \text{if } d \geqslant -1, \\ \Bbbk[t]_{-1-d} & \text{if } d < -1. \end{cases}$$

2. Elliptic curves

Except of $\mathbb{P}^1$, there is only one example of smooth curves, when a description of vector bundles has been obtained, namely that of *elliptic curves*, or smooth curves of genus 1 [1, 22]. For the sake of completeness, we recall their results.

Let X be an elliptic curve. It can always be presented as a 2-fold covering of $\mathbb{P}^1$ with 4 different ramification points of degree 2, which can be chosen as $\{0, 1, \infty, \lambda\}$ (then λ is defined up to the natural action of the symmetric group $\mathbf{S}_3$ generated by the maps $\lambda \mapsto 1 - \lambda$ and $\lambda \mapsto 1/\lambda$). If $\operatorname{char} \Bbbk \neq 2$, it is isomorphic to the plane cubic, whose affine part is given by the equation $y^2 = x(x-1)(x-\lambda)$.

Recall [18, Section IV.4] that in this case the line bundles of a prescribed degree d are in one-to-one correspondence with the points of the curve X. Namely, the set $\operatorname{Pic}_0 X$ of line bundles of degree 0 is in one-to-one correspondence with the points of X: if o is a fixed point, every line bundle of degree 0 is isomorphic to $\mathcal{O}(x - o)$ for a unique point $x \in X$. Therefore, every line bundle of degree d is isomorphic to $\mathcal{O}_X(x + (d-1)o)$ for a uniquely determined point x. Moreover, there is a line bundle $\mathcal{P}$ on $X \times X$ (the *Poincaré bundle*) such that, for every $x \in X$,

$$\mathcal{O}_X(x + (d-1)o) \simeq \mathcal{O}_X(do) \otimes_{\mathcal{O}_X} i_x^*\mathcal{P} \simeq i_x^*\mathcal{P}(d(o \times X)),$$

where i_x is the embedding $X \simeq X \times x \to X \times X$. Thus the line bundles of degree d form a 1-parameter family (parameterised by X).

It so happens that the description of indecomposable vector bundles of an arbitrary rank and degree is quite similar.

Let X be an irreducible projective curve. For an arbitrary coherent sheaf $\mathcal{F}$ on X denote $\mathrm{h}^k(\mathcal{F}) = \dim \mathrm{H}^k(X, \mathcal{F})$, $\chi(\mathcal{F}) = \mathrm{h}^0(\mathcal{F}) - \mathrm{h}^1(\mathcal{F})$ (the *Euler-Poincaré* characteristic) and $g = g(X) = \mathrm{h}^1(\mathcal{O}_X)$ (the *arithmetical genus*, which coincide with the geometrical genus if X is smooth). Define the *rank* of $\mathcal{F}$ as $\dim_{\mathcal{K}}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K})$, where $\mathcal{K}$ is the field of fractions of $\mathcal{O}_X$, the *degree* of $\mathcal{F}$ as $\deg \mathcal{F} = \chi(\mathcal{F}) - \chi(\mathcal{O}_X)\operatorname{rk}(\mathcal{F})$[1] and the *slope* of $\mathcal{F}$ as the ratio $\mu(\mathcal{F}) = \deg \mathcal{F} / \operatorname{rk}(\mathcal{F})$. Note that if $\mathcal{F}$ is a (non-zero) sheaf of rank 0, i.e. a *skyscarper* (a sheaf that is zero ourside finitely many closed points), then $\operatorname{rk} \mathcal{F} = 0$, $\deg \mathcal{F} = \mathrm{h}^0(\mathcal{F}) > 0$, so $\mu(\mathcal{F}) = \infty$. A coherent sheaf is said to be *semistable* (*stable*) if $\mu(\mathcal{F}') \leqslant \mu(\mathcal{F})$ (respectively, $\mu(\mathcal{F}') < \mu(\mathcal{F})$) for every proper subsheaf $\mathcal{F}' \subset \mathcal{F}$. (In particular, all skyscraper sheaves are semistable, but only simple ones, i.e. just $\Bbbk(x)$, $x \in X$, are stable.)

Proposition 2.1. *If $\mathcal{F}$ and $\mathcal{G}$ are coherent sheves on an irreducibles projective curve, and one of them is locally free, then* $\deg(\mathcal{F} \otimes \mathcal{G}) = \operatorname{rk} \mathcal{F} \deg \mathcal{G} + \deg \mathcal{F} \operatorname{rk} \mathcal{G}$, *hence* $\mu(\mathcal{F} \otimes \mathcal{G}) = \mu(\mathcal{F}) + \mu(\mathcal{G})$.

[1] The Riemann–Roch theorem [18] shows that our definition of degree coincide with the usual one for the line bundles.

Proof. Let $\operatorname{rk}\mathcal{F} = r$, $\deg\mathcal{F} = d$, $\operatorname{rk}\mathcal{G} = r'$, $\deg\mathcal{G} = d'$ and $\mathcal{G}$ is locally free. Then $\mathcal{F}\otimes\mathcal{K} \simeq (r\mathcal{O}_X)\otimes\mathcal{K} \simeq r\mathcal{K}$. Consider the subsheaf $\widetilde{\mathcal{F}} \subset r\mathcal{K}$ generated by the images of both $\mathcal{F}$ and $r\mathcal{O}_X$. Then there are exact sequences

$$0 \to \mathcal{S}_1 \to \mathcal{F} \to \widetilde{\mathcal{F}} \to \mathcal{S}_2 \to 0$$

and

$$0 \to r\mathcal{O}_X \to \widetilde{\mathcal{F}} \to \mathcal{S}_3 \to 0,$$

where $\mathcal{S}_i$ are skyscraper sheaves. Since deg is additive in exact sequnces, $\deg\mathcal{O}_X = 0$ $d = \deg\mathcal{S}_1 - \deg\mathcal{S}_2 + \deg\mathcal{S}_3$. Tensoring these sequences by $\mathcal{G}$, we get exact sequences

$$0 \to \mathcal{S}_1\otimes\mathcal{G} \to \mathcal{F}\otimes\mathcal{G} \to \widetilde{\mathcal{F}}\otimes\mathcal{G} \to \mathcal{S}_2\otimes\mathcal{G} \to 0$$

and

$$0 \to r\mathcal{G} \to \widetilde{\mathcal{F}}\otimes\mathcal{G} \to \mathcal{S}_3\otimes\mathcal{G} \to 0,$$

wherefrom

$$\deg(\mathcal{F}\otimes\mathcal{G}) = rd' + \deg(\mathcal{S}_1\otimes\mathcal{G}) - \deg(\mathcal{S}_2\otimes\mathcal{G}) + \deg(\mathcal{S}_3\otimes\mathcal{G}).$$

But, for any skyscraper sheaf $\mathcal{S}$, $\deg\mathcal{S} = \mathrm{h}^0(\mathcal{S}) = \sum_x \dim\mathcal{S}_x$, and, since $\mathcal{G}$ is locally free, $\mathrm{h}^0(\mathcal{S}\otimes\mathcal{G}) = r'\mathrm{h}^0(\mathcal{S})$. Hence,

$$deg(\mathcal{F}\otimes\mathcal{G}) = rd' + r'(\deg\mathcal{S}_1 - \deg\mathcal{S}_2 + \deg\mathcal{S}_3) = rd' + r'd.$$

□

Corollary 2.2. *Let $\mathcal{F}$ be a locally free sheaf, $\mathcal{F} = \mathscr{H}om_X(\mathcal{F},\mathcal{O}_X)$ be its dual sheaf. Then* $\deg\mathcal{F}^\vee = -\deg\mathcal{F}$ *and* $\mu(\mathcal{F}^\vee) = -\mu(\mathcal{F})$.

Proof. Obviously, $\operatorname{rk}(\mathcal{F}^\vee) = \operatorname{rk}(\mathcal{F})$. Recall also that the Serre duality [18, Chapter 3, §7] implies that $\mathrm{H}^i(X,\mathcal{F}^\vee) =\simeq \mathrm{H}^{1-i}(X,\mathcal{F}\otimes\omega_X)^*$, where ω_X is the dualizing sheaf for X and V^* denotes the dual vector space to V. Therefore, if $\operatorname{rk}\mathcal{F} = r$,

$$\begin{aligned}\deg\mathcal{F} &= \chi(\mathcal{F}) + r(g-1) = -\chi(\mathcal{F}^\vee\otimes\omega_X) + r(g-1) = \\ &= -\deg(\mathcal{F}^\vee\otimes\omega_X) + 2r(g-1) = \\ &= -r\deg\omega_X - \deg\mathcal{F} + 2r(g-1) = -\deg\mathcal{F},\end{aligned}$$

since $\operatorname{rk}\omega_X = 1$ and $\deg\omega_X = \chi(\omega_X) + g - 1 = -\chi(\mathcal{O}_X) + g - 1 = 2(g-1)$. □

The definition of the slope also implies immediately the following

Proposition 2.3. *Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be an exact sequence of coherent sheaves. Then $\mu(\mathcal{F}') < \mu(\mathcal{F})$ if and only if $\mu(\mathcal{F}'') > \mu(\mathcal{F})$ and $\mu(\mathcal{F}') > \mu(\mathcal{F})$ if and only if $\mu(\mathcal{F}'') < \mu(\mathcal{F})$.*

Corollary 2.4. *Any stable sheaf $\mathcal{F}$ is a* brick, *i.e.* $\mathrm{Hom}_X(\mathcal{F},\mathcal{F}) = \Bbbk$.

Proof. Let $f : \mathcal{F} \to \mathcal{F}$ be neither zero nor invertible. Then $\operatorname{Im} f \neq \mathcal{F}$, hence $\mu(\operatorname{Im} f) < \mu(\mathcal{F})$, so $\mu(\operatorname{Ker} f) > \mu(\mathcal{F})$, which is impossible. Thus $\operatorname{Hom}_X(\mathcal{F}, \mathcal{F})$ is a field, so it coincides with $\Bbbk$ (since we suppose the latter algebraically closed). $\square$

Let now $g(X) = 1$, thus $\deg \mathcal{F} = \chi(\mathcal{F})$ for every coherent sheaf $\mathcal{F}$. If $\mathcal{F}$ is locally free and $\mathcal{G}$ is arbitrary, then $\mathscr{H}om_X(\mathcal{F}, \mathcal{G}) \simeq \mathcal{F}^\vee \otimes \mathcal{G}$, where $\mathcal{F}^\vee = \mathscr{H}om_X(\mathcal{F}, \mathcal{O}_X)$. Moreover, in this case the dualizing sheaf ω_X is isomorphic to $\mathcal{O}_X$, thus the Serre duality [18, Chapter 3, §7] implies that $\mathrm{H}^0(X, \mathcal{F}^\vee) = \operatorname{Hom}_X(\mathcal{F}, \mathcal{O}_X) \simeq \mathrm{H}^1(X, \mathcal{F})^*$ (V^* denotes the dual space to V) and $\mathrm{H}^1(X, \mathcal{F}^\vee) \simeq \mathrm{H}^0(X, \mathcal{F})$.

Proposition 2.5. *Let $g(X) = 1$ and $\mathcal{F}$ is a locally free sheaf over X.*

1. *If $\mu(\mathcal{F}) < \mu(\mathcal{G})$ for a coherent sheaf $\mathcal{G}$, then $\operatorname{Hom}_X(\mathcal{F}, \mathcal{G}) \neq 0$.*
2. *If $\mu(\mathcal{F}) = \mu(\mathcal{G})$, then $\dim \operatorname{Hom}_X(\mathcal{F}, \mathcal{G}) = \dim \operatorname{Hom}_X(\mathcal{G}, \mathcal{F})$.*
3. *If a vector bundle $\mathcal{F}$ is a brick, it is stable.*
4. *If $\mu(\mathcal{F}) > \mu(\mathcal{G})$ and both $\mathcal{F}$ and $\mathcal{G}$ are semistable vector bundles, then $\operatorname{Hom}_X(\mathcal{F}, \mathcal{G}) = 0$ and $\operatorname{Ext}^1_X(\mathcal{G}, \mathcal{F}) = 0$.*
5. *If X is smooth and $\mathcal{F}$ is indecomposable, then $\mathcal{F}$ is semistable.*

Proof. (1)-(2). Since $\mathscr{H}om_X(\mathcal{F}, \mathcal{G}) \simeq \mathcal{F}^\vee \otimes \mathcal{G}$, we have $\deg(\mathcal{F}^\vee \otimes \mathcal{G}) = \dim \operatorname{Hom}_X(\mathcal{F}, \mathcal{G}) - \dim \operatorname{Ext}^1_X(\mathcal{F}, \mathcal{G})$. If $\mu(\mathcal{F}) < \mu(\mathcal{G})$, i.e $\mu(\mathcal{F}^\vee \otimes \mathcal{G}) > 0$ and $\deg(\mathcal{F}^\vee \otimes \mathcal{G}) > 0$, it implies (1). Moreover, if $\mu(\mathcal{F}) = \mu(\mathcal{G})$, the same observation shows that $\dim \operatorname{Hom}_X(\mathcal{F}, \mathcal{G}) = \dim \operatorname{Ext}^1_X(\mathcal{F}, \mathcal{G})$. But the Serre duality gives that

$$\operatorname{Ext}^1_X(\mathcal{F}, \mathcal{G}) = \mathrm{H}^1(X, \mathcal{F}^\vee \otimes \mathcal{G})) \simeq \operatorname{Hom}_X(\mathcal{F}^\vee \otimes \mathcal{G}, \mathcal{O}_X)^* \simeq \operatorname{Hom}_X(\mathcal{G}, \mathcal{F})^*,$$

wherefrom (2).

(3). Let $\mathcal{F}$ be a brick and $\mathcal{F}' \subseteq \mathcal{F}$ be its proper subsheaf. Then $\operatorname{Hom}_X(\mathcal{F}', \mathcal{F}) \neq 0$, so, if $\mu(\mathcal{F}') \geqslant \mu(\mathcal{F})$, also $\operatorname{Hom}_X(\mathcal{F}, \mathcal{F}') \neq 0$ by (1)-(2). It is impossible, since a map $\mathcal{F} \to \mathcal{F}'$ gives a non-invertible morphism $\mathcal{F} \to \mathcal{F}$.

(4). Note that, since $\omega_X \simeq \mathcal{O}_X$, $\mathscr{E}xt^1_X(\mathcal{G}, \mathcal{O}_X) = 0$, hence $\mathscr{E}xt^1(\mathcal{G}, \mathcal{F}) = 0$ for any locally free sheaf $\mathcal{G}$, so $\operatorname{Ext}^1(\mathcal{G}, \mathcal{F}) \simeq \mathrm{H}^1(\mathscr{H}om_X(\mathcal{G}, \mathcal{F}))$, thus, by the Serre duality, $\operatorname{Ext}^1_X(\mathcal{G}, \mathcal{F}) \simeq \operatorname{Hom}_X(\mathcal{F}, \mathcal{G})^*$. If $f : \mathcal{F} \to \mathcal{G}$ is a nonzero morphism, $\mathcal{H} = \operatorname{Im} f$, then $\mu(\mathcal{F}) \leqslant \mu(\mathcal{H}) \leqslant \mu(\mathcal{G})$, which is impossible. So $\operatorname{Hom}_X(\mathcal{F}, \mathcal{G}) = 0$.

(5). Note first that, since X is smooth, every torsion free coherent sheaf is locally free. Note also that if $\mathcal{F}' \subset \mathcal{F}$ is a proper subsheaf of the same rank, the factorsheaf $\mathcal{S} = \mathcal{F}/\mathcal{F}'$ is skyscraper, hence $\deg \mathcal{S} > 0$ and $\deg \mathcal{F}' < \deg \mathcal{F}$, so $\mu(\mathcal{F}') < \mu(\mathcal{F})$. Let $m = \max \{ \mu(\mathcal{F}') \mid \mathcal{F}' \subseteq \mathcal{F} \}$ and $\mathcal{F}_1 \subset \mathcal{F}$ be a subsheaf of maximal possible rank with $\mu(\mathcal{F}_1) = m$. Then $\mathcal{F}_1$ must be semistable. $\mathcal{F}/\mathcal{F}_1$ is also torsion free: if $\mathcal{M} \neq 1$ were a torsion subsheaf of $\mathcal{F}/\mathcal{F}_1$, its preimage $\mathcal{F}'$ in $\mathcal{F}$ were bigger than $\mathcal{F}_1$ but of the same rank, which is impossible. Moreover, $\mathcal{F}/\mathcal{F}_1$ contains no subheaves with the slope m, since the preimage in $\mathcal{F}$ of such a subsheaf were also of slope m and of bigger rank than $\mathcal{F}_1$. Iterating this procedure, we get a tower of subsheaves $0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}$ with semistable factors $\mathcal{G}_i = \mathcal{F}_i/\mathcal{F}_{i-1}$ and $\mu(\mathcal{G}_i) < m$ for $i > 1$. By (4), $\mathrm{Ext}^1_X(\mathcal{G}_i, \mathcal{F}_1) = 0$ for $i > 1$, so $\mathrm{Ext}^1_X(\mathcal{F}/\mathcal{F}_1, \mathcal{F}_1) = 0$ and, since $\mathcal{F}$ is indecomposable, $\mathcal{F} = \mathcal{F}_1$, hence is semistable. $\square$

Remark. We shall see that the asserion (5) is no more true for *singular* curves of genus 1.

From now on let X be an elliptic curve. Then, to find all indecomposable vector bundles, we only have to comsider semistable vector bundles of some fixed slope μ. Let VB_μ denote the category of all such vector bundles. Obviously, multiplication by a line bundle $\mathcal{L}$ of degree l induces an equivalence $\mathrm{VB}_\mu \simeq \mathrm{VB}_{\mu+l}$, so we only have to consider the case, when $0 \leqslant \mu < 1$. Let first $\mu = 0$.

Theorem 2.6. *For every positive integer r there is a unique (up to isomorphism) indecomposable sheaf $\mathcal{N}_r$ of rank r and degree 0 such that $\mathrm{h}^0(\mathcal{N}_r) \neq 0$. In this case $\mathrm{h}^0(\mathcal{N}_r) = \mathrm{h}^1(\mathcal{N}_r) = 1$ and $\mathcal{N}_r$ has a filtration with all factors isomorphic to $\mathcal{O}_X$. Moreover, $\mathcal{N}_r^\vee \simeq \mathcal{N}_r$.*

Proof. Note that the last statement follows from the other ones, since $\mathrm{H}^0(\mathcal{N}_r^\vee) = \mathrm{Hom}_X(\mathcal{N}_r, \mathcal{O}_X) \neq 0$. We use induction by r. If a morphism $f : \mathcal{O}_X \to \mathcal{F}$ is nonzero, then $\mathrm{Im}\, f \simeq \mathcal{O}_X$, wherefrom $\mathrm{Im}\, f = \mathcal{F}$ if $\mathrm{rk}\, \mathcal{F} = 1$ and $\deg \mathcal{F} = 0$, thus $\mathcal{N}_1 = \mathcal{O}_X$. Suppose that the assertion is true for the sheaves of ranks at most r. Then $\mathrm{Ext}^1_X(\mathcal{N}_r, \mathcal{O}_X) \simeq \mathrm{Hom}_X(\mathcal{O}_X, \mathcal{N}_r)^*$ is 1-dimensional, so there is a unique sheaf (up to isomorphism) $\mathcal{N}_{r+1}$ such that there is a non-split exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{\iota} \mathcal{N}_{r+1} \xrightarrow{\pi} \mathcal{N}_r \to 0. \tag{2.1}$$

It induces an exact sequence

$$\begin{gathered} 0 \to \mathrm{Hom}_X(\mathcal{N}_r, \mathcal{O}_X) \xrightarrow{\pi^0} \mathrm{Hom}_X(\mathcal{N}_{r+1}, \mathcal{O}_X) \to \mathrm{Hom}_X(\mathcal{O}_X, \mathcal{O}_X) \xrightarrow{\delta} \\ \mathrm{Ext}^1_X(\mathcal{N}_r, \mathcal{O}_X) \to \mathrm{Ext}^1_X(\mathcal{N}_{r+1}, \mathcal{O}_X) \xrightarrow{\iota^1} \mathrm{Ext}^1_X(\mathcal{O}_X, \mathcal{O}_X) \to 0, \end{gathered}$$

where $\delta \neq 0$ (since the sequence (2.1) does not split). But both $\mathrm{Hom}_X(\mathcal{O}_X, \mathcal{O}_X)$ and $\mathrm{Ext}^1_X(\mathcal{N}_r, \mathcal{O}_X)$ are 1-dimensional, hence δ is an isomorphism, thus ι^1 and π^0 are isomorphisms as well. Therefore, $\mathrm{h}^0(\mathcal{N}_{r+1}) = \mathrm{h}^1(\mathcal{N}_{r+1}) = 1$. Then $\mathcal{N}_{r+1}$ is indecomposable. Indeed, if $\mathcal{N}_{r+1} = \mathcal{F}' \oplus \mathcal{F}''$, then either $\mathrm{Hom}_X(\mathcal{O}_X, \mathcal{F}') = 0$ or $\mathrm{Hom}_X(\mathcal{O}_X, \mathcal{F}'') = 0$. If $\mathrm{Hom}_X(\mathcal{O}_X, \mathcal{F}'') = 0$, then $\mathrm{Im}\, \iota \subseteq \mathcal{F}'$, so $\mathcal{N}_r \simeq \mathrm{Coker}\, \iota \oplus \mathcal{F}''$ and $\mathcal{F}'' = 0$.

Let now $\mathcal{F} \in \mathrm{VB}_0$ be indecomposable of rank $r+1$ with $\mathrm{h}^0(\mathcal{F}) \neq 0$. Then there is an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{F} \to \mathcal{G} \to 0. \tag{2.2}$$

If $\mathcal{G} \simeq \mathcal{N}_r$, then $\mathcal{F} \simeq \mathcal{N}_{r+1}$. Otherwise $\mathcal{G}$ decomposes: $\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2$ and $\deg \mathcal{G}_1 = \deg \mathcal{G}_2 = 0$: neither of them can be negative, since $\mathcal{F}$ is semistable by Proposition 2.15, $\mu(\mathcal{F}) = 0$ and $\mathcal{G}_1$, $\mathcal{G}_2$ are factorsheaves of $\mathcal{F}$. Moreover, since $\mathcal{F}$ does not split, both $\mathrm{Ext}^1_X(\mathcal{G}_1, \mathcal{O}_X) \neq 0$ and $\mathrm{Ext}^1_X(\mathcal{G}_2, \mathcal{O}_X) \neq 0$, wherefrom $\mathrm{h}^0(\mathcal{G}_1) \neq 0$ and $\mathrm{h}^0(\mathcal{G}_2) \neq 0$. Since $\mu(\mathcal{G}_i) = \mu(\mathcal{O}_X)$, it implies, as in the proof of Proposition 2.5, that $\mathrm{Hom}_X(\mathcal{G}_i, \mathcal{O}_X) \neq 0$, thus $\mathrm{h}^0(\mathcal{F}) > 1$. Therefore, $\mathcal{N}_{r+1}$ is a unique indecomposable sheaf in VB_0 with $\mathrm{h}^0(\mathcal{N}_{r+1}) = 1$.

Dually, there is a unique sheaf $\mathcal{F}$ occuring in a non-split exact sequence

$$0 \to \mathcal{N}_r \xrightarrow{\alpha} \mathcal{F} \to \mathcal{O}_X \to 0,$$

it is indecomposable and $\mathrm{h}^0(\mathcal{F}) = 1$, so $\mathcal{F} \simeq \mathcal{N}_{r+1}$. As $\mathrm{Ext}^2_X = 0$, the embedding α induces a surjection $\alpha^1 : \mathrm{Ext}^1_X(\mathcal{N}_{r+1}, \mathcal{O}_X) \to \mathrm{Ext}^1_X(\mathcal{N}_r, \mathcal{O}_X)$. Since both spaces here are 1-dimensional, α^1 is an isomorphism. Now, in the sequnce (2.2) the sheaf $\mathcal{G}$ splits into direct sums of the sheaves $\mathcal{N}_i$ with $i \leqslant r$ and the corresponding element $\eta \in \mathrm{Ext}^1_X(\mathcal{G}, \mathcal{O}_X)$ is given by a row with components from $\mathrm{Ext}^1(\mathcal{N}_i, \mathcal{O}_X)$, all of them nonzero. If j is the biggest such that $\mathcal{N}_j$ occur as a direct summand of $\mathcal{G}$, there are morphisms $\alpha_i : \mathcal{N}_j \to \mathcal{N}_i$, which induce isomporphisms $\mathrm{Ext}^1_X(\mathcal{N}_j, \mathcal{O}_X) \to \mathrm{Ext}^1_X(\mathcal{N}_i, \mathcal{O}_X)$ for every other i. Then there is an isomorphism β of $\mathcal{G}$ such that all components of $\eta\beta$ are 0, except the j-th one. Therefore, if $\mathcal{N}_j \neq \mathcal{G}$, or, the same, $j \neq r$, $\mathcal{F}$ decomposes, which accomplishes the proof. □

Let now $\mathcal{F}$ be an indecomposable vector bundle. Find a line bundle $\mathcal{L} \subseteq \mathcal{F}$. If $\mathcal{L}$ is maximal with respect to degree, hence also with respect to inclusion, the factor $\mathcal{F}/\mathcal{L}$ is torsion free, hence also a vector bundle, and $\mathrm{Hom}_X(\mathcal{L}, \mathcal{F}/\mathcal{L}) \simeq \mathrm{Ext}^1_X(\mathcal{F}/\mathcal{L}, \mathcal{L})^* \neq 0$. The same is true for every direct summand of $\mathcal{F}/\mathcal{L}$. Since $\mathrm{Hom}_X(\mathcal{L}', \mathcal{L}'') \neq 0$ if $\mathcal{L}', \mathcal{L}''$ are vector bundles with $\deg \mathcal{L}' \leqslant \deg \mathcal{L}''$, it implies

Proposition 2.7. *Let $\mathcal{F}$ be an indecomposable vector bundle, d_1 be the maximal possible degree of line subbundles of $\mathcal{F}$. Then there is a filtration $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}_r = \mathcal{F}$ such that $\mathcal{F}_i/\mathcal{F}_{i-1} = \mathcal{L}_i$ is a line bundle of degree d_i, $d_1 \leqslant d_2 \leqslant \ldots d_r$ and $\mathrm{Hom}_X(\mathcal{L}_1, \mathcal{L}_i) \neq 0$ for all i. Especially, if $\mathrm{h}^0(X, \mathcal{F}) \neq 0$, also $\mathrm{h}^0(X, \mathcal{L}_i) \neq 0$ for all i.*

We call such a filtration a *maximal line filtration*.

Corollary 2.8. *Let $\mathcal{F}$ be an indecomposable vector bundle with $\mathrm{rk}\, \mathcal{F} = r$, $\deg \mathcal{F} = d$ and $\mathrm{h}^0(X, \mathcal{F}) = h > 0$ (the latter always holds if $d > 0$).*

1. *If $0 \leqslant d < r$, $\mathcal{F}$ has a trivial subsheaf $\mathcal{F}_0 \simeq h\mathcal{O}_X$ such that $\mathcal{F}/\mathcal{F}_0$ is also a vector bundle and $\mathcal{F}_0 = \sum_{f:\mathcal{O}_X \to \mathcal{F}} \mathrm{Im}\, f$.*
2. *If $d \geqslant r$, $\mathcal{F}$ has a line filtration with all $d_i > 0$.*
3. *If $d = r$, $\mathcal{F}$ has a filtration with all factors isomorphic to the same line bundle $\mathcal{L}$ of degree 1.*

Proof. (1). Consider a maximal line filtration. Then $d_1 = 0$. Since $\mathrm{h}^0(\mathcal{L}_1) \geqslant d > 0$, it implies that $\mathcal{L}_1 \simeq \mathcal{O}_X$. Moreover, if $f : \mathcal{O}_X \to \mathcal{F}$ is any nonzero morphism, $\mathcal{F}/\,\mathrm{Im}\, f$ is torsion free: otherwise we could take $\mathcal{L}_1$ strictly bigger than $\operatorname{Im} f$, hence, of positive degree. It means that $f(x) : \Bbbk(x) \to \mathcal{F}(x)$ is nonzero for all $x \in X$. Let $f_1, f_2, \dots, f_h$ be a basis of $\mathrm{H}^0(X, \mathcal{F})$. Then $f_1(x), f_2(x), \dots, f_h(x)$ are linear independent for every $x \in X$, therefore, if $\phi : h\mathcal{O}_X \to \mathcal{F}$ has the components $f_1, f_2, \dots, f_h$, also ϕ is an embedding and $\mathcal{F}/\operatorname{Im}\phi$ is locally free. Obviously, $\operatorname{Im} f \subseteq \operatorname{Im}\phi$ for any $f : \mathcal{O}_X \to \mathcal{F}$.

(2)-(3). Again consider a maximal line filtration. If $d_1 = 0$, the same observations show that $\mathcal{F}$ contains $\mathcal{F}_0 \simeq h\mathcal{O}_X$ and $\mathcal{F}/\mathcal{F}_0$ is again a vector bundle. Since $h \geqslant d = r$, $\mathcal{F} = \mathcal{F}_0$, which is impossible. Therefore, $d_1 \geqslant 1$. If $d = r$, it gives that all $d_i = 1$. Since $\operatorname{Hom}_X(\mathcal{L}_1, \mathcal{L}_i) \neq 0$, it implies that $\mathcal{L}_1 \simeq \mathcal{L}_i$ for all i. □

Corollary 2.9. (1) *For every line bundle $\mathcal{L}$ of degree 0 and every positive r there is a unique (up to isomorphism) indecomposable vector bundle $\mathcal{N}_r(\mathcal{L}) = \mathcal{L} \otimes \mathcal{N}_r$ of rank r and degree 0 such that $\operatorname{Hom}(\mathcal{L}, \mathcal{N}_r(\mathcal{L})) \neq 0$. This vector bundle has a filtration with all factors isomorphic to $\mathcal{L}$.*

(2) *Any indecomposable vector bundle of degree 0 is isomorphic to $\mathcal{N}_r(\mathcal{L})$ for some r and $\mathcal{L}$.*

Proof. (1) is evident, since the functor $\mathcal{L} \otimes {}_-$, where $\mathcal{L}$ is a line bundle, is an auto-equivalence of the category $\mathrm{VB}(X)$.

(2). Let $\mathcal{F}$ be an indecomposable vector bundle of rank r and degree 0. Choose a line bundle $\mathcal{L}_1$ of degree 1. Then $\deg \mathcal{L}_1 \otimes \mathcal{F} = r$, therefore, $\mathcal{L}_1 \otimes \mathcal{F}$ has a filtration with all factors isomorphic to a vector bundle $\mathcal{L}_2$ of degree 1, in particular, $\operatorname{Hom}_X(\mathcal{L}_2, \mathcal{L}_1 \otimes \mathcal{F}) \simeq \operatorname{Hom}_X(\mathcal{L}, \mathcal{F}) \neq 0$, where $\mathcal{L} = \mathcal{L}_1^\vee \otimes \mathcal{L}_2$ is of degree 0. So $\mathcal{F} \simeq \mathcal{N}_r(\mathcal{L})$. □

Theorem 2.10. *If $\mathcal{F}$ is an indecomposable vector bundle of rank r and degree $d > 0$. Then $\mathrm{h}^0(\mathcal{F}) = d$, $\mathrm{h}^1(\mathcal{F}) = 0$ and, if $d < r$, there is an exact sequence $0 \to \mathcal{F}_0 \to \mathcal{F} \to \mathcal{F}_1 \to 0$, where $\mathcal{F}_0 = \sum_{f:\mathcal{O}_X \to \mathcal{F}} \operatorname{Im} f \simeq d\mathcal{O}_X$ and $\mathcal{F}_1$ is also indecomposable (of the same degree and rank $r-d$). Moremeover, $\mathcal{F}_1$ defines $\mathcal{F}$ up to isomorphism and every undecomposable vector bundle of rank $r-d$ and degree d is isomorphic to $\mathcal{F}_1$ for some indecomposable $\mathcal{F}$ of rank r and degree d.*

Proof. If $r = 1$ and $d > 0$, then $\mathrm{h}^0(\mathcal{F}) = d$, so we suppose that the same is true for all vector bundles of rank $< r$ and degree $d > 0$. We use Corollary 2.8. If $r \leqslant d$, $\mathcal{F}$ has a filtrations with factors $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_r$, where all $\mathcal{L}_i$ are line bundles with $\deg \mathcal{L}_i > 0$, hence, $\mathrm{h}^1(\mathcal{L}_i) = 0$. Therefore, $\mathrm{h}^1(\mathcal{F}) = 0$ and $\mathrm{h}^0(\mathcal{F}) = d$. If $d < r$ and $h = \mathrm{h}^0(\mathcal{F})$, there is an exact sequence

$$0 \to \mathcal{F}_0 \to \mathcal{F} \to \mathcal{F}_1 \to 0, \tag{2.3}$$

where $\mathcal{F}_0 \simeq h\mathcal{O}_X$ and $\mathcal{F}_1$ is also a vector bundle. It arises from an element $\xi \in \operatorname{Ext}^1_X(\mathcal{F}_1, h\mathcal{O}_X)$, i.e. from a sequence $(\xi_1, \xi_2, \dots, \xi_h)$, where $\xi_i \in \operatorname{Ext}^1_X(\mathcal{F}_1, \mathcal{O}_X)$. As $\mathcal{F}$ is indecomposable, the elements $\xi_1, \xi_2, \dots, \xi_h$

must be linear independent: otherwise one can suppose that one of them is 0, so $\mathcal{F} \simeq \mathcal{F}' \oplus \mathcal{O}_X$. It means that the map $\eta : \Bbbk^h \simeq \operatorname{Hom}_X(h\mathcal{O}_X, \mathcal{O}_X) \to \operatorname{Ext}^1_X(\mathcal{F}_1, \mathcal{O}_X)$ induced by the sequence (2.3) is a monomorphism, since it maps a vector $(\lambda_1, \lambda_2, \dots, \lambda_h)$ to $\sum_{i=1}^h \lambda_i \xi_i$. So $\mathrm{h}^1(\mathcal{F}_1^\vee) = \mathrm{h}^0(\mathcal{F}_1) \geqslant h$. On the other hand, $\mathrm{h}^1(\mathcal{F}_1) \leqslant \mathrm{h}^1(\mathcal{F})$ and $\deg \mathcal{F}_1 = \deg \mathcal{F}$, since $\deg \mathcal{O}_X = 0$. Therefore $\mathrm{h}^i(\mathcal{F}_1) = \mathrm{h}^i(\mathcal{F})$ $(i = 0, 1)$ and η is an isomorphism. If ξ' is another element of $\operatorname{Ext}^1_X(\mathcal{F}_1, h\mathcal{O}_X)$ such that the corresponding extension $\mathcal{F}'$ of $\mathcal{F}_1$ with the kernel $h\mathcal{O}_X$ is indecomposable, there is an automorphism α of $h\mathcal{O}_X$ such that $\alpha\xi = \xi'$, wherefrom $\mathcal{F}' \simeq \mathcal{F}$.

If $\mathcal{F}_1 = \mathcal{F}_1^1 \oplus \mathcal{F}_1^2$, we can choose a decomposition $h\mathcal{O}_X \simeq h_1\mathcal{O}_X \oplus h_2\mathcal{O}_X$ such that ξ is presented by the matrix $\begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix}$, where $\xi_k \in \operatorname{Ext}^1_X(\mathcal{F}_1^k, h_k\mathcal{O}_X)$. Then $\mathcal{F} \simeq \mathcal{F}^1 \oplus \mathcal{F}^2$, where $\mathcal{F}_k$ is defined by ξ_k. Therefore, $\mathcal{F}_1$ is indecomposable. As $\operatorname{rk} \mathcal{F}_1 < \operatorname{rk} \mathcal{F}$ and $\deg \mathcal{F}_1 = d$, we get that $h = \mathrm{h}^0(\mathcal{F}_1) = d$.

If $\mathcal{G}$ is any vector bundle of degree d and rank $r-d$, we already know that $\dim \operatorname{Ext}^1_X(\mathcal{G}, \mathcal{O}_X) = \mathrm{h}^0(\mathcal{G}) = d$. Choose a basis $\eta_1, \eta_2, \dots, \eta_d$ of this space and consider the exact sequence $0 \to d\mathcal{O}_X \xrightarrow{\iota} \tilde{\mathcal{G}} \to \tilde{\mathcal{G}} \to 0$ defined by the element $(\eta_1, \eta_2, \dots, \eta_d) \in \operatorname{Ext}^1_X(\mathcal{G}, d\mathcal{O}_X) \simeq d \operatorname{Ext}^1_X(\mathcal{G}, \mathcal{O}_X)$. Then the homomorphism $\operatorname{Hom}_X(d\mathcal{O}_X, \mathcal{O}_X) \to \operatorname{Ext}^1_X(\mathcal{G}, \mathcal{O}_X)$ arising from this sequence is bijective, $\operatorname{Hom}_X(\mathcal{G}, \mathcal{O}_X) \simeq \mathrm{H}^1(\mathcal{G})^* = 0$, therefore, $\operatorname{Hom}_X(\tilde{\mathcal{G}}, \mathcal{O}_X) = 0$ and $\mathrm{h}^0(\tilde{\mathcal{G}}) = \dim \operatorname{Ext}^1_X(\tilde{\mathcal{G}}, \mathcal{O}_X) = d$ too. If $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}' \oplus \tilde{\mathcal{G}}''$, we may suppose that $\tilde{\mathcal{G}}'$ is indecomposable of positive degree d', hence $\mathrm{h}^0(\tilde{\mathcal{G}}') = d'$ and $\mathrm{h}^0(\tilde{\mathcal{G}}'') = d - d' = \deg \tilde{\mathcal{G}}''$. Therefore, the morphism ι splits as $\iota' \oplus \iota''$, where $\iota' : d'\mathcal{O}_X \to \tilde{\mathcal{G}}'$, $\iota'' : d''\mathcal{O}_X \to \tilde{\mathcal{G}}''$ and $\mathcal{G} \simeq \mathcal{G}' \oplus \mathcal{G}''$, where $\mathcal{G}' = \operatorname{Coker} \iota'$, $\mathcal{G}'' = \operatorname{Coker} \iota''$. This contradiction accomplishes the proof. □

Note that $\operatorname{rk}(\mathcal{F}_1) = r - d$, thus $\mu(\mathcal{F}_1) = d/(r-d) = \mu/(1-\mu)$, where $\mu = \mu(\mathcal{F}) = d/r$.

Corollary 2.11. *If $0 < \mu < 1$, there is an equivalence of categories $\alpha_\mu : \operatorname{ind} \mathrm{VB}_\mu \to \operatorname{ind} \mathrm{VB}_{\mu'}$, where $\mu' = \mu/(1-\mu)$.*

Proof. We only have to construct $\alpha = \alpha_\mu$ on indecomposable vector bundles. Let $\mathcal{F} \in \mathrm{VB}_\mu$ be indecomposable, $r = \operatorname{rk} \mathcal{F}$, $d = \deg \mathcal{F}$. We use the exact sequence (2.3). Recall that $\mathrm{h}^0(\mathcal{F}) = \mathrm{h}^0(\mathcal{F}_1) = d$ and the element $\xi \in \operatorname{Ext}^1_X(\mathcal{F}_1, \mathcal{F}_0) \simeq d \operatorname{Ext}^1_X(\mathcal{F}_1, \mathcal{O}_X)$ defining the sequence (2.3) is presented by a sequence $(\xi_1, \xi_2, \dots, \xi_d)$, where $\xi_1, \xi_2, \dots, \xi_d$ is a basis of $\operatorname{Ext}^1_X(\mathcal{F}_1, \mathcal{O}_X)$. Let $\mathcal{F}'$ be another indecomposable sheaf from VB_μ of rank r' and degree d', $0 \to \mathcal{F}'_0 \to \mathcal{F}' \to \mathcal{F}'_1 \to 0$ be the corresponding exact sequnce defined by an element $\xi' \in \operatorname{Ext}^1_X(\mathcal{F}'_1, \mathcal{O}_X)$. Every homomorphism $\mathcal{F} \to \mathcal{F}'$ maps $\mathcal{F}_0$ to $\mathcal{F}'_0$, so induces a homomorphism $\mathcal{F}_1 \to \mathcal{F}'_1$. Thus we obtain a functor $\alpha : \mathrm{VB}_\mu \to \mathrm{VB}_{\mu'}$.

We will construct an inverse functor $\beta : \mathrm{VB}_{\mu'} \to \mathrm{VB}_\mu$. Namely, we know that every indecomposable vector bundle from $\mathrm{VB}_{\mu'}$ occur as $\mathcal{F}_1$ in a unique sequence if type (2.3). Set $\beta(\mathcal{F}_1) = \mathcal{F}$. Let $\mathcal{F}' = \beta(\mathcal{F}_1')$ and $f_1 : \mathcal{F}_1 \to \mathcal{F}_1'$ be any homomorphism. Consider the corresponding elements ξ and ξ' of Ext-spaces, as above. Since the components of ξ form a basis of $\mathrm{Ext}^1_X(\mathcal{F}_1, \mathcal{O}_X)$, there is a unique homomorphism $f_0 : \mathcal{F}_0 \simeq d\mathcal{O}_X \to \mathcal{F}_0' \simeq d'\mathcal{O}_X$ such that $f_0\xi = \xi' f_1$. It means that f_0 and f_1 arise from a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}_0 \arrow[r] \arrow[d, "f_0"'] & \mathcal{F} \arrow[r] \arrow[d, "f"'] & \mathcal{F}_1 \arrow[r] \arrow[d, "f_1"] & 0 \\ 0 \arrow[r] & \mathcal{F}_0' \arrow[r] & \mathcal{F}' \arrow[r] & \mathcal{F}_1' \arrow[r] & 0 \end{tikzcd}$$

for a unique f. Setting $\beta(f_1) = f$, we accomplish the construction of the functor β. It is obvious that β is indeed inverse to α. $\square$

Corollary 2.12. *For every μ there is an equivalence of categories $\phi_\mu : \mathrm{VB}_\mu \to \mathrm{VB}_0$. Moreover, if $\operatorname{rk}\mathcal{F} = r$ and $\deg\mathcal{F} = d$, then $\operatorname{rk}\phi_\mu(\mathcal{F}) = \gcd(r, d)$.*

Proof. For $\mu = 0$ it is Corollary 2.9. Let $\mu = d/r$, where $\gcd(r, d) = 1$. If $q = \lfloor\mu\rfloor$ and $\mathcal{L} \in \operatorname{ind}(1, -q)$, the functor $\mathcal{L} \otimes \_$ induces an equivalence $\mathrm{VB}_\mu \to \mathrm{VB}_{\mu-q}$, $\mu - q = (d-qr)/r$, $0 \leqslant d-qr < d$ and $\gcd(r, d-qr) = 1$. If $0 < \mu < 1$ use Corollary 2.11 and note that $\mu/(1-\mu) = d/(r-d)$, $r-d < r$ and $\gcd(r, d-r) = 1$. Itereating these steps (in fact, following the Euclidean algorithm for r, d), we get the result. $\square$

Corollary 2.13. *An indecomposable vector bundle $\mathcal{F}$ of rank r and degree d is stable if and only if $\gcd(r, d) = 1$. Then the stable vector bundles of rank r and degree d are in one-to-one correspondence with $\mathrm{Pic}_0 X \simeq X$.*

Proof. Corollary 2.12 implies that we only have to consider the case $\mu(\mathcal{F}) = 0$. Then the claim follows from Theorem 2.6 and Corollary 2.9. $\square$

Exercise 2.14.
(1) Deduce from the construction of the vector bundles $\mathcal{N}_r$ in Theorem 2.7 that there are natural isomorphisms $\mathrm{Hom}_X(\mathcal{N}_r, \mathcal{N}_m) \simeq \mathrm{Hom}_{\Bbbk[t]}(\Bbbk[t]/t^r, \Bbbk[t]/t^m)$.
(2) Prove that $\mathrm{Hom}_X(\mathcal{L} \otimes \mathcal{N}_r, \mathcal{L}' \otimes \mathcal{N}_m) = 0$ if $\mathcal{L}$ and $\mathcal{L}'$ are non-isomorphic line bundles of degree 0.
(3) Deduce from (1) and (2) that VB_0 is equivalent to the category VB_∞ of skyscraper sheaves over X.

Oda [22] has shown that these constructions can be performed in such a way that we get indeed *families* of indecomposable vector bundles of any fixed rank and degree parametrized by the points of X. Here is his result (we do not present its proof).

We denote by nx the closed subscheme of X defined by the sheaf of ideals $\mathcal{O}_X(-nx)$ and by i_{nx} the embedding $X \times nx \to X \times X$.

Theorem 2.15. *For every pair of coprime integers (r, d) with $r > 0$ there is a vector bundle $\mathcal{P}_{r,d}$ over $X \times X$ such that every indecomposable vector bundle over X of rank nr and degree nd, where n is a positive integer, is isomorphic to $p_{1*}i^*_{nx}\mathcal{P}_{r,d}$ for a uniquely determined point $x \in X$. Moreover, $\mathcal{P}_{r,d+mr} \simeq \mathcal{P}_{r,d}(m(o \times X))$ and $\mathcal{P}_{1,0} \simeq \mathcal{P}$.*

3. Sandwich procedure

From now on, we suppos that X is a *singular curve* (reduced and connected, but maybe reducible). Let $S = \operatorname{Sing} X$ be the set of singular point of X, $\pi : \tilde{X} \to X$ be the *normalization* of X, i.e. birational finite morphism such that $\tilde{X}$ is smooth (recall that it is defined up to isomorphism), $\widetilde{S} = \pi^{-1}(S)$. We denote $\mathcal{O} = \mathcal{O}_X$, $\widetilde{\mathcal{O}} = \pi_*(\mathcal{O}_{\tilde{X}})$. Then $\widetilde{\mathcal{O}}$ is a coherent sheaf of rings over X and $\mathcal{O}$ is identified with its subsheaf. Moreover, $\widetilde{\mathcal{O}}/\mathcal{O}$ is a skyscraper sheaf of $\mathcal{O}$-modules supported by the finite set S. Let $\mathcal{J} = \operatorname{Ann}(\widetilde{\mathcal{O}}/\mathcal{O})$, the *conductor* of $\widetilde{\mathcal{O}}$ in $\mathcal{O}$ (the maximal sheaf of $\widetilde{\mathcal{O}}$-ideals contained in $\mathcal{O}$). Given any vector bundle $\mathcal{F}$ over X, we can consider the sheaf $\widetilde{\mathcal{F}} = \widetilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{F} \simeq \pi_*\pi^*\mathcal{F}$. Obviously, $\widetilde{\mathcal{F}} \supseteq \mathcal{F} \supseteq \mathcal{J}\mathcal{F} = \mathcal{J}\widetilde{\mathcal{F}}$. Moreover, $\widetilde{\mathcal{F}}/\mathcal{J}\widetilde{\mathcal{F}}$ and $\mathcal{F}/\mathcal{J}\mathcal{F}$ are skyscraper sheaves supported by S, so they are uniquely defined by their stalks at the points $x \in S$. Let $\mathsf{A}_x = (\mathcal{O}/\mathcal{J})_x$ and $\mathsf{B}_x = (\widetilde{\mathcal{O}}/\mathcal{J})_x$. They are finite dimensional commutative $\Bbbk$-algebras and $\mathsf{A}_x \subseteq \mathsf{B}_x$. Set also $\mathsf{F}_x = (\mathcal{F}/\mathcal{J}\mathcal{F})_x$, $\widetilde{\mathsf{F}}_x = (\widetilde{\mathcal{F}}/\mathcal{J}\widetilde{\mathcal{F}})_x$. They are free modules of rank $r = \operatorname{rk}\mathcal{F}$ over the algebras, respectively, A_x and B_x, and $\mathsf{F}_x \subseteq \widetilde{\mathsf{F}}_x$. In what follows, it is convenient to identify the sheaves $\mathcal{F}/\mathcal{J}\mathcal{F}$ and $\widetilde{\mathcal{F}}/\mathcal{J}\widetilde{\mathcal{F}}$ with the modules $\mathsf{F} = \bigoplus_{x\in S}\mathsf{F}_x$ and $\widetilde{\mathsf{F}} = \bigoplus_{x\in S}\widetilde{\mathsf{F}}_x$ over the algebras, respectively, $\mathsf{A} = \bigoplus_{x\in S}\mathsf{A}_x$ and $\mathsf{B} = \bigoplus_{x\in S}\mathsf{B}_x$. Thus, any vector bundle $\mathcal{F}$ over X defines the triple $T(\mathcal{F}) = (\widetilde{\mathcal{F}}, \mathsf{F}, \iota_{\mathcal{F}})$, where $\iota_{\mathcal{F}} : \mathsf{F} \to \widetilde{\mathsf{F}}$ is the natural embedding.

Note that, since π is finite and birational, the kernel of the natural morphism $\pi^*\pi_*\mathcal{O}_{\tilde{X}} \to \mathcal{O}_{\tilde{X}}$ is a skyscraper sheaf (zero outside S). Hence, for every coherent sheaf of $\mathcal{O}_{\tilde{X}}$-modules $\mathcal{H}$, the kernel of the map $\pi^*\pi_*\mathcal{H} \to \mathcal{H}$ is skyscraper too, wherefrom, for every torsion free sheaf of $\mathcal{O}_{\tilde{X}}$-modules

$$\operatorname{Hom}_{\tilde{X}}(\mathcal{H}, \mathcal{H}') \simeq \operatorname{Hom}_{\tilde{X}}(\pi^*\pi_*\mathcal{H}, \mathcal{H}') \simeq \operatorname{Hom}_X(\pi_*\mathcal{H}, \pi_*\mathcal{H}').$$

Especially, π_* induces an equivalence between the category of vector sheaves of constant rank over $\tilde{X}$ and that of locally free coherent sheaves of $\widetilde{\mathcal{O}}$-modules. Thus, in what follows, we always identify these two categtories. In particular, we identify the sheaf $\widetilde{\mathcal{F}}$ above with $\pi^*\mathcal{F}$.

Now we define the *sandwich* category (or the *category of triples*) $\mathscr{T}(X)$ as follows:

- The *objects* of $\mathscr{T}(X)$ are triples $(\mathcal{G}, \mathsf{F}, \iota)$, where $\mathcal{G}$ is a locally free sheaf of $\widetilde{\mathcal{O}}$-modules, or, the same, a vector bundle of constant rank r over $\tilde{X}$, F is a free A-module of rank r and ι is an embedding $\mathsf{F} \to \mathsf{G}$, where $\mathsf{G} = \mathcal{G}/\mathcal{J}\mathcal{G}$ (considered as A-module), such that $\operatorname{Im}\iota$ generates G as B-module.
- A *morphism* $f : (\mathcal{G}, \mathsf{F}, \iota) \to (\mathcal{G}', \mathsf{F}', \iota')$ is a pair (f_l, f_r), where $f_l \in \operatorname{Hom}_{\widetilde{\mathcal{O}}}(\mathcal{G}, \mathcal{G}')$, $f_r \in \operatorname{Hom}_{\mathsf{A}}(\mathsf{F}, \mathsf{F}')$, such that the diagram

$$\begin{array}{ccc} \mathsf{F} & \xrightarrow{\iota} & \mathsf{G} \\ {\scriptstyle f_r}\downarrow & & \downarrow{\scriptstyle \bar{f}_l} \\ \mathsf{F}' & \xrightarrow[\iota']{} & \mathsf{G}' \end{array} \tag{3.1}$$

commutes, where $\bar{f}_r$ is induced by f_r.

Theorem 3.1 (Sandwich Theorem). *Mapping $\mathcal{F} \mapsto T(\mathcal{F})$ induces an equivalence of categories* $\mathrm{VB}(X) \to \mathscr{T}(X)$.

Proof. Obviously, the triple $T(\mathcal{F}) = (\widetilde{\mathcal{F}}, \mathsf{F}, \iota_{\mathcal{F}})$ belongs to $\mathscr{T}(X)$. If $f \in \operatorname{Hom}_X(\mathcal{F}, \mathcal{F}')$, it induces morphisms $\widetilde{f} : \widetilde{\mathcal{F}} \to \widetilde{\mathcal{F}}'$ and $\bar{f} : \mathsf{F} \to \mathsf{F}'$ so that the corresponding diagram of type (3.1) commutes, therefore the pair $T(f) = (\widetilde{f}, \bar{f})$ is a morphism $T(\mathcal{F}) \to T(\mathcal{F}')$. Hence, T is a functor $\mathrm{VB}(X) \to \mathscr{T}(X)$. We are going to construct an inverse functor $R : \mathscr{T}(X) \to \mathrm{VB}(X)$.

Let $\mathsf{T} = (\mathcal{G}, \mathsf{F}, \iota)$ be a triple from $\mathscr{T}(X)$ and $\operatorname{rk}\mathcal{G} = r$. Consider the preimage $\mathcal{F}$ of $\operatorname{Im}\iota \simeq \mathsf{F}$ in $\mathcal{G}$. Then $\widetilde{\mathcal{O}}\mathcal{F} = \mathcal{G}$, hence, $\mathcal{J}\mathcal{F} = \mathcal{J}\mathcal{G}$ and the natural map $p : \widetilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{G}$ is an epimorphism. Consider a stalk $\mathcal{F}_x$. If $x \notin S$, it coinsides with $\mathcal{G}_x$, so is free of rank r over $\mathcal{O}_x = \widetilde{\mathcal{O}}_x$. If $x \in S$, there is an epimorphism $r\mathcal{O}_x \to \mathsf{F}_x = \mathcal{F}_x/\mathcal{J}_x\mathcal{F}_x$. Since $\mathcal{J}_x \subseteq \operatorname{rad}\mathcal{O}_x$, it lifts to an epimorphism $\phi : r\mathcal{O}_x \to \mathcal{F}_x$. The latter induces an epimorphism $\widetilde{\phi} : r\widetilde{\mathcal{O}}_x \to \widetilde{\mathcal{O}}_x \otimes_{\mathcal{O}_x} \mathcal{F}_x$. Combined with p, it gives an epimorphism $p_x\widetilde{\phi} : r\widetilde{\mathcal{O}}_x \to \mathcal{G}_x$. Since both modules here are free of the same rank, it is an isomorphism. Since $\widetilde{\phi}$ is surjective, both $\widetilde{\phi}$ and p_x are actually isomorphisms. Therefore, ϕ is injective, hence, an isomorphism too, and $\mathcal{F}$ is locally free of rank r. Set $R(\mathsf{T}) = \mathcal{F}$. If $f = (f_l, f_r)$ is a morphism $\mathsf{T} \to \mathsf{T}' = (\mathcal{G}', \mathsf{F}', \iota')$, it obviously induces a morphism $R(f) : \mathcal{F} \to \mathcal{F}'$, where $\mathcal{F}'$ is the preimage of $\operatorname{Im}\iota'$ in $\pi_*\mathcal{G}'$. So get a functor $R : \mathscr{T}(X) \to \mathrm{VB}(X)$. Obviously, $RT(\mathcal{F}) \simeq \mathcal{F}$. Moreover, the isomorphism $p : \widetilde{\mathcal{F}} \to \mathcal{G}$ above induces a functorial isomorphism of the triples $TR(\mathsf{T}) = T(\mathcal{F}) = (\widetilde{\mathcal{F}}, \mathsf{F}, \iota_{\mathcal{F}}) \simeq \mathsf{T}$, so R is indeed inverse to T. $\square$

4. Bimodule categories

The sandwich procedure of the preceeding section can be reformulated in terms of *bimodule categories*, usual and widely used in the representation theory of algebras. Recall the corresponding deinitions. We consider modules and bimodules over categories rathcker than over rings. We always suppose a considered category $\mathscr{A}$ *$\Bbbk$-linear* and *fully additive*. The first notion means that all sets of morphisms $\mathscr{A}(A, A')$ are vector spaces over $\Bbbk$, while the multiplication is $\Bbbk$-bilinear. The second one means that this category is *additive*, i.e. contains all finite direct sums, moreover, every idempotent morphsim $e : A \to A,\ e^2 = e$ in it *splits*, i.e. comes from a direct sum decomposition $A \simeq A_1 \oplus A_2$ as $e = i_1 p_1$, where $i_1 : A_1 \to A$ is the canonical injection and $p_1 : A \to A_1$ is the canonical projection. All functors are supposed to be $\Bbbk$-linear.

An *$\mathscr{A}$-module* is, by definition a ($\Bbbk$-linear) functor $\mathscr{M} : \mathscr{A} \to \mathsf{Vec}$, the category of vector spaces over $\Bbbk$. Such modules form a ($\Bbbk$-linear) category $\mathscr{A}$-Mod. If $\mathscr{M}$ is an $\mathscr{A}$-module, $v \in \mathscr{M}(A)$ and $a \in \mathscr{A}(A, A')$, we write av instead of $\mathscr{M}(a)v$; it is an element from $\mathscr{M}(A')$. If $\mathscr{B}$ is another category, an *$\mathscr{A}$-$\mathscr{B}$-bimodule* is, by definition, a ($\Bbbk$-bilinear) functor $\mathscr{W} : \mathscr{A}^{\mathrm{op}} \times \mathscr{B} \to \mathsf{Vec}$, where $\mathscr{A}^{\mathrm{op}}$ denotes the *dual category* to $\mathscr{A}$. Again, if $v \in \mathscr{W}(A, B)$, $a \in \mathscr{A}(A', A)$, $b \in \mathscr{B}(B, B')$, we write bva instead of $B(a, b)v$; it is an element from $\mathscr{W}(A', B')$.

Let now $\mathscr{W}$ be an $\mathscr{A}$-bimodule (i.e. an $\mathscr{A}$-$\mathscr{A}$-bimodule). The *bimodule category* $\mathsf{El}(\mathscr{V})$ is defined as follows.

- $\operatorname{Ob} \mathsf{El}(\mathscr{V}) = \bigcup_{A \in \operatorname{Ob} \mathscr{A}} \mathscr{V}(A, A)$.
- A *morphism* $f : v \to v'$, where $v \in \mathscr{V}(A, A)$, $v' \in \mathscr{V}(A', A')$, is a morphism $a : A \to A'$ such that $av = v'a$ (note that both elements belong to $\mathscr{V}(A, A')$).
- The product coincide with the product in $\mathscr{A}$.

One easily verifies that $\mathsf{El}(\mathscr{V})$ is indeed a $\Bbbk$-linear, fully additive category.

We usually deal with the situation, when both the category $\mathscr{A}$ and the bimodule $\mathscr{W}$ are *locally finite dimensional* (*lofd*), i.e. all vector spaces $\mathscr{A}(A, A')$, and $\mathscr{W}(A, B)$ are finite dimensional (over $\Bbbk$). Since we supposed $\mathscr{A}$ fully additive, it implies that every object $A \in \mathscr{A}$ decomposes into a finite direct sum $A \simeq \bigoplus_{k=1}^{n} A_k$, where all objects $A_1, A_2, \dots, A_n$ are *local*, i.e. their endomorphism algebras $\mathscr{A}(A_k, A_k)$ contain no idempotents, hence, are local [14]. Such decomposition is unique (up to isomorphisms of summands and their permutations) [15]. (For instance, it is the case, when $\mathscr{A} = \operatorname{Coh}(X)$, the category of coherent sheaves over a projective variety X, or its subcategory of vector bundles $\operatorname{VB}(X)$.) Then we denote by $\operatorname{ind} \mathscr{A}$ a (fixed) set of representatives of isomorphism classes of indecomposable objects from $\mathscr{A}$, as well as a full subcategory of $\mathscr{A}$ having $\operatorname{ind} \mathscr{A}$ as the set of objects. The bimodule $\mathscr{W}$ is completely defined (up to isomorphism) by its restriction onto $\operatorname{ind} \mathscr{A}$ and $\operatorname{ind} \mathscr{B}$. Namely, let $A = \bigcup_{j=1}^{t} n_j A_j$, where $A_j \in \operatorname{ind} \mathscr{A}$, $A_k \neq A_j$ if $k \neq j$, $B = \bigcup_{i=1}^{s} m_i B_i$ be the analogous decomposition of B. Then $\mathscr{W}(A, B)$ is identified with the set of block matrices (W_{ij}), where W_{ij} is an $m_i \times n_j$ matrix with the elements from $\mathscr{W}(A_j, B_i)$. Analogous identifications can be done for morphisms between the objects of $\mathscr{A}$ so that their multiplication as well as their action on $\mathscr{W}$ correspond to the usual multiplication of matrices. We will use such identifications all the time. That is why a bimodule category is sometimes called the *category of matrices over a bimodule*.

Perhaps, the most important case for applications is that of *bipartite* bimodules. It arises as follows. Every $\mathscr{A}$-$\mathscr{B}$-bimodule $\mathscr{W}$ can be considered as a bimodule over the direct product $\mathscr{A} \times \mathscr{B}$: we set $\mathscr{W}((A, B), (A', B')) = \mathscr{W}(A, B')$. If we do so, the objects from $\mathsf{El}(\mathscr{W})$ are just elements from the spaces $\mathscr{W}(A, B)$ ($A \in \operatorname{Ob}\mathscr{A}$, $B \in \operatorname{Ob}\mathscr{B}$), while a morphism $f : v \to v'$, where $v \in \mathscr{V}(A, B)$, $v' \in \mathscr{V}(A', B')$, is a pair of morphisms (a, b), $a : A \to A'$, $b : B \to B'$ such that $bv = v'a$. As we have mentioned, usually the bimodules arising from the theory of representations, vector bundles and other "external" topics are bipartite. Nevertheless, in most calculations the general case cannot be avoided (we shall see examples below).

Consider now the situation of Section 3. Let $\mathscr{A} = \mathsf{A}$-pro be the category of finitely generated projective A-modules (i.e. direct summands of free modules of finite rank), $\mathscr{B} = \mathrm{VB}(\tilde{X})$ and $\mathscr{W}(\mathsf{F}, \mathcal{G}) = \operatorname{Hom}_{\mathsf{A}}(\mathsf{F}, \mathcal{G}/\mathcal{J}\mathcal{G})$ (as before, we identify $\mathcal{G}$ with the sheaf of $\widetilde{\mathcal{O}}$-modules $\pi_*\mathcal{G}$). Thus $\mathscr{W} = \mathscr{W}_X$ is a bipartite $\mathscr{A}$-$\mathscr{B}$-bimodule. Every triple $\mathsf{T} = (\mathcal{G}, \mathsf{F}, \iota)$ can be identified with the object $\iota \in \mathscr{W}(\mathsf{F}, \mathcal{G})$ of $\mathsf{El}(\mathscr{W})$. Moreover, this identification defines a full embedding $\mathcal{T}(X) \to \mathsf{El}(\mathscr{W})$. We always identify $\mathcal{T}(X)$ with the image of this embedding and actually work with the elements of the bimodule $\mathscr{W}$ presented as matrices. The objects from $\mathsf{El}(\mathscr{W})$ isomorphic to the image of a triple from $\mathscr{T}$ will be called *strict*.

Example 4.1 (Projective Configurations). We call X a *projective configuration* if all components $X_1, X_2, \dots, X_s$ of $\tilde{X}$ are isomorphic to $\mathbb{P}^1$, while all singular points $x \in S$ are *simple nodes*, i.e. $\pi^{-1}(x) = \{\, x', x'' \,\}$, a reduced subvariety of $\tilde{X}$ consisting of two points. Then $\mathcal{J}_x = \mathfrak{m}_x$, the maximal ideal of $\mathcal{O}_x$, $\mathsf{A}_x = \Bbbk(x)$ and $\mathsf{B}_x = \Bbbk(x') \times \Bbbk(x'')$, where $\Bbbk(x)$ is embedded diagonally. Thus $\operatorname{ind}\mathscr{A} = \{\, \Bbbk(x) \mid x \in S \,\}$. On the other hand, $\operatorname{ind}\mathscr{B} = \{\, \mathcal{O}_i(d) \mid 1 \leqslant i \leqslant s,\ d \in \mathbb{Z} \,\}$, where $\mathcal{O}_i = \mathcal{O}_{X_i}$. Moreover, $(\mathcal{O}_i(d)/\mathcal{J}\mathcal{O}_i(d))_x \simeq \Bbbk(x') \times \Bbbk(x'')$ as well, since a shift does not affect skyscraper sheaves. Therefore, elements of $\mathscr{W}$ are presented as block matrices W with blocks $W(i, d, y)$, where $1 \leqslant i \leqslant s$, $d \in \mathbb{Z}$ and $y \in X_i \cap \pi^{-1}(S)$. It is convenient to suppose that all preimages from $\pi^{-1}(S)$ lie in the affine part $U_0 = \{\, (\lambda : \mu) \mid \lambda \neq 0 \,\}$ of $\mathbb{P}^1$, so can be identified with the points of $\mathbb{A}^1$.

There are no nonzero morphisms between different objects of $\mathscr{A}$, and $\operatorname{End}(\Bbbk(x)) = \Bbbk$. Let $x' \in X_{i_1}$, $x'' \in X_{i_2}$ (maybe $i_1 = i_2$, but $x' \neq x''$), $A = m\Bbbk(x)$, $A' = n\Bbbk(x)$ and $a : A' \to A$ be presented by an $m \times n$

matrix $\alpha = (a_{kl})$. Let also $W \in \mathsf{El}(\mathscr{W})$. Then Wa is obtained from W by replacing the matrices $W(i_1, d, x')$ and $W(i_2, d, x'')$ for all values of d, respectively, by $W(i_1, d, x')\alpha$ and $W(i_2, d, x'')\alpha$.

On the other hand, there are no morphisms $\mathcal{O}_i(d') \to \mathcal{O}_j(d)$ for $i \neq j$ or $d < d'$ while $\mathrm{Hom}(\mathcal{O}_i(d'), \mathcal{O}_i(d)) = \Bbbk[t]_{d-d'}$ if $d' \leqslant d$ (see Proposition 1.3). Note that $\mathcal{O}_i(d)/\mathfrak{m}_y\mathcal{O}_i(d) \simeq \mathcal{O}_i/\mathfrak{m}_y = \Bbbk(y)$. If $y \in U_0$, the morphism $\mathcal{O}_i(d') \to \mathcal{O}_i(d)$ given by a polynomial $f(x)$ induces the multiplication by $f(y)$ on these factors. Let $B = \bigoplus_d n_d\mathcal{O}_i(d)$, $B' = \bigoplus_d m_d\mathcal{O}_i(d)$ and $b : B \to B'$ be presented by a block matrix $(\beta_{dd'})$, where $\beta_{dd'}$ is an $m_d \times n_{d'}$ matrix with elements from $\Bbbk[t]_{d-d'}$ (zero if $d < d'$). Then bW is obtained from W by replacing each $W(i, d, y)$ (for all $y \in X_i$) by $\sum_{d' \leqslant d} \beta_{dd'}(y)W(i, d', y)$.

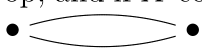
Recall that a homomorphism ι in a triple $(\mathcal{G}, \mathsf{F}, \iota)$ from $\mathcal{T}(X)$ must be a monomorphism and such that $\mathrm{Im}\,\iota$ generates G as B-module. We propose to the reader to prove that these conditions hold if and only if for every point $y \in \widetilde{S}$ the "*big y-block*"

$$W(y) = \begin{pmatrix} \vdots \\ W(i, -1, y) \\ W(i, 0, y) \\ W(i, 1, y) \\ \vdots \end{pmatrix},$$

where $y \in X_i$, is invertible. If ι is not a monomorphism, but still $\mathrm{Im}\,\iota$ generates G, its preimage in $\mathcal{G}$ is no more locally free, but it remains torsion free and all torsion free coherent sheaves of $\mathcal{O}_X$-modules are obtained in this way (check it!). For the corresponding set of matrices it means that the row of all big y -blocks are linear independent.

We will use the calculations of this example in the next sections. Given a projective configuration X, we define its *intersection graph* $\Delta(X)$ as follows:

- The *vertices* of $\Delta(X)$ are the components $X_1, X_2, \ldots, X_s$ of $\tilde{X}$.
- The *edges* of $\Delta(X)$ are the singular points $x \in S$.
- An edge x is incident to a vertex X_i if and only if $\pi^{-1}(x) \cap X_i \neq \emptyset$; especially, if $\pi^{-1}(x) \subset X_i$ for some i, the edge x is actually a *loop* at the vertex X_i.

For instance, if X is a non-degenerate plane quadric, it consists of two projective lines with a transversal intersection. Hence, $\Delta(X)$ has two points and one edge joining them. If X is a nodal plane cubic given (in an affine part of the projective plane) by the equation $y^2 = x^3 + x^2$, $\Delta(X)$ consists of one vertex and one loop, and if X consist of a quadric and a line not tangent to it, $\Delta(X)$ is  .

5. Bunches of chains

We consider a special class of bimodule problems, the so called *bunches of chains* (see, for instance, [9], or [12, Appendix B]). They will be used in the next two sections, where we describe vector bundles over two types of projective configurations. Bunches of chains also arise in many other problem, so we believe that the acquaintence with them can be useful for any working mathematician. Nevertheless, if a reader does not want to deal with technical details, we propose him either to understand at least the results (omitting the proofs) or just to take on trust the descriptions of the next sections.

A *bunch of chains* is given by $2n$ disjoint chains (linear ordered sets without common elements) $\mathfrak{E}_1, \mathfrak{E}_2, \dots, \mathfrak{E}_n$ and $\mathfrak{F}_1, \mathfrak{F}_2, \dots, \mathfrak{F}_n$ and an equivalence relation $\sim$ on the union $\mathfrak{A} = \mathfrak{E} \cup \mathfrak{F}$, where $\mathfrak{E} = \bigcup i = 1^n \mathfrak{E}_i$, $\mathfrak{F} = \bigcup i = 1^n \mathfrak{F}_i$, such that the set $\{\, b \in \mathfrak{A} \mid a \sim b \,\}$ has at most 2 elements for every $a \in \mathfrak{A}$. We call elements of $\mathfrak{A}$ *letters* and $\mathfrak{A}$ the *alphabet.* We also define a symmetric relation $-$ on $\mathfrak{A}$ setting $e - f$ and $f - e$ if $e \in \mathfrak{E}_i,\ f \in \mathfrak{F}_i$ (with the same i). Let $\mathfrak{B} = \mathfrak{A}/\sim$, the set of equivalence classes of letters. We define the category $\mathscr{A}_0$ and the $\mathscr{A}_0$-bimodule $\mathscr{W}$ as follows.

- $\operatorname{Ob} \mathscr{A} = \mathfrak{B}$.
- For every pair elements a, b from the same chain such that $a \leqslant b$ choose a new symbol α_{ba}.
- If $\bar{a}, \bar{b} \in \mathfrak{B}$, the vector space $\mathscr{A}(\bar{a}, \bar{b})$ has a base
  $$\left\{\, \alpha_{ba} \mid a \in \bar{a},\ b \in \bar{b},\ a < b \,\right\}$$
  whenever $\bar{a} \neq \bar{b}$; if $\bar{a} = \bar{b}$, we also add to this base the unit morphism $1_{\bar{a}}$.
- $\alpha_{cb}\alpha_{ba} = \alpha_{ca}$; all other possible products of the base elements are zero.
- For every pair $e \in \mathfrak{E},\ f \in \mathfrak{F}$ such that $e - f$, we choose a new symbol γ_{ef}.
- The vector space $\mathscr{W}(a, b)$ has a basis
  $$\left\{\, \gamma_{ef} \mid e \in \bar{b} \cap \mathfrak{E},\ f \in \bar{a} \cap \mathfrak{F},\ e - f \,\right\};$$
  it is zero if there are no such pairs.
- $\alpha_{e'e}\gamma_{ef} = \alpha_{e'f}$ if $e' < e$; $\gamma_{ef}\alpha_{ff'} = \gamma_{ef'}$ if $f < f'$; all other possible products of the base elements are zero.

We define $\mathscr{A}$ as the *additive closure* of the category $\mathscr{A}_0$, i.e. its objects are formal direct sums $a_1 \oplus a_2 \oplus \dots a_n$ of objects from $\mathscr{A}$ (not necesserily different) and morphisms from such a sum to another one $b_1 \oplus b_2 \oplus \dots b_m$ are $m \times n$ matrices (ϕ_{ij}), where $\phi_{ij} \in \mathscr{A}_0(a_j, b_i)$. We also extend the $\mathscr{A}_0$-bimodule $\mathscr{W}$ to an $\mathscr{A}$-bimodule, which we the denote by the same letter $\mathscr{W}$ (its elements can also be considered as matrices). The bimodule category $\mathsf{El}(\mathscr{W})$ is just what they call the *category of representations of this bunch of chains.*

Passing to the language of matrices, we present any object from $\mathsf{El}(\mathscr{W})$ as a set of matrices $W(a,b)$ ($a \in \mathfrak{E}$, $b \in \mathfrak{F}$, $a - b$) of size $n_a \times n_b$ such that $n_a = n_{a'}$ if $a \sim a'$: the elements of this matrix are just coefficients near γ_{ab}. Two sets of matrices define isomorphic objects if and only if one of them can be transformed to the other by a sequence of the following transformations:

(1) Replace $W(a,b)$ by $S_b^{-1} W(a,b) S_a$, where S_a are invertible matrices and $S_a = S_{a'}$ if $a \sim a'$.
(2) Replace $W(a,b)$ by $W(a,b) + W(a',b) S_{a'a}$ for some $a' > a$ and some matrix $S_{a'a}$.
(3) Replace $W(a,b)$ by $W(a,b) + S_{bb'} W(a,b')$ for some $b' < b$ and some matrix $S_{bb'}$.

Note that there are no relations between the matrices $S_{a'a}$ with different a, a' even if some of them are equivalent. It is convenient to gather all matrices $W(a,b)$ with $a \in \mathfrak{E}_i$, $b \in \mathfrak{F}_i$ (fixed i) and write them like one block matrix:

$$\left\downarrow \overrightarrow{\begin{pmatrix} W(a_1,b_1) & W(a_1,b_2) & \dots & W(a_1,b_n) \\ W(a_2,b_1) & W(a_2,b_2) & \dots & W(a_2,b_n) \\ \dots & \dots & \dots & \dots \\ W(a_m,b_1) & W(a_m,b_2) & \dots & W(a_m,b_n) \end{pmatrix}} \right. ,$$

where $a_1 < a_2 < \cdots < a_m$, $b_1 > b_2 > \cdots > b_n$. The arrwos show that we can add the rows from the upper stripes to those of the lower stripes, as well as the columns from the left-hand stripes to those of the right-hand stripes. We also can make elementary transformations inside the stripes so that they are the same in the equivalent stripes (if one of them is horizontal and the other one is vertical, "the same" means, of course, "contragredient").

To describe indecomposable objects from $\mathsf{El}(\mathscr{W})$, we introduce some combinatorics.

(1) A *word* is a sequnce $w = a_1 r_1 a_2 r_2 \dots a_{n-1} r_{n-1} a_n$, where $a_i \in \mathfrak{X}$, $r_i \in \{\sim, -\}$, such that $a_i r_i a_{i+1}$ in $\mathfrak{X}$, $r_i \neq r_{i+1}$ and $a_i \neq a_{i+1}$ for each $1 \leqslant i < n$. Such a word is said to be *full* if either $r_1 = \sim$ or $a_1 \not\sim b$ for any $b \neq a_1$ and also either $r_{n-1} = \sim$ or $a_n \not\sim b$ for any $b \neq a_n$. We call n the *length* of the word w and write $n = l(w)$. The *inverse word* w^* is defined as $w^* = a_n r_{n-1} a_{n-1} \dots r_2 a_2 r_1 a_1$.
(2) A word w is called *cyclic*, if $r_1 = r_{n-1} = \sim$ and $a_n - a_1$ in $\mathfrak{X}$. For such a cyclic word we set $r_n = -$, $a_{i+n} = a_i$ and $r_{i+n} = r_i$ (e.g. $a_{n+1} = a_1$) and define its *cyclic shift* $w^{(k)}$ as the word $a_{2k+1} r_{2k} a_{2k+2} \dots r_{2k-1} a_{2k}$, where $0 \leqslant k < n/2$ (note that the length of a cyclic word is always even). A cyclic word w of length n is called *periodic* if $w^{(k)} \neq w$ for $0 < k < n/2$. (Actually, if $w^{(k)} = w$, then $k \mid n/2$.) We define the *sign* $\delta(w,k)$ of such a

shift as $(-1)^h$, where h is the number of indices $0 \leqslant j < k$ such that either $\{ a_{2j+1}, a_{2j} \} \subseteq \mathfrak{E}$ or$\{ a_{2j+1}, a_{2j} \} \subseteq \mathfrak{F}$.

(3) Given a full word w, we define the *string* (or *string representation*) as the element $\mathbf{S}(w) \in \mathscr{W}(A, A)$, where $A = \bigoplus_{i=1}^{n} \bar{a}_i$, $\bar{a}_i$ being the equivalence class of a_i, given by the $n \times n$ matrix (σ_{ij}), where

$$\sigma_{ij} = \begin{cases} \gamma_{a_i, a_{i+1}} & \text{if } j = i+1,\ a_i \in \mathfrak{E},\ a_{i+1} \in \mathfrak{F},\ r_i = -, \\ \gamma_{a_{i+1}, a_i} & \text{if } i = j+1,\ a_j \in \mathfrak{F},\ a_{j+1} \in \mathfrak{E},\ r_j = -, \\ 0 & \text{otherwise.} \end{cases}$$

(4) Given a *non-periodic* cyclic word w, a positive integer $m \in \mathbb{N}$ and a non-zero element $\lambda \in \Bbbk$, we define the *band* (or *band representation*) as the element $\mathbf{B}(w, m, \lambda) \in \mathscr{W}(A, A)$, where $A = \bigoplus_{i=1}^{n} m\, \bar{a}_i$, given by the $n \times n$ matrix (β_{ij}), where

$$\beta_{ij} = \begin{cases} \gamma_{a_i, a_{i+1}} I_m & \text{if } j = i+1,\ a_i \in \mathfrak{E},\ a_{i+1} \in \mathfrak{F},\ r_i = -, \\ \gamma_{a_{i+1}, a_i} I_m & \text{if } i = j+1,\ a_j \in \mathfrak{F},\ a_{j+1} \in \mathfrak{E},\ r_j = -, \\ \gamma_{a_1, a_n} J_m(\lambda) & \text{if } i = 1,\ j = n,\ a_1 \in \mathfrak{E},\ a_n \in \mathfrak{F}, \\ \gamma_{a_n, a_1} J_m(\lambda) & \text{if } i = n,\ j = 1,\ a_1 \in \mathfrak{F},\ a_n \in \mathfrak{F}, \\ 0 & \text{otherwise.} \end{cases}$$

Here I_m denotes the unit $m \times m$ matrix and $J_m(\lambda)$ is the $m \times m$ Jordan block with the eigenvalue λ.

Now we can formulate the main result.

Theorem 5.1. (1) *Strings and bands are indecomposable objects in* $\mathsf{El}(\mathscr{W})$ *and every indecomposable object is isomorphic to a string or a band.*

(2) $\mathbf{S}(w) \simeq \mathbf{S}(w')$ *if and only if either* $w' = w$ *or* $w' = w^*$.

(3) $\mathbf{B}(w, m, \lambda) \simeq \mathbf{B}(w', m', \lambda')$ *if and only if* $m = m'$, *either* $w' = w^{(k)}$ *or* $w' = w^{*(k)}$ *for some* $0 \leqslant k < n/2$, *and* $\lambda' = \lambda^{\delta(w,k)}$.

Proof. The proof settles on a *reduction algorithm* for representations of bunches of chains. Namely, we reduce one block $W(a, b)$ of the matrix presenting an element from $\mathsf{El}(\mathscr{W})$ to a normal form, and only consider the elements, where these blocks have this normal form. Then we check that the category of such elements is equivalent to a full subcategory of $\mathsf{El}(\mathscr{W}')$, where $\mathscr{W}'$ is again a bimodule arising from (another) bunch of chains. After it has been shown, the proof follows by an easy induction.

First we order the triples (i, a, b), where $a \in \mathfrak{E}_i$, $b \in \mathfrak{F}_i$: $(i, a, b) < (i', a', b')$ if either $i < i'$, or $i = i'$, $a < a'$, or $i = i'$, $a = a'$, $b' < b$ (note the last inequality!). Denote by $\mathsf{El}[i, a, b]$ the full subcategory of $\mathsf{El}(\mathscr{W})$ consisting of all elements presented by the block matrices W with $W(a', b') = 0$ for every triple $(i', a', b') < (i, a, b)$. There are two possible cases.

Case 1. $a \not\sim b$.
Then the matrix $W(a,b)$ allows all elementary transformations (they are of type (1) above), so it can be redeuced to the diagonal form $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$, where I is a unit matrix. Since $a < a'$ and $b > b'$ for all nonzero matrices $W(a',b')$ with $a',b' \in \mathfrak{E}_i$, using transformations of types (2-3) we can make zero all rows and columns of these matrices corresponding to those containing units in $W(a,b)$ so that the whole i-th block will be:

$$\begin{array}{|cc|ccc|} \hline I & 0 & 0 & \dots & 0 \\ 0 & 0 & * & * & * \\ \hline 0 & * & & & \\ \vdots & * & & & \\ 0 & * & & & \\ \hline \end{array}$$

We denote by $W'(a,f)$ and $W'(e,b)$ the blocks marked here by stars and set $W'(a,b) = 0$. If $a \not\sim a'$ and $b \not\sim b'$ for any $a' \neq a$ and $b' \neq b$, W has direct summands of the sort $\gamma_{ab} \in \mathscr{W}(a,b)$ (they are strings $\mathbf{S}(a-b)$). If $a \sim a'$, $a' \neq a$, or $b \sim b'$, $b' \neq b$, or both, we add new elements $[a'b]$ or (and) $[ab']$ to the bunches that contain, respectively, a' and b', so that $a' < [a'b] < a''$ for each $a'' > a'$, $b' < [ab'] < b''$ for each $b'' > b'$, and $[ab'] \sim [ba']$ if both exist. If, say, $a' \in \mathfrak{E}$, we denote by $W'(a',f)$ and $W'([a'b],f)$ the parts of the block $W(a',f)$ corresponding, respectively, to zero and nonzero rows of $W(a,b)$; we use analogous notations in other possible cases for a' and b'. For instance, if both $a' \in \mathfrak{E}$ and $b' \in \mathfrak{F}$ exist and $a' - b'$, the block $W(a,b)$ subdivides into 4 blocks:

$$\begin{pmatrix} W'(a',b') & W'([a'b],b') \\ W'(a',[ab']) & W'([a'b],[ab']) \end{pmatrix}.$$

Thus we obtain a new bunch of chains, so a new bimodule $\mathscr{W}'$, and a map of the objects from $\mathsf{El}[i,a,b]$ to $\mathsf{El}(\mathscr{W}')$. One easily see that this map prolongs to morphisms too, and we obtain a functor $\rho : \mathsf{El}[i,a,b] \to \mathsf{El}(\mathscr{W}')$, whose image belongs to $\mathsf{El}[i_1,a_1,b_1]$ for a $(i_1,a_1,b_1) > (i,a,b)$.. Moreover, there is a natural functor $\mathsf{El}[i_1,a_1,b_1] \to \mathsf{El}[i,a,b]$: we just restore the matrix $W(a,b)$, so that $\rho\rho'(W') = W'$ and $\rho'\rho(W) = W$ whenever the latter has no direct summands γ_{ab}. There is also a correspondence between full words. Namely, if $b \not\sim b'$ for $b' \neq b$, we replace in any word w all occurences of $a' \sim a - b$ or $b - a \sim a'$ by $[a'b]$; analogous is the procedure if $a \not\sim a'$. If $a \sim a'$ and $b \sim b'$, we replace all occurences of $a' \sim a - b \sim b'$ (of $b' \sim b - a \sim a'$) by $[a'b] \sim [ab']$ (by $[ab'] \sim [a'b]$).

Case 2. $a \sim b$.
Now the matrix $W(a,b)$ is transformed by conjugations: $W(a,b) \mapsto S_a^{-1}W(a,b)S_a$, so we may suppose that it is a direct sum of Jordan blocks $J_m(\lambda)$. Using transformations of types (2)-(3), we split out

direct summands $J_m(\lambda)\gamma_{ab}$ for $\lambda \neq 0$ (they are bands $\mathbf{B}(a \sim b, m, \lambda)$). So we only have to consider the case, when $W(a,b) = \bigoplus_m k_m J_m(0)$:

$$W(a,b) = \begin{array}{|c|ccc|ccc|c|}
\hline
0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\hline
0 & 0 & I & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\hline
0 & 0 & 0 & 0 & I & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & I & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\hline
\end{array}$$

Using transformations (2)-(3) one can make all rows and columns having nonzero entries in $W(a,b)$ zero outside $W(a,b)$:

$$W(a,b) = \begin{array}{|c|ccc|ccc|c|c|}
\hline
0 & 0 & 0 & 0 & 0 & 0 & \dots & * \\
\hline
0 & 0 & I & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & * \\
\hline
0 & 0 & 0 & 0 & I & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & 0 & I & \dots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & * \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\hline
* & * & 0 & * & 0 & 0 & \dots & * \\
\hline
\end{array}$$

We exclude a and b from $\mathfrak{E}_i$ and $\mathfrak{F}_i$ replacing them by the elements a_m and b_m ($m \in \mathbb{N}$) such that $a_i < a_j$ and $b_i < b_j$ for $j > i$, while the order relations between a_i and $a' \neq a$ (b_i and $b' \neq b$) are the same as between a and a' (b and b'). We aslo set $a_i \sim b_i$. Take for $W'(a_i, f)$ and $W(e, b_i)$ the parts of $W(a, f)$ and $W(e, b)$ corresponding to the zero rows or, respectively, columns of m-dimensional Jordan blocks from $W(a,b)$ (marked by stars above). It defines a functor $\rho : \mathsf{El}[i,a,b] \to \mathsf{El}(\mathscr{W}')$ for a new bunch of chains, so that its image belongs to $\mathsf{El}[i_1, a_1, b_1]$ with $(i,a,b) < (i_1, a_1, b_1)$. Again there is a functor ρ' : $\mathsf{El}[i_1, a_1, b_1] \to \mathsf{El}[i,a,b]$ such that $\rho\rho'(W) = W$, while $\rho'\rho(W)$ and W can only differ by direct summands $J_m(\lambda)\gamma_{ab}$ with $\lambda \neq 0$. There is also a natural correspondence between words. (We propose the reader to restore it.)

Now the proof of the theorem becomes an obvious induction. $\square$

Exercise 5.2. Let Q be the quiver (oriented graph)

$$\begin{array}{ccc}
1 & \xrightarrow{a} & 2 \\
{\scriptstyle b}\big\downarrow & & \big\downarrow{\scriptstyle c} \\
3 & \xrightarrow[d]{} & 4
\end{array}$$

Recall that a *representation* of Q is, by definition, a diagram of vector spaces and linear maps

$$\begin{tikzcd} V_1 \arrow[r, "A"] \arrow[d, "B"'] & V_2 \arrow[d, "C"] \\ V_3 \arrow[r, "D"'] & V_4 \end{tikzcd}$$

Two diagram define *equivalent* representations if they differ by automorphisms of the vector spaces V_i.

Construct a bunch of chains such that the category of its representations coincide with that of the representations of Q and use it to describe all indecomposable representations of Q.

6. Easy case: A-configurations

The simplest case of projective configurations is that *of type* A, when the intersection graph $\Delta(X)$ is a chain (a graph of type A_s)

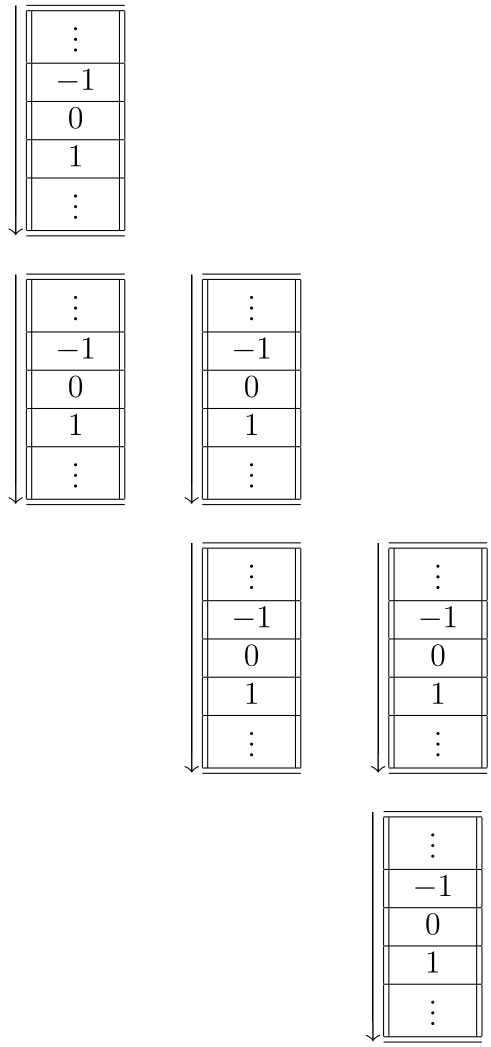
$$\bullet - \bullet - \cdots - \bullet$$

Then we can suppose that $S = \{ x_1, x_2, \dots, x_{s-1} \}$ and $x'_i \in X_i$, while $x''_i \in X_{i+1}$. It is convenient to write the matrices $W(i, d, y)$ as is shown in Figure 1 (for $s = 4$). Here the blocks correspond to the matrices $W(i, d, y)$ (the values of d are written inside blocks). The blocks with the same i and y form a big block shown by double lines; the blocks with the same i and d form horizontal stripes; the block with $y \in \{ x', x'' \}$ for some singular point x form a vertical stripe. The verical arrows stress that homomorphisms add the “upper” blocks (with smaller d) to the “lower” ones, but not vice versa (see the formula for aW at the end of the Example 4.1). Note also that there are at most two points y_1, y_2 in $\pi^{-1}(S) \cap X_i$ and for every $k > 0$ there is a polynomial $f(t) \in \Bbbk[t]_k$ with arbitrary prescribed values of $f(y_1)$ and $f(y_2)$. Therefore, two sets of matrices define isomorphic objects from $\mathsf{El}(\mathscr{W})$ if and only if one of them can be transformed to the other one by a sequence of the following transformations:

(1) Elementary transformations inside horizontal and vertical stripes (the same for all blocks of the stripe!).
(2) Adding a multiple of a row from an “upper” block to a row of a “lower” block (inside the same big block).

Obviously, we are in the situation of a bunch of chains. Namely, we have the pairs of chains $\mathfrak{E}_i = \{ e_{id} \mid d \in \mathbb{Z} \}$ $\mathfrak{F}_i = \{ f_i \}$, where $1 \leqslant i \leqslant s$, and $\mathfrak{E}'_i = \{ e'_{id} \mid d \in \mathbb{Z} \}$ $\mathfrak{F}'_i = \{ f_i \}$, where $1 \leqslant i < s$; the order in $\mathfrak{E}_i$ and $\mathfrak{E}'_i$ is that of integers (the second indices). Actually, the chains $\mathfrak{E}_i$ and $\mathfrak{F}_i$ describe the blocks $W(i, x'_i, d)$, while $\mathfrak{E}'_i$ and $\mathfrak{F}'_i$ describe the blocks $W(i+1, x''_i, d)$. The equivalence relation $\sim$ is given by the rules $e_{id} \sim e'_{i-1,d}$ $(1 < i \leqslant s)$ and $f_i \sim f'_i$. Note that in this case there are no cyclic words; morover, in every word there can be at most

Figure 1.



one occurence of a letter from $\mathfrak{E}_i$ (or $\mathfrak{F}_i$) for any fixed i. It gives the following description of indecomposable objects in $\mathsf{El}(\mathscr{W})$.

Theorem 6.1. *Let X be a projective configuration of type* A*, $1 \leqslant k \leqslant l \leqslant s$ and d_i $(k \leqslant i \leqslant l)$ be some integers. Denote by $\iota(d_k,\dots,d_l)$ the object from $\mathscr{W}(\mathsf{F},\mathcal{G})$, where $\mathcal{G} = \bigoplus_{i=k}^{l} \mathcal{O}_i(d_i)$, $\mathsf{F} = \bigoplus_{i=k}^{l-1} \Bbbk(x_i)$ and $\iota(d_k,\dots,d_l)$ maps every $\Bbbk(x_i)$ $(k \leqslant i < l)$ diagonally in $\Bbbk(x_i') \times \Bbbk(x_i'')$. Then $\iota(d_k,\dots,d_l)$ are indecomposable, pairwise non-isomorphic objects from $\mathsf{El}(\mathscr{W})$ and every indecomposable object is isomorphic to one of them. $\iota(d_k,\dots,d_l)$ belongs to $\mathcal{T}(X)$ if and only if $k=1$ and $l=s$.*

We denote the vector bundle corresponding to $\iota(d_1,d_2,\dots,d_s)$ by $\mathcal{L}(d_1,d_2,\dots,d_s)$ (note that it is a line bundle). Note that $\mathcal{O}_X \simeq \mathcal{L}(0,0,\dots,0)$.

Corollary 6.2. *The line bundles* $\mathcal{L}(d_1, d_2, \dots, d_s)$ *form a complete list of pairwise nonisomorphic indecomposable vector bundles on a projective configuration of type* A_s.

This description of vector bundles allows to establish a lot of their properties. For instance, one can calculate their cohomologies. To do it, we need the following definitions. Given a sequence of integeres $\mathbf{d} = (d_1, d_2, \dots, d_s)$, we call a *positive subsequence* of it a subsequence $\mathbf{d}' = (d_{k+1}, d_{k+2}, \dots, d_{k+l})$ $(0 \leqslant k < s,\ 1 \leqslant l \leqslant s-k)$ such that $d_{k+i} \geqslant 0$ for $1 \leqslant i \leqslant l$, but $d_k < 0$ if $k > 0$ and $d_{k+l+1} < 0$ if $l = s-k$. For such a positive part we define its *effective length* $L(\mathbf{d}')$ as

$$L(\mathbf{d}') = \begin{cases} l+1 & \text{if } d_{k+i} > 0 \text{ for some } 1 \leqslant i \leqslant l,\ k+i \notin \{1, s\}, \\ l & \text{otherwise.} \end{cases}$$

We also set $a^+ = (a + |a|)/2$ (*positive part* of a) and $a^- = (|a| - a)/2$ (*negative part* of a).

Proposition 6.3. *Let* $\mathcal{L} = \mathcal{L}(\mathbf{d})$, *where* $\mathbf{d} = (d_1, d_2, \dots, d_s)$, $\mathbf{P}$ *be the set of all positive sunsequences of* $\mathbf{d}$. *Then*

$$\mathrm{h}^0(\mathcal{L}) = \sum_{i=1}^{s} (d_i + 1)^- \sum_{\mathbf{d}' \in \mathbf{P}} L(\mathbf{d}'),$$

$$\mathrm{h}^1(\mathcal{L}) = \sum_{i=1}^{s} (d_i + 1)^- + (s-1) - \sum_{\mathbf{d}' \in \mathbf{P}} L(\mathbf{d}').$$

Proof. Consider the exact sequence $0 \to \mathcal{L} \to \widetilde{\mathcal{L}} \xrightarrow{\sigma} \mathcal{S} \to 0$, where $\mathcal{S} = \widetilde{\mathcal{L}}/\mathcal{L}$. It induces an exact sequence

$$0 \to \mathrm{H}^0(\mathcal{L}) \to \mathrm{H}^0(\widetilde{\mathcal{L}}) \xrightarrow{\sigma_*} \mathrm{H}^0(\mathcal{S}) \to \mathrm{H}^1(\mathcal{L}) \to \mathrm{H}^1(\widetilde{\mathcal{L}}) \to 0,$$

since $\mathcal{S}$ is a skyscraper sheaf, so $\mathrm{H}^1(\mathcal{S}) = 0$. Hence, $\mathrm{h}^0(\mathcal{L}) = \mathrm{h}^0(\widetilde{\mathcal{L}}) - \dim \operatorname{Im} \sigma_*$ and $\mathrm{h}^1(\mathcal{L}) = \mathrm{h}^0(\widetilde{\mathcal{L}}) + \mathrm{h}^0(\mathcal{S}) - \dim \operatorname{Im} \sigma_*$. We know that $\mathrm{h}^0(\mathcal{O}_{\mathbb{P}^1}(d)) = (d+1)^+$ and $\mathrm{h}^1(\mathcal{O}_{\mathbb{P}^1}(d)) = (d+1)^-$ [18]. Therefore, $\mathrm{h}^0(\widetilde{\mathcal{L}}) = \sum_{i=1}^{s} (d_i + 1)^+$ and $\mathrm{h}^1(\widetilde{\mathcal{L}}) = \sum_{i=1}^{s} (d_i + 1)^-$. Moreover, $\mathrm{H}^0(\mathcal{S}) \simeq \bigoplus_{x \in S} \mathcal{S}_x \simeq \bigoplus_{i=1}^{s-1} (\Bbbk(x_i') \times \Bbbk(x_i''))/\Bbbk(x_i)$, so $\mathrm{h}^0(\mathcal{S}) = \#(S) = s-1$. We denote by v_i' and v_i'' the images in $\mathrm{H}^0(\mathcal{S})$ of the units from $\Bbbk(x_i')$ and $\Bbbk(x_i'')$; then $v_i' = -v_i''$. We also set $v_0'' = v_s' = 0$.

Denote by σ_i the restriction of σ onto $\mathrm{H}^0(\mathcal{O}_i(d_i))$. If $d_i < 0$, $\sigma_i = 0$. If $d_i = 0$, $\operatorname{Im} \sigma_i$ is generated by $v_i' + v_{i-1}''$. If $d_i > 0$ then $\operatorname{Im} \sigma_i$ is generated by v_i' and v_{i-1}'' if $i \notin \{1, s\}$, by v_1' if $i = 1$, and by v_{s-1}'' if $i = s$. It implies that $\dim \operatorname{Im} \sigma_* = \sum_{\mathbf{d}' \in \mathbf{P}} L(\mathbf{d}')$, so proves the claims. $\square$

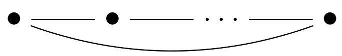
Exercise 6.4. Let $\mathcal{L}_k = \mathcal{L}(\mathbf{d}_k)$, $k = 1, 2$. Prove that

(1) $\mathcal{L}_1 \otimes_{\mathcal{O}} \mathcal{L}_2 \simeq \mathcal{L}(\mathbf{d}_1 + \mathbf{d}_2)$.
(2) $\mathscr{H}om_X(\mathcal{L}_1, \mathcal{L}_2) \simeq \mathcal{L}(\mathbf{d}_2 - \mathbf{d}_1)$.

Exercise 6.5. Show that every torsion free coherent sheaf over X is isomorphic to $\mathcal{L}_{kl}(d_k, d_{k+1}, \dots, d_l) = i_{kl}^*\mathcal{L}$, where i_{kl} is the embedding of the union of components $X_{kl} = \pi(\bigcup_{i=k}^{l} X_i) \to X$ and $\mathcal{L} = \mathcal{L}(d_k, d_{k+1}, \dots, d_l)$ is a line bundle on X_{kl}.

7. $\widetilde{\mathrm{A}}$-configurations

Now consider the case of projective configurations *of type* $\widetilde{\mathrm{A}}$, when the intersection graph is a cycle (a graph of type $\widetilde{\mathrm{A}}_s$):



Then we suppose that $S = \{\, x_1, x_2, \dots, x_s \,\}$, $x_i' \in X_i$, $x_i'' \in X_{i+1}$, where we always set $X_{s+i} = X_i$ and $x_{s+i} = x_i$, especially $X_{s+1} = X_1$. In it is this case convenient to write the matrices $W(i, d, x)$ as is shown in Figure 2 (again for $s = 4$) with the same notations and agreements as in the preceeding section. In particular, the isomorphisms of objects from $\mathsf{El}(\mathsf{W})$ correspond to the same transformations (1)-(2) from page 24. Again we get a problem on bunches of chains $\mathfrak{E}_i = \{\, e_{id} \mid d \in \mathbb{Z} \,\}$, $\mathfrak{F}_i = \{\, f_i \,\}$ and $\mathfrak{E}_i' = \{\, e_{id}' \mid d \in \mathbb{Z} \,\}$, $\mathfrak{F}_i' = \{\, f_i \,\}$ $(1 \leqslant i \leqslant s)$, with the equivalence relations $e_{id} \sim e_{i-1,d}'$ and $f_i \sim f_i'$. (Here we also replace $s+i$ by i, especially, 0 by s and $s+1$ by 1.) Note that this time any full word starts as $a \sim b$ for some letters a, b, since every letter has an equivalent pair. Then the string defined by this word has a zero row or column (corresponding to a), therefore cannot arise from a vector bundle. Thus all vector bundles over X come from bands.

One easily sees that any cyclic word is of the sort (up to symmetry and cyclic shift):

$$e_{sd_1}' \sim e_{1d_1} - f_1 \sim f_1' - e_{2d_2}' \sim e_{2d_2} - f_2 \sim f_2' - \dots - e_{s-1,d_{rs}}' \sim e_{sd_{rs}} - f_s \sim f_s,$$

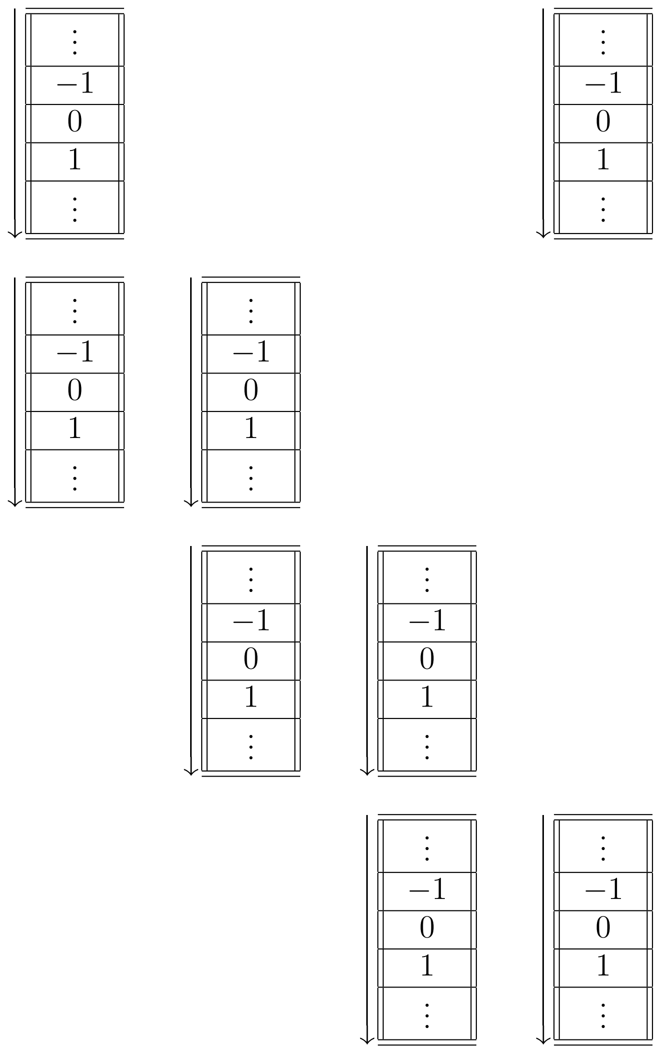
so is given by a sequence of integers $\mathbf{d} = (d_1, d_2, \dots, d_{rs})$. The length of this sequence is a multiple of s and it is defined up to a *cyclic s-shift*: $\mathbf{d} = (d_1, d_2, \dots, d_{rs}) \mapsto \mathbf{d}^{(ks)} = (d_{ks+1}, d_{ks+2}, \dots, d_{ks})$. This sequence also cannot be periodic, which means that $\mathbf{d} \neq \mathbf{d}^{(ks)}$ if $0 < k < r$. So we get a description of vector bundles over X.

Theorem 7.1. (1) *Any indecomposable vector bundle over a projective configuration X of type $\widetilde{\mathrm{A}}_s$ is defined by a triple $(\mathbf{d}, m, \lambda)$, where $\mathbf{d} = (d_1, d_2, \dots, d_{rs})$ is a sequence of integers such that $\mathbf{d} \neq \mathbf{d}^{(ks)}$ for $0 < k < r$, $m \in \mathbb{N}$ and $\lambda \in \Bbbk^\times$.*
We denote this vector bundle by $\mathcal{B}(\mathbf{d}, m, \lambda)$. Obviously, the rank of this vector bundle is r.

(2) $\mathcal{B}(\mathbf{d}, m, \lambda) \simeq \mathcal{B}(\mathbf{d}', m', \lambda')$ *if and only if $m = m'$, $\lambda' = \lambda$ and $\mathbf{d}'$ is a cyclic s-shift of $\mathbf{d}$: $\mathbf{d}' = \mathbf{d}^{(ks)}$ for some k.*

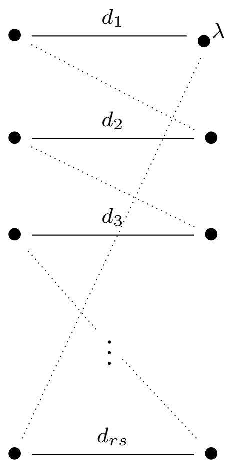
Note that $\mathcal{O}_X \simeq \mathcal{B}(\bar{0}, 1, 1)$, where $\bar{0} = (0, 0, \dots, 0)$.

FIGURE 2.



The structure of these vector bundles can be illustrated by rather simple pictures. Let first $m = 1$. Then $\mathcal{B}(\mathbf{d}, 1, \lambda)$ is glued from the line bundles on the components X_i as is shown in Figure 3. Here the i-th horizontal line symbolize a line bundle over X_i of the superscripted degree, its left and right ends are the basic elements of these bundles at the points x'_i and x''_{i-1}, the dotted lines show which of them must be glued. All gluings are trivial, except that going from the uppermost right point to the lowermost left one, where we glue one vector to the other multiplied by λ. If $m > 1$, one has to take m copies of each vector bundle from this picture, make again trivial all gluings except the last one, where identifications must be made using the Jordan $m \times m$ cell with eigenvalue λ.

FIGURE 3.



Note that this time there are indecomposable vector bundles of arbitrary ranks. Moreover, if we fix $\mathbf{d}$ and m, the vector bundles $\mathcal{B}(\mathbf{d}, m, \lambda)$ form a 1-parameter family with the base $\Bbbk^\times = \mathbb{A}^1 \setminus \{0\}$.

Again, we can use this description to get more information about vector bundles. For instance, we can calculate their cohomologies. Since these calculations are quite analogous to those of Proposition 6.3, we propose them as an exercise.

Exercise 7.2. For a sequence of integers $\mathbf{d} = (d_1, d_2, \dots, d_n)$, define its *cyclic positive part* as a sequence $\mathbf{d}' = (d_{k+1}, d_{k+2}, \dots, d_{k+l})$, where $0 \leqslant k < n$, $1 \leqslant l < n$ if $k \neq 0$, $1 \leqslant l \leqslant n$ if $k = 0$, and we set $d_{n+i} = d_i$, such that $d_i \geqslant 0$ for all $k < i \leqslant k + l$, but, if $l < n$, both $d_k < 0$ and $d_{k+l+1} < 0$. Define the *effective length* $L(\mathbf{d}')$ of such a cyclic positive part as l if either $l = n$ or $\mathbf{d}' = (0, 0, \dots, 0)$ and $l + 1$ otherwise. At last, set $\delta(\mathbf{d}, \lambda) = 1$ if $\lambda = 1$ and $\mathbf{d} = (0, 0, \dots, 0)$ and $\delta(\mathbf{d}, \lambda) = 0$ otherwise. Then, for every indecomposable vector bundle $\mathcal{B} = \mathcal{B}(\mathbf{d}, m, \lambda)$ of Theorem 7.1,

$$\mathrm{h}^0(X, \mathcal{B}) = m\Big(\sum_{i=1}^{rs} (d_i + 1)^+ - \sum_{\mathbf{d}' \in \mathbf{P}} L(\mathbf{d}')\Big) + \delta(\mathbf{d}, \lambda),$$

$$\mathrm{h}^1(X, \mathcal{B}) = m\Big(\sum_{i=1}^{rs} (d_i + 1)^- + rs - \sum_{\mathbf{d}' \in \mathbf{P}} L(\mathbf{d}')\Big) + \delta(\mathbf{d}, \lambda),$$

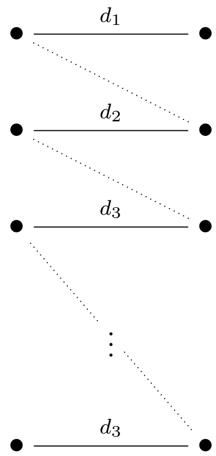
where $\mathbf{P}$ is the set of cyclic positive parts of $\mathbf{d}$. In particular,

$$\chi(\mathcal{B}) = \chi(\widetilde{\mathcal{B}}) - mrs, \quad \text{where} \quad \widetilde{\mathcal{B}} = \widetilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{B} \simeq \pi_* \pi^* \mathcal{B}.$$

Note that $\chi(\mathcal{B}) = \chi(\widetilde{\mathcal{B}}) - rs = \sum_{i=1}^{r} s(d_i - 1)$, in particular $\chi(\mathcal{O}_X) = 0$, so the arithmetic genus of X equals $1 - \chi(\mathcal{O}_X) = 1$.

Exercise 7.3. Show that every torsion free sheaf $\mathcal{F}$ over X is uniquely defined by a pair $(k, \mathbf{d})$, where $0 \leqslant k < s$ and $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ (with any length n). Its structure is described by Figure 4. Here the i-th

FIGURE 4.



horizontal line symbolizes a line bundle over X_{m+i} of the superscripted degree, its left and right ends are the basic elements of these bundles at the points x'_{m+i} and x''_{m+i-1}, the dotted lines show which of them must be glued and all gluings are trivial. Calculate $\mathrm{rk}(\mathcal{F})$, $\mathrm{h}^i(\mathcal{F})$ and $\dim \mathcal{F}(x)$ for all points $x \in X$.

Exercise 7.4. Let X be a nodal cubic, i.e. $s = 1$, $\tilde{X} = \mathbb{P}^1$ and X has a unique singular point x, which is a simple node. Prove that if a vector bundle $\mathcal{B}(\mathbf{d}, m, \lambda)$ is stable, then $m = 1$ and all components of $\mathbf{d}$ are from the set $\{\, d, d+1 \,\}$ for some $d \in \mathbb{Z}$. Moreover, $\mathcal{B}(\mathbf{d}, 1, \lambda)$ is stable if and only if $\mathcal{B}(\mathbf{d} - d\bar{1}, 1, 1)$ is stable, where $\bar{1} = (1, 1, \ldots, 1)$ (so all components of $\mathbf{d} - d\bar{1}$ are 0 or 1).

HINT: If $d_i \leqslant d_j - 2$, there is a nonzero morphism $f : \widetilde{\mathcal{O}}(d_i) \to \widetilde{\mathcal{O}}(d_j)$ such that $f(x') = f(x'') = 0$. It induces a non-invertible endomorpism of $\mathcal{B}(\mathbf{d}, m, \lambda)$. If all $d_i \in \{\, d, d+1 \,\}$, then $\mathrm{End}(\mathcal{B}(\mathbf{d}, m, \lambda) \simeq \mathrm{End}(W)$, where $W = \mathsf{B}(w, m, \lambda)$ is the corresponding band from $\mathsf{El}(\mathscr{W})$.

A description of stable vector bundles for a nodal cubic has been obtained by Burban [4, 3] and Mozgovoy [21]. We shall propose another approach, elaborated mainly by L. Bodnarchuk, in Section 9.

Example 7.5. We end this section with an example of indecomposable sheaf over a nodal cubic that is not semistable. Let $\mathcal{B} = \mathcal{B}((0, 5), 1, 1)$; thus $\mathrm{rk}\, \mathcal{B} = 2$, $\deg \mathcal{B} = 5$ and $\mu(\mathcal{B}) = 5/2$. Let also $\mathcal{L}$ be a line bundle of degree 3 (i.e. $\mathcal{L} = \mathcal{B}(3, 1, \lambda)$ for some λ). There is a nonzero morphism $f : \widetilde{\mathcal{O}}_X(3) \to \widetilde{\mathcal{O}}_X(5)$ such that $f(x') = f(x'') = 0$. It induces a nonzero morphism $\mathcal{L} \to \mathcal{B}$. Thus $\mathcal{B}$ contains a subsheaf isomorphic to $\mathcal{L}$, which is of slope 3, so $\mathcal{B}$ is not semistable.

8. Wild cases

It so happens that the considered cases of projective line, ellitic curve and projective configurations of types A and $\widetilde{\mathrm{A}}$ exhaust all cases when a complete description of vector bundles can be obtained in a more or less observable form. All other projective curves are, as they say, *vector bundle wild*. Non-formally, it means that a description of vector bundles over such a curve contains a description of representations of all finitely generated algebras over the field $\Bbbk$. A formal definition is:

Definition 8.1. A curve X is called *vector bundle wild* if for every finitely generated algebra Λ over the field $\Bbbk$ there is an exact functor Φ : Λ-mod $\to$ VB(X), where Λ-mod is the category of finite dimensional Λ-modules, such that $\Phi(M) \simeq \Phi(N)$ implies $M \simeq N$ and if M is indecomposable, so is also $\Phi(M)$.

We call such a functor Φ a *representation embedding.*

Theorem 8.2. *Let X be a projective curve, which is neither a projective line, nor an elliptic curve, nor a projective configuration of type* A *or* $\widetilde{\mathrm{A}}$. *Then X is vector bundle wild.*

Proof. First note that actually we only have to find a representation embedding for a unique algebra, namely for the free (non-commutative) $\Bbbk$-algebra with 2 generators $\Sigma = \Bbbk\langle\, z_1, z_2 \,\rangle$. It follows from

Lemma 8.3. *For every finitely generated $\Bbbk$-algebra Λ there is a representation embedding $\Phi : \Lambda\text{-mod} \to \Sigma\text{-mod}$.*

Proof. Let $a_1, a_2, \dots, a_n$ be generators of Λ. A finite dimensional Λ-module M is given by a sequence $A_1, A_2, \dots, A_n$ of square matrices over $\Bbbk$ satisfying certain relations. We write $M = M\,(A_1, A_2, \dots, A_n)$. Then a homomorphism $M\,(A_1, A_2, \dots, A_n) \to M\,(B_1, B_2, \dots, B_n)$ is given by a matrix C such that $CA_i = B_iC$ for all i; $M\,(A_1, A_2, \dots, A_n)$ is decomposable if and only if there is a matrix E such that $EA_i = A_iE$ for all i, $E^2 = E$ and E is neither zero nor unit matrix. In particular, a Σ-module is given by a pair of matrices Z_1, Z_2 and this pair can be arbitrary. Given a module $M = M\,(A_1, A_2, \dots, A_n)$, define $\Phi(M)$ as the Σ-module given by the pair

$$Z_1 = \begin{pmatrix} A_1 & I & 0 & \dots & 0 & 0 \\ 0 & A_2 & I & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & A_{n-1} & I \\ 0 & 0 & 0 & \dots & 0 & A_n \end{pmatrix},$$

and $Z_2 = \operatorname{diag}(\lambda_1 I, \lambda_2 I, \dots, \lambda_n I)$, where $\lambda_1 \dots \lambda_5$ are different elements from $\Bbbk$ and I is the unit matrix of the same size as A_i. Then one easily shows that any homomorphism $\Phi(M) \to \Phi(M')$, where $M' = M(A_1', A_2', \dots, A_n')$, is given by a block diagonal matrix $\operatorname{diag}(C, C, \dots, C)$, where $C \in \operatorname{Hom}_\Lambda(M, M')$ (check it!). Obviously, it implies that Φ is a representation embedding. $\square$

Corollary 8.4. *A curve X is vector bundle wild if and only if there is a representation embedding Σ-mod $\to \mathrm{VB}(X)$.*

The proof of Theorem 8.1 consists of considering several cases.

Case 1. X is a smooth curve of genus $g > 1$.

Note that, since $g > 1$, $\chi(\mathcal{L}) < 0$ for a line bundle of degree 0, so $\mathrm{h}^1(\mathcal{L}) > 0$. For instance, $\operatorname{Ext}^1_X(\mathcal{O}(x), \mathcal{O}(y)) \simeq \mathrm{H}^1(\mathcal{O}(y-x)) \neq 0$, where $\mathcal{O} = \mathcal{O}_X$. On the other hand, $\operatorname{Hom}_X(\mathcal{O}(x), \mathcal{O}(y)) \simeq \mathrm{H}^0(\mathcal{O}(y-x)) = 0$ if $x \neq y$, and $\operatorname{Hom}_X(\mathcal{L}, \mathcal{L}) = \Bbbk$ for every line bundle $\mathcal{L}$.

Let x_1, x_2, x_3, x_4, x_5 be different points of X, ξ_{ij} be a nonzero element from $\operatorname{Ext}^1_X(\mathcal{O}(x_i), \mathcal{O}(x_j))$. For every pair of matrices $Z_1, Z_2 \in \operatorname{Mat}(n \times n, \Bbbk)$ defining a Σ-modules $M = M(Z_1, Z_2)$ denote by $\xi(Z_1, Z_2)$ the element from $\operatorname{Ext}^1_X(n\mathcal{O}(x_3) \oplus n\mathcal{O}(x_4) \oplus n\mathcal{O}(x_5), n\mathcal{O}(x_1) \oplus n\mathcal{O}(x_2))$ given by the matrix

$$\Xi(Z_1, Z_2) = \begin{pmatrix} \xi_{31} I_n & \xi_{41} I_n & \xi_{51} I_n \\ \xi_{32} I_n & \xi_{42} I_n & \xi_{52} I_n \end{pmatrix}$$

and by $\Phi(M) = \mathcal{F}(Z_1, Z_2)$ the corresponding extension of $n\mathcal{O}(x_3) \oplus n\mathcal{O}(x_4) \oplus n\mathcal{O}(x_5)$ by $n\mathcal{O}(x_1) \oplus n\mathcal{O}(x_2)$. Since $\operatorname{Hom}_X(\mathcal{O}(x_i), \mathcal{O}_X(x_j)) = 0$ for $i \neq j$, every morphism $\sigma : \mathcal{F}(Z_1, Z_2) \to \mathcal{F}(Z_1', Z_2')$ maps $n\mathcal{O}(x_1) \oplus n\mathcal{O}(x_2)$ to $n'\mathcal{O}(x_1) \oplus n'\mathcal{O}(x_2)$, therefore, induces morphisms $\sigma_1 : n\mathcal{O}(x_1) \oplus n\mathcal{O}(x_2)) \to n'\mathcal{O}(x_1) \oplus n'\mathcal{O}(x_2)$ and $\sigma_2 : n\mathcal{O}(x_3) \oplus n\mathcal{O}(x_4) \oplus n\mathcal{O}(x_5) \to n'\mathcal{O}(x_3) \oplus n'\mathcal{O}(x_4) \oplus n'\mathcal{O}(x_5)$ such that $\sigma_1 \xi(Z_1, Z_2) = \xi(Z_1', Z_2')\sigma$, or, the same, $C_1 \Xi(Z_1, Z_2) = \Xi(Z_1', Z_2') C_2$ if σ_i is presented by the matrix C_i. It immediately implies that both C_1 and C_2 are block diagonal: $C_1 = \operatorname{diag}(C, C)$ and $C_2 = \operatorname{diag}(C, C, C)$, where $C \in \operatorname{Hom}_\Sigma(M(Z_1, Z_2), M(Z_1', Z_2'))$. Therefore, Φ defines a representation embedding Σ-mod $\to \mathrm{VB}(X)$.

So we have now to consider singular curves.

Case 2. One of the components $\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_s$ of $\tilde{X}$ is not rational.

Suppose that $\tilde{X}_1$ is of genus $g \geqslant 1$. As X is connected, there is a singular point p that belongs to X_1. We suppose that x has at least 2 preimages on $\tilde{X}$. If it only has one, the algebra $\widetilde{\mathsf{F}}_p$ is not semi-simple, which simplifies the calculations. Let $\{ p_1, p_2, \dots, p_t \}$ be all preimages of p, with $p_1 \in \tilde{X}_1$, and let Y be the component that contains p_2 (we allow $Y = \tilde{X}_1$). Let $\{ p_{t+1}, \dots, p_l \}$ be all other points from $\widetilde{S}$. Choose 4 different points x_i $(i = 1, \dots, 4)$ on $\tilde{X}_1 \setminus \widetilde{S}$ and another regular point y on Y. For every Σ-module $M = M(Z_1, Z_2)$ with Z_i of size $n \times n$, consider the object $W = W(Z_1, Z_2)$ from $\mathsf{El}(\mathscr{W})$ such that $W \in \mathscr{W}(4n\mathsf{A}, n\mathcal{A})$, where $\mathcal{A} = \bigoplus_{k=1}^4 \widetilde{\mathcal{O}}(x_k + ky)$, all components of W in $\mathrm{Hom}_{\mathsf{A}}(4n\mathsf{A}, n\mathcal{A}_{p_i}/\mathcal{J}\mathcal{A}_{p_i})(1 < i \leqslant l)$ are unit matrices, and its component in $\mathrm{Hom}_{\mathsf{A}}(4n\mathsf{A}, n(\mathcal{A}_{p_1}/\mathcal{J}\mathcal{A}_{p_1}))$ equals

$$\begin{pmatrix} 0 & 0 & 0 & I \\ 0 & 0 & I & I \\ 0 & I & I & Z_1 \\ I & 0 & I & Z_2 \end{pmatrix}.$$

Using the equalities $\mathrm{Hom}_{\mathcal{O}_{X_1}}(\mathcal{O}_{X_1}(x_k), \mathcal{O}_{X_1}(x_j)) = 0$ if $k \neq j$ and $\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y(ky), \mathcal{O}_Y(jy)) = 0$ if $k > j$, one can check that it gives a representation embedding Σ-mod $\to \mathrm{VB}(X)$.

So from now on we suppose that all components of $\tilde{X}$ are $\mathbb{P}^1$.

Case 3. B is not semisimple.

Choose a point $p \in S$ such that B_p is not semisimple and a nonzero element $\theta \in \mathrm{rad}\,\mathsf{B}_p$. Set $\mathcal{A} = 4\widetilde{\mathcal{O}} \oplus 4\widetilde{\mathcal{O}}(x) \oplus 4\widetilde{\mathcal{O}}(2x) \oplus \widetilde{\mathcal{O}}(3x)$, where $x \notin \widetilde{S}$ belongs to the same component as p. For any pair (Z_1, Z_2) of square matrices of size n consider the matrices

$$A_1 = \begin{pmatrix} Z_1 & Z_2 \\ I_n & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix}, \quad A_3 = \begin{pmatrix} I_n \\ 0 \end{pmatrix}$$

and define the element $W = W(Z_1, Z_2) \in \mathscr{W}(7n\mathsf{A}, n\mathcal{A})$ such that all its components except that in $\mathrm{Hom}_{\mathsf{A}}(7n\mathsf{A}_p, \mathcal{A}_p/\mathcal{J}\mathcal{A}_p)$ equal unit matrices, while the last is

$$\begin{pmatrix} I_{2n} & 0 & 0 & 0 & \theta I_{2n} & 0 & 0 \\ 0 & I_{2n} & \theta I_{2n} & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{2n} & \theta A_1 & \theta A_2 & 0 & 0 \\ 0 & 0 & 0 & I_{2n} & 0 & \theta A_2 & 0 \\ 0 & 0 & 0 & 0 & I_{2n} & \theta A_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{2n} & \theta A_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_n \end{pmatrix}.$$

Again a straightforward calculation shows that we obtain a representation embedding Σ-mod $\to \mathrm{VB}(X)$.

So we can suppose that all singular points of X are *ordinary multiple points*, i.e. such that the number of different tangent directions at each of them equals the multiplicity of that point.

Case 4. There is a singular point p of multiplicity $m > 2$.

Let p be a point of multiplicity $l \geqslant 3$, which we suppose an ordinary multiple point, $p_1, p_2, \dots, p_l$ be its preimages on $\tilde{X}$. Denote by Y_i the component of $\tilde{X}$ containing p_i (some of them may coincide). Choose points $y_i \in Y_i \setminus \widetilde{S}$ and set $\mathcal{A} = \bigoplus_{k=1}^{4} \widetilde{\mathcal{O}}(ky_1 + ky_2 + ky_3)$. For each pair of $n \times n$ matrices (Z_1, Z_2) consider the object $W \in \mathscr{W}(4n\mathsf{A}, n\mathcal{A})$ that has unit matrices as all its components except those at the points p_1 and p_2, the last two being respectively

$$\begin{pmatrix} 0 & 0 & 0 & I_n \\ 0 & 0 & I_n & 0 \\ 0 & I_n & 0 & 0 \\ I_n & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} I_n & I_n & Z_1 & Z_2 \\ 0 & I_n & I_n & I_n \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_n \end{pmatrix}.$$

It defines a representation embedding Σ-mod $\to \mathrm{VB}(X)$ (check it!).

Thus, now we only have to study *projective configurations*, and to accomplish the proof of Theorem 8.2, we must prove that if a projective configuration X is not vector bundle wild, every component of $\tilde{X}$ has at most 2 points from $\widetilde{S}$. (If the last conditions holds, X is of type A or $\widetilde{\mathrm{A}}$.) The proof of this statement is the most technical and complicated, though not too interesting, so we omit it, referring to [12, 10]. □

9. Stable vector bundles

Though a complete description of vector bundles, say, over the cuspidal cubic is in some sense unrealizable, one can try to describe, at least, *stable vector bundles*. Actually, for the cuspidal cubic it can be done using the technique of bimodule categories, as was shown in [2] (see also [3]).

Let X be a cuspidal cubic given (in an affine part of the projective plane) by the equation $y^2 = x^3$. It has a uniue singular point $p = (0, 0)$ and a normalization $\pi : \mathbb{P}^1 \to X$ such that $\pi^{-1}(p)$ consists of a uniqe point too, which we also denote by p. Moreover, the conductor $\mathcal{J} = \widetilde{\mathcal{O}}(2p)$, so $\mathsf{A} = \mathcal{O}/\mathcal{J} = \Bbbk$ and $\mathsf{B} = \widetilde{\mathcal{O}}/\mathcal{J} \simeq \Bbbk[t]/t^2$, where t is a local parameter on $\mathbb{P}^1$ at the point p. Since $\widetilde{\mathcal{O}}/\mathcal{O} \simeq \Bbbk$, $\chi(\mathcal{O}) = \chi(\widetilde{\mathcal{O}}) - 1 = 0$, so $g(X) = 1$. In particular, Corollary 2.4 and Proposition 2.5 can be applied, so *stable vector bundle* coincide with *bricks*, i.e. vector bundle that only have scalar endomorphisms.

The main result of this section is

Theorem 9.1. (1) *If $\mathcal{F}$ is a stable vector bundle, its rank and degree are coprime.*
(2) *If $\gcd(r, d) = 1$, there is a 1-parameter family $\mathcal{B}(r, d; \lambda)$ of stable vector bundles of rank r and degree d parametrized by $\Bbbk$.*

Note that in this case $\operatorname{Pic}_0 X \simeq \Bbbk$, so the description is just alike that for elliptic curves.

Proof. It is based on the technique of sandwiche procedure and bimodule categies of Sections 3 and 4. We always identify $\widetilde{\mathcal{O}}(d)/\mathcal{J}\widetilde{\mathcal{O}}(d)$ with B. If a morphism $\widetilde{\mathcal{O}}(d) \to \widetilde{\mathcal{O}}(d')$ is given by a polynomial $f(t)$, the induced map $\mathsf{B} \to \mathsf{B}$ is also the multiplication by $f(t)$, or, the same, by its linear part. Therefore, if we consider the bimodule $\mathscr{W}$ of Section 4 arising from the sandwich procedure for X, its elements from $\mathscr{W}(r\mathsf{A}, \bigoplus_n m_n\mathcal{O}(n))$ can be presented as block matrices (actually columns of blocks) $W = (W(n))$, where $W(n)$ is of size $m_n \times r$ with components from B. A morphism $f : W \to W' \in \mathscr{W}(r'\mathsf{A}, \bigoplus_d m'_d\mathcal{O}(n))$ is presented by a pair of matrices (f_1, f_0): $f_0 \in \mathrm{Mat}(r' \times r, \Bbbk)$ and f_1 is a block matrix $f_1 = (\phi_{n'n})$, where $\phi_{n'n}$ is of size $m'_{n'} \times m_n$ with components from $\Bbbk[t]_{n'-n}$ (zero if $n' < n$). These matrices must satisfy the equations $W'(n')f_1 = \sum_n \phi_{n'n}W(n)$. Moreover, W correspond to a vector bundle if and only if it is invertible. We denote by $\mathscr{E}$ the full subcategory of $\mathsf{El}(\mathscr{W})$ consisting of the invertible matrices W. Since $\mathrm{VB}(X) \simeq \mathscr{T}(X) \simeq \mathscr{E}$, the stable vector bundles over X are in one-to-one correspondence with bricks in the category $\mathscr{E}$, and we will study such bricks. Note that, if $W \in \mathscr{E}$, $r = \sum_n m_n$, it coinsides with the rank of the corresponding vector bundle $\mathcal{F}$, while $\deg \mathcal{F} = d = \sum_n nm_n$. We set $\mathrm{rk}\, W = r$ and $\deg W = d$ and call them the *rank* and the *degree* of the block matrix W.

Just as in Exercise 7.4, one easily proves

Proposition 9.2. *If $W \in \mathscr{E}$ is a brick, $m_i = 0$, except at most two of them: m_n and, maybe, m_{n+1}.*

Proof. Exercise! □

So from now on we always suppose that W has at most two blocks, $W(n)$ and, maybe, $W(n + 1)$, of sizes, respectively, $k \times r$ and $l \times r$. Then $r = \mathrm{rk}\, W = k + l$ and $d = \deg W = nk + (n + 1)l$. Note that k, l and n are uniquely defined by r and d, namely, $n = \lfloor d/r \rfloor$, $l = d - rn$ and $k = r - l$. Note also that $\gcd(k, l) = \gcd(r, d)$. We denote the full subcategory of $\mathscr{E}$ consisting of such matrices by $\mathscr{E}_n$. Obviously, all these categories are equivalent, so we can (and will) only consider $\mathscr{E}_0$ (i.e. only $W(0)$ and, maybe, $W(1)$ actually occur).

It is convenient to write $W = W_0 \oplus tW_1$. As we have notices in Section 4, the whole matrix W, arising from a vector bundle $\mathcal{F}$ must be invertible, or, the same, W_0 must be invertible. Then W_0 can be transformed to the unit matrix:

$$W_0 = \left(\begin{array}{cc} I_k & 0 \\ \hline 0 & I_l \end{array}\right), \qquad \text{correspondingly,} \qquad W_1 = \left(\begin{array}{cc} A_1 & A_0 \\ \hline A_3 & A_2 \end{array}\right),$$

where the horizontal line show the division of W into $W(0)$ (upper) and $W(1)$ (lower). So we can now fix W_0 and study the matrix W_1. One can check that in this case the matrices f_0, f_1 defining morphisms are of the form

$$f_0 = \begin{pmatrix} C_1 & 0 \\ C_0 & C_2 \end{pmatrix}, \quad f_1 = \begin{pmatrix} C_1 & 0 \\ C_0 + \beta t & C_2 \end{pmatrix},$$

where C_i and β are matrices over $\Bbbk$. f is an isomorphism if and only if C_1 and C_2 are invertible. Taking C_1, C_2 unit, $C_0 = 0$ and $\beta = -A_3$, we can make A_3 zero. It does not imply the form of the endimorphism matrices, but now β is uniquely defined by the other matrices. Therefore, such a vector bundle $\mathcal{F}$ is defined by the triple (A_0, A_1, A_2). Moreover, a morphism $\mathcal{F} \to \mathcal{F}'$, where $\mathcal{F}'$ correspond to the triple (A_0', A_1', A_2') is given by a triple (C_0, C_1, C_2) such that

$$(9.1) \quad C_1A_0 = A_0'C_2,\ C_1A_1 = A_1'C_1 + A_0'C_0,\ C_2A_2 + C_0A_0 = A_2C_2.$$

Such triples can also be considered as objects of a bimodule category $\mathsf{El}(\mathscr{V})$. Namely, $\mathscr{V}$ is the bimodule over the path category $\mathcal{P}$ of the quiver (oriented graph) $1 \xrightarrow{c_0} 2$ such that $\mathscr{V}(1,1), \mathscr{V}(2,2)$ and $\mathscr{V}(2,1)$ are 1-dimensional, generated, respectively, by a_1, a_2 and a_0, $\mathscr{V}(1,2) = 0$, and the action is given by the rules $c_0a_0 = a_2,\ a_0c_0 = a_1,\ c_0a_1 = a_2c_0 = 0$ (check it!).

If $l = 0$, i.e. $W = W(0)$, the matrices A_0, A_2 disappear and A_1 can be transformed to the Jordan normal form. If W is a brick, A_1 consists of a unique 1×1 block (λ): otherwise W has nontrivial endomorphisms. We denote the corresponding vector bundle by $\mathcal{B}(1, 0; \lambda)$ (by $\mathcal{B}(1, n; \lambda)$ if we start from $\mathcal{E}_n$).

Suppose that $l \neq 0$, so $W(1)$ and A_0 are not empty. Then A_0 can be transformed to the diagonal form, thus suppose that $A_0 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$. The following easy observation is the clue one.

Lemma 9.3. *If W is a brick,* $\operatorname{rk} A_0 = \min(k, l)$.

Proof. If it is not the case, the last (l-th) column and the last (k-th) row of A_0 are zero. Then, setting $C_1 = C_2 = 0,\ C_0 = e_{lk}$ (the matix unit), we get a nontrivial endomorphism of W. □

Therefore, if $k = l$, $A_0 = I_k$. Setting $C_1 = I_k,\ C_2 = I_l,\ C_0 = -A_2$, we make A_2 zero. Then A_1 can be transformed to the Joprdan form and again must consist of a unique 1×1 block, thus $k = l = 1$. We denote the corresponding vector bundle by $\mathcal{B}(2, 1; \lambda)$ (by $\mathcal{B}(2, 2n+1; \lambda)$ if we start from $\mathcal{E}_n$).

Suppose that $k < l$, so $A_0 = \begin{pmatrix} I_k & 0 \end{pmatrix}$. Using automorphisms, we can make $A_1 = 0$ and $A_2 = \begin{pmatrix} B_1 & B_0 \\ 0 & B_2 \end{pmatrix}$, where B_1 is of size $k \times k$ and B_2 is of size $(l-k) \times (l-k)$ (check it!). One can verify that mapping the triple (A_0, A_1, A_2) to (B_0, B_1, B_2) we obtain a fully faithful functor $\mathsf{El}(\mathscr{V}) \to \mathsf{El}(\mathscr{V})$ (check it!). Since the pair of sizes (k, l) has changed into $(k, l-k)$, we can proceed by induction. The calculations for the case $k > l$ are quite the same; as the result we get a triple with sizes $(k-l, l)$. It accomplishes the proof. □

Exercise 9.4. (1) Using the recursive procedure above, show that for every brick $W \in \mathcal{E}$ the corresponding triple (A_0, A_1, A_2) is such that $\operatorname{rk} \begin{pmatrix} A_1 & A_0 \end{pmatrix} = k$ and $\operatorname{rk} \begin{pmatrix} A_0 \\ A_2 \end{pmatrix} = l$.

(2) Deduce from (1) that for $\mathcal{B} = \mathcal{B}(r, d, \lambda)$

$$\mathrm{h}^0(\mathcal{B}) = \begin{cases} d & \text{if } d > 0, \\ 1 & \text{if } d = \lambda = 0, \\ 0 & \text{otherwise.} \end{cases} \mathrm{h}^1(\mathcal{B}) = \begin{cases} -d & \text{if } d < 0, \\ 1 & \text{if } d = \lambda = 0, \\ 0 & \text{otherwise.} \end{cases}$$

(Use the same arguments as in Proposition 6.3.)

Exercise 9.5. Following a similar procedure, prove the analogues of Theorem 9.1 and Exercise 9.4 for a nodal cubic. Namely, let X be a nodal cubic. Then

(1) *If $\mathcal{F}$ is a stable vector bundle, its rank and degree are coprime.*
(2) *If* $\gcd(r, d) = 1$, *there is a* 1*-parameter family* $\mathcal{B}(r, d; \lambda)$ *of stable vector bundles of rank r and degree d parametrized by* $\Bbbk^\times$.
(3) *For* $\mathcal{B} = \mathcal{B}(r, d; \lambda)$

$$\mathrm{h}^0(\mathcal{B}) = \begin{cases} d & \text{if } d > 0, \\ 1 & \text{if } d = 0,\ \lambda = 1 \\ 0 & \text{otherwise.} \end{cases} \mathrm{h}^1(\mathcal{B}) = \begin{cases} -d & \text{if } d < 0, \\ 1 & \text{if } d = 0,\ \lambda = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathrm{Pic}_0 \simeq \Bbbk^\times$ for a nodal cubic [23], this description is again similar to that for elliptic curves.

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